Asset Pricing with Heterogeneous Recursive Preferences

Katharina Greulich

First version: April 2007
Current version: July 2007

This research has been carried out within the NCCR FINRISK project on “Macro Risk, Systemic Risks and International Finance”
Asset Pricing with Heterogeneous Recursive Preferences

Katharina Greulich

First draft: 04/07
Current draft: 07/07

Abstract
We introduce heterogeneity in agents’ risk aversion into a general equilibrium asset pricing framework with recursive preferences. Agents trade in a stock, whose dividend is the only source of consumption, and in a short-term bond in zero net supply. In equilibrium the less risk averse agents are leveraged in the stock, and their share in the economy’s wealth is positively correlated with the dividend shock. Loosely speaking, "average risk aversion" declines when dividend growth is strong, which implies lower expected excess returns. At the same time the price-dividend ratio rises. Thus, in line with the data, a high price dividend ratio predicts low future excess returns. Moreover, predictability of excess returns displays the empirically observed pattern of $R^2$’s rising with horizon. We manage to generate $R^2$’s of similar magnitudes as in the data at all horizons.

Keywords: Asset pricing, heterogeneous agents, equity premium, predictability

1 Introduction

By now a host of stylized facts about asset prices and their dynamics have been established that are hard to match jointly in consumption based asset pricing models. To mention just the ones that will be of concern to this work: Excess returns are high despite the relative smoothness of consumption and at the same time the riskless real rate of interest is very low and stable. These are the famous equity premium and risk free rate puzzles.\(^1\) Another well known property of stock prices is their ‘excess volatility’, i.e. the fact that prices are subject to swings much greater than what seems explicable from changing cash flow and interest rate forecasts.\(^2\) While these puzzles refer to unconditional properties of asset prices and returns, a number of conditional properties of asset prices and returns has been established too. There is now strong evidence that equity premium and Sharpe ratio are high in recessions and low in booms. Moreover, variables such as the price-dividend ratio are subject to similar swings and can serve to predict excess returns. The predictive power of

\(^{\ast}\)Support from the National Centre of Competence in Research “Financial Valuation and Risk Management” (NCCR FINRISK) and the Research Priority Program on Finance and Financial Markets of the University of Zurich is gratefully acknowledged.

\(^{1}\)Contact details: IEW, University of Zurich, Bmlisalpstrasse 10, CH-8066 Zurich. email: greulich@iew.uzh.ch.

\(^{2}\)These terms were coined by Mehra & Prescott (1985) and Weil (1989) respectively.

the price-dividend ratio as measured by the $R^2$ ranges from about 5% at a yearly horizon to more than 50% when excess returns are compounded over 10 or more years.  

There is an extensive literature addressing some or all of these puzzles. However, it seems fair to say that no fully satisfactory framework has been provided to date. Two lines of attack have been pursued in many contributions: One is to endow models with some feature that allows to separate agents’ attitude towards intertemporal substitution from their attitude towards risk. To this end authors have either employed the recursive utility framework that goes back to Kreps & Porteus (1978) or various forms of consumption habits. The other approach has been to introduce market incompleteness, acknowledging the fact that individual consumption is much riskier than aggregate consumption. Campbell & Cochrane (1999) in the habit tradition and Constantinides & Duffie (1996) with incomplete markets are the two most successful contributions to date in that they replicate a large variety of asset pricing facts, but - as stated frankly in Cochrane (2005) - both of them are deliberately reverse-engineered and should be regarded as clever proofs of existence of a solution rather than as the ultimate economic stories settling the issues.

In this paper we therefore take another step towards explaining the aforementioned puzzles through an economically plausible mechanism. We introduce heterogeneity in agents’ risk aversion into a general equilibrium asset pricing framework with recursive preferences. Our economy is endowed with one unit of a stock that produces a stochastic dividend in each period. Two types of agents trade in this stock and in a riskless short-term bond that is in zero net supply such as to adjust their exposure to return risk to their risk preferences. In equilibrium the less risk averse agents are leveraged in the stock, and their share in the economy’s wealth is positively correlated with the dividend shock. Loosely speaking, ”average risk aversion” declines when dividend growth is strong, which implies lower expected excess returns. At the same time the price-dividend ratio rises, provided the intertemporal elasticity of substitution is greater than one. Thus, in line with the data, a high price dividend ratio predicts low future excess returns. Moreover, predictability of excess returns extends over many periods and displays the empirically observed pattern of $R^2$s rising with horizon. Quantitatively, we manage to generate $R^2$s of similar magnitudes as in the data

---

3 See for example Fama & French (1988) and Fama & French (1989).
4 Of course there are other less common approaches, like for example the recent departures from log-normal, random walk dividends as in Bansal & Yaron (2004) and Weitzman (2007).
5 E.g. Epstein & Zin (1989) and Weil (1989). In this type of utility aggregator intertemporal substitutability and risk aversion are governed by separate parameters.
6 Early contributions in this latter vein are Abel (1990), Abel (1999), Campbell & Cochrane (1999) and Constantinides (1990). Habits of the ratio type ($c/h$) make the interest rate less sensitive to risk aversion. Habits of the difference type ($c−h$) allow to choose a low value of the risk aversion parameter that is compatible with a low interest rate by raising the curvature of the marginal utility for a given risk aversion parameter.
at all horizons.

The model just described can only be solved numerically. However, we can analytically solve a set of simpler models that isolate certain features of the full model. These models provide a lot of intuition and help to pin down precisely which aspects of the full model drive which features of the solution. We thus show that both heterogeneous risk aversion and recursive utility are essential ingredients in generating these results. The first simplified model illustrates how heterogeneous risk aversion causes time variation in equity premium and price-dividend ratio. The second model shows why recursive utility allows the price-dividend ratio to predict excess returns with the right (negative) sign, while under standard CRRA utility, which ties together risk aversion and intertemporal willingness to substitute, it would not be possible to get the sign of predictability right unless risk aversion is unreasonably low. Finally, a third simplified version of our model helps to explain why the $R^2$s of our long-horizon regressions rise strongly with horizon, as in the data. For this to happen shocks need to have very persistent effects. We demonstrate that when there is uncertainty about dividends in only one period, the effects of this shock persist into the entire future. In a model with a habit utility function, for example, this would not be the case.

Nevertheless, in our full dynamic framework shocks do not fully persist in asset prices either. The reason is that all the action in the model takes place in the transition to the steady-state in which the less risk averse agents own the entire economy and heterogeneity disappears with all its effects on asset price dynamics. The wealth share of the less risk averse agent grows because he earns a higher expected return rate on his portfolio. However, the return differential between the two agents’ portfolios declines as the less risk averse type’s wealth share increases because his leverage and the equity premium decrease. Thus, once the less risk averse agent is sufficiently rich, his wealth share increases ever more slowly and converges to one. As this convergence becomes strong the paths of the economy after a shock and after no shock become less and less distinguishable and it seems as if the shock had faded away.

As mentioned before, the related literature is huge. We therefore content ourselves with relating our work to a few recent contributions that are similar either technically or in focus. The model of Chan & Kogan (2002) is the one closest to ours. It also features heterogeneity in agents’ risk aversion but utility functions are of the CRRA-type, defined over consumption relative to a slow moving external habit stock. Qualitatively their results are similar, but the predictive power of the

---

8It would be easy to eliminate this feature of the model in favor of a steady state with heterogeneity, in which asset prices and expected returns would continue to vary. One option would be to regard agents as dynasties in which with a certain probability children have different risk preferences than their parents. However, for the sake of clarity of exposition, we prefer to stick to the bare-bones version of the model at this point and leave this extension to future work.
price-dividend ratio for excess returns is negligible in their model and hardly increases with horizon. Moreover, with the help of our simplified models we are able to be much more precise about the intuitions for our results.

To the best of our knowledge the only other contribution to asset pricing featuring heterogeneous risk-aversion in combination with recursive utility is Coen-Pirani (2005). His interest, however, is not in explaining broad sets of asset pricing facts. Rather his point is to show that margin requirements do not necessarily increase the volatility of stock prices. The use of recursive utility seems to be motivated mostly by technical convenience.

Another contribution employing recursive utility but closer to ours in thrust is Bansal & Yaron (2004). The model features a homogeneous agent and separate, exogenous processes for consumption and dividend growth. One main insight is that if these processes contain a small persistent component, which cannot be rejected from the data, moderate degrees of risk aversion are sufficient to generate a high equity premium, and the price-dividend ratio becomes volatile. The other insight is that fluctuating volatility in the driving processes further increases the risk premium and the volatility of the price-dividend ratio because under recursive utility volatility risk is priced. Moreover, the equity premium and the price of risk then become time-varying and predictable from the price-dividend ratio.

Finally, Campbell & Cochrane (1999) is a classic that cannot be left unmentioned. Featuring habit utility of the difference type \( (c - h) \), it manages to generate counter-cyclical excess returns, predictability, and other features by making effective risk aversion counter-cyclical. In terms of its quantitative success at reproducing stylized asset pricing facts it is still the benchmark. However, the structure imposed on the way the habit evolves is very particular. Moreover, it does not solve the equity premium puzzle either in the sense that the risk aversion required to match the equity premium is extremely high.

The remainder of this paper is organized as follows: Section 2 presents our dynamic heterogeneous model. In section 3 we take a step back and analyze our three simplified models in order to gain insights into the forces at work in our full model. Section 4.2 contains our numerical results for the full model. We first discuss the calibration of the model (section 4.1), then its qualitative properties (section 4.2.1), and finally present our quantitative results with a particular focus on long horizon predictability (section 4.2.2). Section 5 concludes.
2 The model

We consider an infinite-horizon exchange economy with two types of agents who differ exclusively in their risk aversion. Time is discrete. The wealth of the economy consists in one unit of a stock that produces a stochastic dividend in the form of a perishable consumption good each period. The dividend process constitutes the only source of uncertainty in this model. To be more specific:

The market structure: Agents can trade in two assets. One is the stock, whose total supply is one unit. Its ex-dividend value is equal to its (ex-dividend) price $P$. The other one is a riskless bond in zero net supply. Dividends $D$ grow stochastically. In logs, they follow a random walk with drift

$$\Delta d' = \log(D'/D) = \mu + \epsilon,$$

where $\epsilon \sim N(0, \sigma^2)$ and i.i.d. over time.

The individual’s maximization problem: Agents of both types aggregate utility from consumption streams according to the recursive preference specification of Epstein & Zin (1989) and Weil (1989). They thus solve the following problem:

$$V_i(W_i, \Gamma) = \max_{\{C_i, \phi_i\}} \left\{ \left[ (1 - \delta)C_i^{1-1/\psi} + \delta[E(V_i(W_i', \Gamma')^{1-\gamma_i})]^{1/(1-\gamma_i)} \right]^{1/\psi} \right\}$$

$$\text{s.t. } W_i' = \left[ \phi_i \frac{P(\Gamma')}{{P(\Gamma)}} + D' + (1 - \phi_i)R'(\Gamma) \right] (W_i - C_i)$$

$$\Gamma' = G(\Gamma)$$

$$W_i' \geq W$$

In words, agents optimally choose their consumption $C_i$ and their portfolio share of the stock, $\phi_i$, conditional on their own wealth $W_i$ and the aggregate state $\Gamma$, respecting their intertemporal budget constraint. This budget constraint states that an agent’s wealth at the beginning of a period $W_i'$ is equal to his investment in the previous period, $W_i - C_i$, times the gross portfolio return of agent $i$ given his portfolio choice $\phi_i$. $W \leq 0$ is some lower bound on wealth which serves to rule out Ponzi schemes. Agents take the price of the stock $P$ and the gross riskless rate of interest $R$ as given and understand how they are determined in equilibrium as functions of the aggregate state $\Gamma$, which in equilibrium evolves according to the law of motion $G$.

We look for solutions where the state vector of the economy only contains the natural states $(D, S_L, B_L)$, with $S_L' = \phi_L(W_L - C_L)/P$ and $B_L' = (1 - \phi_L)(W_L - C_L)$ being the beginning of period.
stock and bond holdings of the agent type with low risk aversion that result from the previous period’s portfolio choice.\footnote{For the numerical solution we will be able to compact this state vector into a single dimension, which is the wealth distribution. See appendix A.}

Note that only $\gamma$, the coefficient of relative risk aversion, is indexed by $i$. This is where the two types differ. The elasticity of intertemporal substitution, $\psi$, on the other hand, is common to the two groups. It is well known that portfolio choice is largely governed by the parameter $\gamma$, while $\psi$ primarily determines the intertemporal consumption profile.\footnote{See Campbell & Viceira (2002), ch. 2.} Thus, our specification of heterogeneity will lead to heterogeneous portfolios. Savings decisions, by contrast, will differ between agents only to the extent that portfolio heterogeneity leads to different portfolio returns.

The distinction between risk aversion and elasticity of intertemporal substitution, which recursive utility allows us to make, is crucial for our results even at a qualitative level. It not only enables us to simultaneously match the equity premium and the risk free rate without resorting to extremely high discount factors.\footnote{We do need a high value of the risk aversion parameter, however.} More importantly, choosing risk aversion and intertemporal elasticity of substitution separately is essential for the price-dividend ratio to predict equity returns with the correct - negative - sign. We will see this in detail in section 3.2. There it will also be shown how things would go wrong with CRRA utility.\footnote{This is not to say that agents have to be homogeneous in their elasticity of intertemporal substitution. Within limits they may also differ in $\psi$. What we need is to be able to choose average risk aversion and average intertemporal elasticity of substitution separately. The role of these choices will be discussed in greater detail in sections 3.2 and 4.1.}

**Competitive equilibrium:** Due to the homotheticity of the utility function and the absence of non-tradable assets, the consumption, stock holdings, and bond holdings of all agents of a certain type are proportional to their wealth level. We can therefore aggregate across all agents of a type such as to ignore individual wealth levels and focus on the wealth held by each group. Equivalently we can think of each group as consisting of a single agent who behaves competitively.

Equilibrium requires market clearing in the markets for consumption, stocks, and bonds. We index by $L$ the variables referring to the type with the lower coefficient of risk aversion and by $H$ those referring to the more risk averse type. The market clearing conditions at each state are then

\begin{align*}
C_L + C_H &= D \quad \text{and} \quad (2) \\
\phi_L(W_L - C_L) + \phi_H(W_H - C_H) &= P. \quad (3)
\end{align*}

Equation (2) ensures clearing of the goods market. Equation (3) is the condition for stock mar-
ket clearing. Bond market clearing follows by Walras’ Law. Note that $W_i$ denotes wealth at the beginning of the period, i.e. after dividends have been paid and before consumption. Therefore, $W_L + W_H = P + D$. But investment takes place after consumption of the dividend, so total investment in the stock has to equal $P$, the ex-dividend value of the stock.

We are now ready to formally define a competitive equilibrium for this economy:

**Definition 1** A **recursive competitive equilibrium** in this economy is a set of consumption and portfolio rules \( \{C_i(\Gamma), \phi_i(\Gamma)\}_{i=L,H} \), as well as a price function \( P(\Gamma) \), an interest rate function \( R'(\Gamma) \) and a law of motion for the state variables \( G(\Gamma) \) such that

1. the allocation solves the individual optimization problem \( \text{[2]} \) for each agent type,
2. the market clearing conditions \( \text{[2]} \) and \( \text{[3]} \) are satisfied, and
3. \( G(\Gamma) \) is consistent with individual choices.

3 Three auxiliary models

The model we have just laid out requires numerical techniques for its solution. In order to gain some analytical insights and intuition for the forces at work, we turn to three simpler models as a first step in our analysis. The first of these auxiliary models is a two period model with heterogeneous agents, which will show how heterogeneous risk aversion leads to time-variation in the equity premium. The second model is the representative agent version of our infinite horizon model, which has closed form solutions for asset prices and returns that illustrate the relationship between equity premium and price-dividend ratio. Finally, as a third simplification, we analyze the model of section \( \text{[2]} \) with dividend risk in only one period. This will shed light on the persistence properties of our model.

3.1 A two-period model with heterogeneous agents

We consider a version of our model with only two investment periods and consumption taking place only after the second investment returns are realized. Even this little two-period model replicates several of the stylized asset pricing facts mentioned in the introduction: The equity premium varies ‘counter-cyclically’ in the sense that high growth rates are associated with low subsequent equity premia. Moreover, the stock price is ‘excessively’ volatile relative to a representative agent model, in which it would be constant. And finally, high prices ‘predict’ low future excess returns.\(^{13}\)

\(^{13}\)This last point may be a bit of a stretch, since there is not much of a future.
The stock can now best be thought of as a tree that grows for two periods before it is chopped down and consumed. The tree’s initial size is one. After the first period it has grown to $X_1$, which is stochastic. Its final size is $X_1 \cdot X_2$, where $X_2$ is again stochastic and independent of $X_1$. Initially, individuals choose portfolios of the tree and a riskless bond in zero net supply, given their initial wealth shares $w_{L,0}$ and $1 - w_{L,0}$. After the realization of $X_1$ they choose portfolios for the second period. The time line in figure 1 illustrates the sequence of events. Since there is no consumption at $t = 0,1$, equilibrium only needs to determine the relative price between stocks and bonds, which will be denoted by $P_t$. The gross return to the bond is normalized to 1.

We do not specify any particular utility function but rather make a few plausible assumptions about the chosen portfolio shares $\phi_{i,t}$:

**Assumption 2** Agents’ optimal portfolio shares of the stock are differentiable in their arguments risk aversion, $\gamma_i$, and equity premium, $EP_t$, with derivatives $\frac{\partial \phi_{i,t}}{\partial \gamma_i} < 0$ (for $EP_t > 0$) and $\frac{\partial \phi_{i,t}}{\partial EP_t} > 0$.

The first assumption implies that for a given equity premium the less risk averse agent type has a greater share of stock in his portfolio. The second one will deliver the comparative statics with respect to the equity premium. Under these assumptions we can prove the following lemma, from which all the properties of asset prices mentioned above follow:

**Lemma 3** If the initial wealth share of the less risk averse agent type, $w_{L,0}$, is sufficiently large or sufficiently small, a high growth rate in the first period, $X_1$, raises the subsequent stock price $P_1$ and lowers the subsequent equity premium $EP_1$.

**Proof.** We derive how the equity premium during the second investment period depends on the realization of $X_1$. The equity premium at this point is equal to the expected return to the stock, $E_1(\frac{X_2}{P_1}) - 1$, due to the normalization of the riskless rate. Hence, trivially, the stock price $P_1$ and the equity premium move in opposite directions. We proceed backwards in two steps. We first turn to the equilibrium at $t = 1$, i.e. when agents make their second portfolio choice, and establish that the equity premium for the second period depends negatively on the L-type’s wealth share at this point, $w_{L,1}$. Then we show that, at least under certain circumstances, $w_{L,1}$ rises with the first period growth rate $X_1$.

Market clearing in the stock market at $t = 1$ requires

$$\phi_1(\gamma_L, EP_1)w_{L,1} + \phi_1(\gamma_H, EP_1)(1 - w_{L,1}) = 1,$$

where $w_{L,1} = \frac{W_{L,1}}{P_1X_1}$ is the wealth share of type $L$. The crucial question at this stage is, what happens
as we vary $w_{L,1}$. By assumption we have $\phi_{L,1} > \phi_{H,1}$ for any given equity premium. Thus a rise in $w_{L,1}$ goes along with an excess demand for stocks, which is eliminated by a rise in $P_1$ that reduces the equity premium such as to lower both types’ portfolio share of stocks. Intuitively, the shift of wealth towards the $L$-type lowers ‘average’ risk aversion in the economy and therefore the equity premium.

Next we show how $w_{L,1}$ depends on the realization of $X_1$, the growth rate of the stock in the first period. Notice that

$$w_{L,1} = \left[ \phi_{L,0} + (1 - \phi_{L,0}) \frac{P_0}{P_1 X_1} \right] w_{L,0}$$

(5)

Moreover, market clearing at $t = 0$ and assumption imply $\phi_{L,0} > 1 > \phi_{H,0}$. Thus, the second term in brackets in equation (5) is negative and $w_{L,1}$ rises with $X_1$, unless $P_1$ falls with an elasticity greater than one in absolute value. Substituting for $w_{L,1}$ in equation (4) from equation (5) and differentiating, we can calculate this elasticity as

$$\varepsilon(P_1, X_1) = \frac{(\phi_{L,1} - \phi_{H,1})(\phi_{L,0} - 1)w_{L,0} P_0}{(w_{L,1} \frac{d\phi_{L,1}}{dEP_1} + (1 - w_{L,1}) \frac{d\phi_{H,1}}{dEP_1})EP_1 - (\phi_{L,1} - \phi_{H,1})(\phi_{L,0} - 1)w_{L,0} \frac{P_0}{P_1 X_1}}$$

(6)

The numerator and the second term in the denominator are equal and positive, and the first term in the denominator is positive as well. Thus we will find $\varepsilon(P_1, X_1) > 0$ if and only if $(w_{L,1} \frac{d\phi_{L,1}}{dEP_1} + (1 - w_{L,1}) \frac{d\phi_{H,1}}{dEP_1})EP_1 > (\phi_{L,1} - \phi_{H,1})(\phi_{L,0} - 1)w_{L,0} \frac{P_0}{P_1 X_1}$. This will for sure be the case for $w_{L,0}$ sufficiently close to zero or one, for in either case the right hand side of this condition goes to zero, while the left hand side is strictly positive as long as the $L$-type is not risk neutral. (Note that as $w_{L,0} \to 1, \phi_{L,0} \to 1$.) Thus, to recap, for $w_{L,0}$ small enough or large enough the first period growth rate $X_1$ affects the stock price $P_1$ positively and the equity premium $EP_1$ negatively. ■

The proof has shown clearly that at the heart of the variability of the equity premium and the (excess) volatility of the stock price there is a relative wealth effect between the agent types. It exists because they choose different portfolios. For comparison, if agents were not heterogenous, i.e. if we had $\phi_{L,t} = \phi_{H,t}$, we would have $\varepsilon(P_1, X_1) = 0$, and consequently the equity premium $EP_1$ would be constant.

Alternatively, we can interpret what is going on in this model as consumption insurance. The less risk averse agent type insures the more risk averse type. His share of final consumption will hence be higher the bigger total consumption $X_1 \cdot X_2$. Notice that the expectation of final consumption at $t = 1$, $E_1(X_1 \cdot X_2)$, rises with the first period growth rate $X_1$. Consequently, $L$-types should expect
a higher share of final consumption after high realizations of $X_1$ than after low ones. This is the case if their share of wealth, $w_{L,1}$, is higher after high $X_1$.

3.2 The infinite horizon model with a representative agent: comparative statics

In the two period model we saw how with heterogeneous agents shocks effectively shift the degree of risk aversion of the economy. In this section we will use this insight to make a short-cut that permits an analytical investigation into the nature of predictability in the infinite horizon model. While our dynamic heterogeneous agent model of section 2 is not amenable to analytical solution, its representative agent version features simple closed form solutions for asset prices and returns. We state these and do comparative statics with respect to the coefficient of relative risk aversion. Effectively, we ignore the non-linearities that prevent aggregation in the heterogeneous agent model and discuss the effect of a shock that once and for all shifts the wealth distribution such as to precipitate a certain change in average risk aversion. This analysis illustrates that working with recursive utility rather than with the more standard CRRA form is crucial in order for the price-dividend ratio to predict excess returns with a negative sign, as in the data.

If we eliminate heterogeneity from the model of section 2 but maintain the assumption that dividend growth is i.i.d. we can derive analytical expressions for asset prices and returns. The equity premium equals

$$EP = \gamma \sigma^2.$$  \hspace{1cm} (7)

The gross risk free rate is determined as

$$R = \delta^{-1} \exp \left[ \frac{1}{\psi} (\mu + \frac{1}{2} \sigma^2) - \frac{1}{2} (1 + \frac{1}{\psi}) \gamma \sigma^2 \right].$$  \hspace{1cm} (8)

And finally, the price-dividend ratio can be expressed as

$$PD = \frac{\hat{\delta}}{1 - \hat{\delta}},$$  \hspace{1cm} (9)

where

$$\hat{\delta} = R^{-1} \exp[-EP] \exp[\mu + \frac{1}{2} \sigma^2] = \delta \exp[(1 - \frac{1}{\psi})(\mu + \frac{1}{2} \sigma^2)] \exp[(\frac{1}{\psi} - 1) \frac{1}{2} \gamma \sigma^2].$$

---

14If $X_1$ did not contain any information about the distribution of final consumption this would not be the case.

15At this level of generality we cannot rule out the alternative that $w_{L,1}$ is actually lower and subsequent portfolio returns are higher. But this case seems rather unintuitive and never arose in the numerical solutions to the dynamic model.
Equations (7) to (9) show that equity premium, risk free rate and price-dividend ratio are all constant in the representative agent model, i.e. there is no predictability. Nevertheless, their comparative statics with respect to the coefficient of relative risk aversion, \( \gamma \) mimic how predictability and the like arise in the heterogeneous agent version.

Turning to these comparative statics, the equity premium clearly increases in risk aversion \( \gamma \), while the risk free rate falls. The latter is a consequence of the increasing precautionary savings motive. Being the expected discounted sum of all future dividends, discounted at the risk adjusted rate and normalized by the current dividend, the price-dividend ratio depends negatively on both the risk free rate and the equity premium. An increase in \( \gamma \) lowers the risk free rate but raises the equity premium, as equations (7) and (8) reveal. The reaction of the price-dividend ratio is therefore ambiguous. Whether the fall in \( R \) or the rise in \( EP \) dominates depends on the intertemporal elasticity of substitution, \( \psi \). The rise in the equity premium dominates and the price-dividend ratio falls if and only if \( \psi > 1 \). This is the case we have to focus on if we want to generate predictability with the right sign in our heterogeneous agent model. For, empirically, a high price-dividend ratio predicts low excess returns.\(^{16}\)

Such a high value for the intertemporal elasticity of substitution seems to fly in the face of micro-evidence suggesting that \( \psi \) should be close to zero. (Hall 1988) However, the choice of \( \psi > 1 \) has other arguments in its favor beyond the pattern of excess return predictability. For example, we know that the interest rate is very stable. But low values of the intertemporal elasticity of substitution make the interest rate very reactive to small variations in expected consumption growth. (Cochrane (2005), chapter 21) Taking the same short-cut that we have taken above for the case of changes in \( \gamma \) again for changes in the growth rate \( \mu \), this can also be seen from equation (8). Also, at \( \psi < 1 \) we would have that the price-dividend ratio is lower the stronger growth, which is counterfactual. (See equation [9].) In sum, in the case of the intertemporal elasticity of substitution, like in that of the coefficient of risk aversion, it seems that micro-evidence and the values implied by asset prices are plainly incompatible. In the branch of the asset pricing literature that focuses on conditional moments of asset prices rather than on the equity premium puzzle the micro-evidence is therefore typically ignored. High elasticities of intertemporal substitution (e.g. Bansal & Yaron (2004)) and high degrees of risk aversion (e.g. Campbell & Cochrane (1999)) are employed as need be to match empirical properties of asset prices.

What would happen with CRRA-utility? Imposing \( \gamma = 1/\psi \) in equation (9) and differentiating

\(^{16}\)In a different model Bansal & Yaron (2004) work with an elasticity of intertemporal substitution greater one for the same reason.
with respect to the risk aversion coefficient shows that the price-dividend ratio predicts excess returns with the right sign only if $\gamma\sigma^2 < \mu + \sigma^2$, i.e. only if the equity premium is smaller than a number that is hardly bigger than the average growth rate of the economy. This would not come out of any reasonable calibration, for the right hand side of this condition will typically be around 1-3%, while estimates of the equity premium are always in excess of 4%. Thus, to recap, we need to employ recursive utility with an intertemporal elasticity of substitution in excess of one in order to generate a negative relationship between the price-dividend ratio and future excess returns, as observed empirically.

3.3 Uncertainty in a single period

We now return to our dynamic heterogeneous agent model of section 2, however, there will be uncertainty about dividends in one period only. The purpose of this simplification is to show analytically that shocks have permanent effects on the allocation of consumption between agents and thus on the wealth distribution. We first argue that after the resolution of uncertainty agents consume the same fraction of the dividend forever. Then we show that this fraction depends on the realization of the dividend in the period of uncertainty. In this simplified setting we cannot make meaningful statements about the effect of the shock on asset prices because after the resolution of uncertainty the stock is a safe asset whose price is not influenced by agents’ risk attitudes. However, we take the shift in the wealth distribution as an indication that in a model with repeated uncertainty we would observe permanent asset price effects through the shift in average risk aversion.

For the sake of precision and clarity we frame our discussion in the following lemma, which will be proved in turn:

**Lemma 4** Let dividends be log-normally distributed in period $s$ and let them grow at a constant rate $\mu$ thereafter. Then the realization of the dividend shock, $D_s$, permanently affects the distribution of consumption and hence of wealth between agents.

**Proof.** For all $t \geq s$ markets are trivially complete and stochastic discount factors equalize across

---

17 The expected growth rate would be $\mu + \frac{1}{2}\sigma^2$, but $\sigma^2$ is an order of magnitude smaller than $\mu$ and hence does not matter much.

18 Notice that our reasoning is true only for pure CRRA utility. CRRA with a habit can yield the right correlation between price-dividend ratio and equity premium because the habit introduces an element of mean reversion: a good shock raises the surplus of consumption over its habit level but creates the expectation of a reduction in the surplus as the habit adjusts. This lowers the interest rate and raises the price dividend ratio. If at the same time a force such as agent heterogeneity lowers average risk aversion the resulting correlation between price-dividend ratio and equity premium is negative as in the data. This is the - unstated - intuition behind the results of Chan & Kogan (2002).
agents. The stochastic discount factor of agent type $i$ is

$$m_{i,t+1} = \delta \left( \frac{U_{i,t+1}}{E_t(U_{i,t+1})} \right)^{1/\gamma_i} \left( \frac{C_{i,t+1}}{C_{i,t}} \right)^{-1/\psi}. \quad (10)$$

For all $t \geq s$ the terms involving $U$ cancel and the condition $m_{L,t+1} = m_{H,t+1}$ collapses to

$$\frac{C_{L,t+1}}{C_{L,t}} = \frac{C_{H,t+1}}{C_{H,t}}, \quad \forall t \geq s. \quad (11)$$

Hence, agents consume the same fraction of the dividend forever from period $s$ on. Call the fraction consumed by the less risk averse type $c_L = C_L/D$.

It remains to be shown that $c_L$ depends on $D_s$. To do so we will use the two pricing conditions for the stock and the bond, which are

$$E_t(m_{i,t+1} \frac{P_{t+1} + D_{t+1}}{P_t}) = 1 \quad \text{and} \quad (12)$$

$$E_t(m_{i,t+1} R_{t+1}) = 1, \quad (13)$$

and the fact that they have to be satisfied for both agents. Given the assumption that dividends grow at a constant rate $\mu$ after period $s$, we can write utility as $U_{i,s} = \left( \frac{1-\delta}{1-\delta \exp[\mu]} \right)^{\frac{s}{\gamma_i}} C_{i,s}$, and the stochastic discount factor simplifies to $m_{i,s} = k_i C_{i,s}^{1-\gamma_i}$, where $k_i$ summarizes all terms that are determined before time $s$. Also, the stock price $P_s$ will be proportional to $D_s$.

From now on we proceed by contradiction and show that the distribution of consumption cannot be independent of $D_s$. Suppose to the contrary that irrespective of the realization of $D_s$ the less risk averse type has consumption share $c_L = c_L$. Then setting the left hand sides of equations (12) and (13) equal for both agent types and canceling some terms we get

$$k_L c_L^{1-\gamma_L} E_{s-1}(D_s^{-\gamma_L}) = k_H (1 - c_L)^{-\gamma_H} E_{s-1}(D_s^{-\gamma_H}) \quad \text{and} \quad \frac{k_L c_L^{1-\gamma_L} E_{s-1}(D_s^{1-\gamma_L})}{k_H (1 - c_L)^{-\gamma_H} E_{s-1}(D_s^{1-\gamma_H})},$$

which implies

$$E_{s-1}(D_s^{-\gamma_L}) = E_{s-1}(D_s^{-\gamma_H}). \quad (14)$$

Using the log-normality of $D_s$, this expression can be simplified to yield

$$\gamma_L = \gamma_H$$
which contradicts our assumption that agents are heterogeneous and $\gamma_L < \gamma_H$.

Thus $c_L$ depends on $D_s$. Moreover, the wealth of agent $i$ at time $s$ is

$$W_{i,s} = \frac{1}{1-\delta} U_{i,s}^{1/\psi} C_{i,s}^{1/\psi} = \frac{C_{i,s}}{1 - \delta \exp[\mu]}$$

which implies that the wealth share of the less risk averse agent type is simply $w_L = c_L$. This completes the proof.

At this level of generality it seems impossible to make stronger statements. Under the assumption that the dividend process can take only two realizations such as to make markets complete at all times\footnote{Appendix B discusses conjectures about the general relationship between our model and its complete markets version.} one can further show that the wealth and consumption shares of the L-type increase in $D_s$, which is in line with our findings for the two period model of section 3.1.

As mentioned before, the wealth shift that is precipitated by the shock in this simple model has no consequences for asset prices because after the shock there is no more uncertainty and hence risk aversion does not affect the pricing of the stock. Nevertheless, to the extent that the strong persistence property that we have found carries over to the full model, it is good news for the potential of the model to generate long horizon predictability. For it suggests that a shock will change expected excess returns not only today but for the entire future. For comparison, under CRRA with habits, as employed for example in Chan & Kogan (2002), a one-off shock to the level of dividends would have only transitory effects because everything mean-reverts as the habit catches up with consumption.

4 Numerical analysis

The three auxiliary models of section 3 have provided us with intuition and analytical insights into the mechanisms at work in our model. Against this backdrop we can now present the numerical results for the full dynamic model. These will be both qualitative and quantitative in nature. We first discuss the calibration we choose.

4.1 Calibration

We fix the parameters of our model on the basis of the annual US-data for the years 1890-1995 that is also used in Campbell (2003)\footnote{This data can be downloaded from http://kuznets.fas.harvard.edu/~campbell/data.html.}. The consumption data refers to real per-capita consumption of 19
non-durables and services, stock market data is based on the S&P 500, and the real interest rate is derived by deflating six-months commercial paper, bought in January and rolled over in June\textsuperscript{22}.

The first important choice we have to make regards the parameters of the dividend growth process. While in reality dividends are only a small, but very volatile component of consumption, in our model dividends and aggregate consumption are identical. Unlike contributions like Campbell & Cochrane (1999), which work with a representative agent, we have to impose this equilibrium condition and cannot simply price the stock given separate processes for consumption and dividends\textsuperscript{23}.

In keeping with the related literature\textsuperscript{24} we opt for using consumption data for the main calibration of the model. (See the first column of table 1.) As a check we also use dividend data. (See the second column of table 1.) Unfortunately, however, we cannot solve the model with dividend data for comparable levels of agent heterogeneity. The dilemma in the choice between consumption and dividend data is the following: If we calibrate the dividend process to consumption data, it will be very hard to get the volatilities of price-dividend ratio and returns even approximately right because consumption is much less volatile than dividends\textsuperscript{25}. On the other hand, if we base our calibration on dividend data, consumption becomes far too volatile, such as to seemingly resolve the equity premium puzzle at low levels of risk aversion.

We next explain how we fix the preference parameters of our model. A summary of the resulting parameter values is contained in table 1. We choose average risk aversion\textsuperscript{26}, time preference rate, and elasticity of intertemporal substitution such as to match average excess returns, the average

\begin{table}[h]
\centering
\begin{tabular}{|l|l|c|c|}
\hline
\textbf{Parameter} & \textbf{Variable} & \textbf{Benchmark calibration} & \textbf{Dividend calibration} \\
\hline
Mean consumption growth & $\mu$ & 1.7\% p.a. & 1.25\% p.a. \\
Std. dev. of consumption growth & $\sigma$ & 3.3\% p.a. & 12.8\% p.a. \\
Discount factor & $\delta$ & .96 & .96 \\
Intertemporal elasticity of substitution & $\psi$ & 1.5 & 1.5 \\
Average risk aversion & $\bar{\gamma}$ & 42.74 & 3.31 \\
L-type’s risk aversion* & $\gamma_L$ & 36 & 2.335 \\
H-type’s risk aversion* & $\gamma_H$ & 97.51 & 5.683 \\
\hline
\end{tabular}
\caption{Benchmark Parameter Values}
\end{table}

\textsuperscript{*}These risk aversion levels translate into the above average risk aversion if on average the L-type owns 75\% of the wealth.

\textsuperscript{22}For greater detail consult the file readmeus.txt in the data set.
\textsuperscript{23}An - albeit complicated - solution would be to work with leveraged equity and bonds in endogenous positive supply. This avenue will be followed in future work.
\textsuperscript{24}E.g. Chan & Kogan (2002).
\textsuperscript{25}In fact, matching the empirical values of these volatilities would be of questionable desirability since it would mean to introduce too much heterogeneity. This criticism applies for example to Chan & Kogan (2002).
\textsuperscript{26}How average risk aversion relates to $\gamma_L$ and $\gamma_H$ will be discussed below.
safe rate, and the average price-dividend ratio in our data set. Instead of iteratively simulating our model to match the empirical values, we take the short-cut of inferring the parameters from the representative agent version of section 3.2, which yielded analytical solutions for the equity premium, the risk free rate, and the price-dividend ratio. Specifically, average risk aversion, call it $\bar{\gamma}$, is chosen to match the equity premium. Since in our data we measure the average log excess returns we have to adapt equation (7), which refers to levels. The resulting expression is $(\bar{\gamma} - 1/2)\sigma^2$. The risk free rate and the price-dividend ratio cannot be determined separately, as equations (8) and (9) reveal. We therefore fix $\delta$ at .96 per year, in keeping with a lot of macroeconomic literature, and choose $\psi$ such as to approximately match both $R$ and $PD$. The resulting values are reported in table 2 together with other unconditional moments. Admittedly, this method only permits an approximate matching of the asset pricing data in our full model, however, as we will see the deviations are an order of magnitude below the variation in parameter estimates from the different data sets that have been used in the literature.

The representative agent model only helps to choose average risk aversion. How does this concept relate to the individual risk aversions of the two agent types? It turns out that defining average risk aversion as the harmonic mean of type specific risk aversions, weighted by the types’ wealth shares, yields a good approximation of the equity premium. To see why this is so think of the representative agent model as the limit of our heterogeneous agent model as $\gamma_L$ and $\gamma_H$ converge. In this limit consumption and returns are log-normal and i.i.d., such that the rules of myopic portfolio choice apply and we have

$$\phi_i = \frac{EP}{\gamma_i \sigma^2}. \quad (15)$$

In the neighborhood of the homogeneous agent limit returns will not be too far from log-normal and i.i.d., so this portfolio rule will still be approximately true. Use it in the condition for market clearing in the stock market, $\phi_L w_L + \phi_H (1 - w_L) \equiv \overline{\phi} = 1$, and cancel terms to get

$$\frac{1}{\gamma_L} w_L + \frac{1}{\gamma_H} (1 - w_L) = \frac{1}{\bar{\gamma}}. \quad (16)$$

Equation (16) suggests that, once we have fixed $\bar{\gamma}$, there are still two free parameters to pin down. However, the wealth distribution between agents is not really a choice. Rather, we can only fix the initial wealth distribution. As the model is fed with shocks, it evolves endogenously, and that in a non-stationary manner. The non-stationarity of the model will be discussed below in section

---

28To be precise, $w_L$ must be the post-consumption wealth distribution.
4.2 In choosing the free risk aversion parameter we strive to generate enough agent heterogeneity to quantitatively match the long horizon predictability we find in our data. The resulting values, $\gamma_L = 36$ and $\gamma_H = 98.14$ for our benchmark calibration, may seem very far apart, suggesting that we impose a huge amount of heterogeneity. However, judging from the resulting portfolios this is not true, as we will see in section 4.2.2. The values for the calibration to dividend data are as far apart as numerically possible.

To generate the artificial asset pricing data necessary for calculating moments and running predictive regressions, we use repeated simulations of our model at quarterly frequency over a period of 100 years, roughly corresponding to the length of the data set. We alternatively tried simulating at annual or monthly frequencies, neither of which changes our results in a significant way. The higher the frequency, the smaller is the departure from dynamically complete markets, but the higher the computational effort. This is why we decided in favor of quarters as a middle way.

4.2 Results

We present the results for our full model in two steps. Section 4.2.1 is of a mostly qualitative nature and serves to illustrate the wealth effect in our full model. The quantitative assessment of our model is deferred to section 4.2.2 where we present and discuss statistics obtained by simulating the model.

4.2.1 Heterogeneity and the wealth effect

In the following we elicit how and to what extent the insights from the three auxiliary models carry over to the dynamic heterogeneous agent case. To this end we graphically illustrate how various aspects of the solution such as the price-dividend ratio and the equity premium depend on the distribution of wealth between the two agent types. The wealth distribution, summarized by the wealth share of the less risk averse agent type $w_L$, is the only state variable in our numerical solution. Appendix A provides details on our choice of state space and the numerical solution.

Figure 2 traces out the equity premium and the price-dividend ratio as functions of the share of wealth held by the agent type with low risk aversion for our benchmark calibration. As predicted by the two-period model of section 3.1 the more wealth is held by the less risk averse agents, the lower is the risk premium. The price-dividend ratio, by contrast, rises with the wealth share of the L-types. This reflects our choice of an intertemporal elasticity of substitution greater than one, as discussed in the context of the representative agent model of section 3.2. Note that in the limit as $w_L$ approaches zero or one the values of equity premium and price-dividend ratio equal their
representative agent counterparts for risk aversions $\gamma_H$ and $\gamma_L$ respectively. This is not surprising because at these limits heterogeneity disappears and we are in the respective representative agent worlds.

Correspondingly, at the extremes of the set of wealth distributions, the portfolio share of the stock of the agent type who holds all the wealth approaches one. This is illustrated in figure 3. At the same time the other agent type’s portfolio share approaches $\frac{\gamma_j}{\gamma_i}$, where $j$ is the wealthy agent. At interior wealth distributions, on the other hand, there is true heterogeneity, and the less risk averse agents hold leveraged positions in the stock, while the more risk averse ones hold a mixture of stock and bonds. Thus, the less risk averse agents take on a disproportionate amount of risk, in line with the insurance considerations that we discussed in section 3.1. Combining figures 2 and 3 we can now infer for the dynamic model the result that we proved in lemma 3 for the two period illustration: Since the less risk averse agents are leveraged in the stock (figures 3 and 4), their wealth share increases after good shocks, which lowers expected future excess returns and raises the price-dividend ratio (figure 2). Note that in speaking of wealth shares, we mean the wealth distribution before consumption. A good shock shifts pre-consumption wealth towards the low risk aversion types, and pre-consumption $w_L$ is negatively (positively) related to the equity premium (the price-dividend ratio). Hence we can be sure that consumption choices, which were absent in the two period model, do not pervert the effect.

In section 3.2 we invoked the simple solutions to the representative agent version of our model and did comparative statics with respect to the coefficient of risk aversion, arguing that the effects would be similar to those of varying the wealth distribution in the heterogeneous agent version. We now investigate how far this similarity goes. To this end we compare the equity premium, the Sharpe ratio, and the volatility of excess returns (all in dependence on $w_L$) to their counterparts in the representative agent model for different risk aversions. Figure 5 provides a graphical comparison. The solid lines represent the graphs for the heterogeneous agent model, the dashed ones are the result of specifying risk aversion according to equation (16) in the representative agent version of section 3.2.

In the first panel the two graphs are virtually identical and hence overlap. Thus for the Sharpe premium, the L-type’s leverage does not monotonically decrease in his wealth share even though the equity premium does so. This feature could be due to the deviations from i.i.d. log-normality of returns and consumption that are caused by heterogeneity. (See discussion below.) The amount of stock held by the L-type, on the other hand, increases smoothly with his wealth share. (See figure 4.) We checked very carefully that the non-monotonicity in the portfolio graph is not due to numerical inaccuracies. In particular, we tried approximating different objects. Also, we made sure that the risk of bankruptcy, which theoretically exists in discrete time unlike in the continuous time limit, does not cause the hump. To this end we solved the model at monthly and smaller trading intervals. The hump does not go away but if anything becomes slightly bigger as we get closer to continuous time.

This follows from equation (15), which implies $\phi_i = \frac{\gamma_j}{\gamma_i} \phi_j$, and the fact that $\phi_j = 1$.

To illustrate what we do: E.g. the dashed line for the equity premium represents equation (7) with equation (16).
ratio, i.e. for the price of risk, we practically have aggregation. That is to say, the price of risk in the heterogeneous agent economy with a wealth distribution described by \( w_L \) and type specific risk aversion levels \( \gamma_L \) and \( \gamma_H \) is identical to the price of risk in a representative agent economy with risk aversion \( \gamma = \left[ \frac{1}{\gamma_L} w_L + \frac{1}{\gamma_H} (1 - w_L) \right]^{-1} \). The equity premium, on the other hand, is always higher in the heterogeneous agent economy, as the second panel shows. This can be explained from the graph for the volatility of excess returns at the bottom of the figure. Heterogeneity introduces extra volatility in the economy, which implies a higher equity premium for a given price of risk. In the representative agent economy the risk free rate is constant and so is the price-dividend ratio, such that the volatility of excess returns is equal to the volatility of dividends. Heterogeneity introduces volatility into the price dividend ratio (and to a minor extent into the risk free rate), which makes excess returns more volatile than dividends.\(^{32}\)

Excess return volatility in the heterogeneous agent model is hump shaped, again with the limits at the two extremes of the wealth distribution being identical to the respective representative agent versions. It is a good indicator of the strength of the wealth shifts between the agents in different regions of the wealth distribution. For, as mentioned previously, return volatility in excess of the volatility of dividend growth is due to volatility in the price-dividend ratio, which in turn comes about by shifts in wealth between the two agent types. (See figure 2, panel 1.) Thus a given dividend shock has the strongest effect on the wealth distribution in its intermediate range. To see why this is so recall figure 3, which depicts the share of the stock in agents’ portfolios across the range of wealth distributions. When the L-type owns almost all wealth his portfolio displays only little leverage. Hence shocks do not affect the wealth distribution very much. At the other extreme, when the L-type is very poor, he is very leveraged and thus very exposed to return risk, however, since he has so little wealth, the wealth distribution does not move very much in absolute terms either. Hence, in order to have a lot of effective heterogeneity for a given choice of risk aversion levels we should focus on intermediate, but lower rather than higher, values for \( w_L \). This insight will guide us in the choice of initial wealth distribution for our simulations.

In section 3.2 we also argued that it was crucial to choose a value of the intertemporal elasticity of substitution in excess of one in order for the price-dividend ratio to predict excess returns with the right sign. Figure 6 confirms that this insight from the representative agent model carries over to the model with heterogeneity. The three panels graph the price dividend ratio, the equity premium and plugged in for \( \gamma \), i.e. \( EP = \left[ \frac{1}{\gamma_L} w_L + \frac{1}{\gamma_H} (1 - w_L) \right]^{-1} \sigma^2 \).

\(^{32}\)Still, at least for our calibration to consumption data and for the degree of heterogeneity we have imposed, the volatility of excess returns is far below its empirical counterpart of 18.5% per year. This is why the Sharpe ratio is far too high when the equity premium is in the right range.
the volatility of excess returns as functions of the wealth distribution. Our benchmark calibration is depicted in dots. For comparison we plot the same functions for elasticities of substitution of one (in dashes) and .75 (solid lines). For the latter the price-dividend ratio falls as the wealth share of the less risk averse agents rises, while the curve for the equity premium retains its negative slope across all values of $\psi$. Thus a positive shock that increases $w_L$ will lower both price-dividend ratio and future excess returns, contrary to the evidence on predictability. The fact that shocks and price dividend ratio are negatively correlated when the elasticity of intertemporal substitution is less than one also explains why in this case the volatility of excess returns is actually lower than without heterogeneity. When $\psi$ is exactly one the price-dividend ratio is constant. Correspondingly, the volatility of excess returns is constant too at the level of the volatility of dividend growth and the behavior of the equity premium is indistinguishable from its counterpart in our ‘mock-heterogeneity’ model in which we vary risk aversion in the representative agent model.

A fundamental difference between the dynamic heterogeneous agent model and its simplified versions of sections 3.2 and 3.3 regards stationarity. As mentioned previously, our model is non-stationary in the sense that the wealth distribution drifts over time. For realistic calibrations the wealth share of the less risk averse agents approaches one in the long run, and the only stable stationary steady state is at $w_L = 1$, i.e. in the representative agent limit with $\gamma = \gamma_L$. Why is this so? Since the L-types have a higher share of stock in their portfolios and stocks command a return premium, they earn a higher return on their invested wealth. Their propensity to consume out of wealth, on the other hand, is lower precisely because their portfolio returns are higher and the elasticity of intertemporal substitution is greater than one. Both forces lead the less risk averse agents to accumulate wealth faster than the more risk averse ones. However, the average rate of change of the wealth share of the less risk averse agent type is not constant. Rather, the pattern is hump shaped, similarly to that of the volatility of excess returns. The reasons are also similar: While the return advantage of the less risk averse agents tends to decline as their wealth share grows, this effect is more than compensated at low levels of $w_L$ by the increase in wealth on which this return advantage acts.

33 Nevertheless, even in this special case there is no obvious aggregation result for the true heterogeneous agent model. The correspondence between the ‘mock’ and ‘true’ versions is exact only for the price dividend ratio and stock returns. To see this first note that with a unit elasticity of substitution income and substitution effects exactly cancel and each agent’s consumption is a constant fraction $1 - \delta$ of his wealth. Market clearing for the consumption good then implies a constant price-dividend ratio, which in turn implies that stock returns are log-normal and i.i.d. Nevertheless, due to the time variation in the interest rate portfolio choice is not myopic.

34 For certain combinations of (very low) risk aversion coefficients it would be possible for the more risk averse agents to dominate in the long run. In those cases, even though the less risk averse agents will always earn higher expected simple returns on wealth, their log returns will be lower. See Coen-Pirani (2004) for details.

35 With CRRA utility this effect would be even more pronounced because the less risk averse agents would also be more willing to substitute intertemporally.
The drift in the wealth distribution also has consequences for the persistence properties of our model. Recall that in our simplified model of section 3.3 where dividends were shocked only once, those shocks had fully persistent effects on the economy. The drift in the model with recurrent dividend uncertainty takes away some of that persistence. To see this, note that, like the wealth share of the L-type, all asset price moments converge to their counterparts in the representative agent model with \( \gamma = \gamma_L \) in the long run. Like the wealth distribution, they move strongly as long as effective heterogeneity is big, i.e. in that intermediate range of \( w_L \) referred to previously, and then change ever more slowly to approach their steady state values asymptotically. Now consider a positive shock. All this shock does is to take the economy a little closer to its long run steady-state. This changes asset prices and their moments noticeably at first, but the effect decreases over time and vanishes asymptotically as the economy approaches \( w_L = 1 \). Figure 7 illustrates this behavior through a set of impulse responses. To obtain them we simulated the economy, starting at \( w_L = 0.1 \), setting all shocks to zero except for one after 10, 50 or 90 quarters, which we set to one standard deviation. The responses are graphed as the difference between time series with and without the respective impulse. This representation shows very clearly how the effect vanishes over time. Moreover, shocks that occur early on are more persistent than later ones and can even build up. This is because they occur at a time when the moderating effects of convergence are not dominant yet.

It is clearly an extreme implication of our model that predictability is an entirely transitory phenomenon. It could be avoided by viewing agents as dynasties whose future generations will have different risk preferences with a certain probability. This modification would introduce an element of mean reversion and hence allow for an interior stochastic steady state in which the wealth distribution and hence the equity premium and the price-dividend ratio still respond to dividend shocks. It would come at the price of taking away some persistence and thus some long horizon predictability. For the sake of clarity of exposition and in the light of a recent trend to view the past decades of asset pricing data featuring a high but declining equity premium as a transitory period rather than as a steady state\(^{36}\), we prefer to evaluate the model during the transition and regard the extension to dynasties as an extension left for future work.
Note: The model is simulated at a quarterly frequency. Statistics are calculated from time-averaged data at an annual frequency. All returns are annual percentages. Small letter are logs.

Table 2: Unconditional statistics of simulated and historical data

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Data</th>
<th>Model ( w_{L,0} = 0.1 )</th>
<th>Model ( w_{L,0} = 0.2 )</th>
<th>Model ( w_{L,0} = 0.3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E(\Delta c) )</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
<td>1.7</td>
</tr>
<tr>
<td>( \sigma(\Delta c) )</td>
<td>3.3</td>
<td>3.3</td>
<td>3.3</td>
<td>3.3</td>
</tr>
<tr>
<td>( E(r_f) )</td>
<td>1.7</td>
<td>1.3</td>
<td>1.4</td>
<td>1.5</td>
</tr>
<tr>
<td>( \sigma(r_f) )</td>
<td>5.1</td>
<td>0.9</td>
<td>0.7</td>
<td>0.5</td>
</tr>
<tr>
<td>( E(r - r_f) )</td>
<td>4.6</td>
<td>4.9</td>
<td>4.7</td>
<td>4.6</td>
</tr>
<tr>
<td>( \sigma(r - r_f) )</td>
<td>18.5</td>
<td>3.6</td>
<td>3.5</td>
<td>3.5</td>
</tr>
<tr>
<td>( E(PD) )</td>
<td>22.7</td>
<td>23.4</td>
<td>23.5</td>
<td>23.6</td>
</tr>
<tr>
<td>( \sigma(PD) )</td>
<td>6.2</td>
<td>1.0</td>
<td>0.8</td>
<td>0.7</td>
</tr>
<tr>
<td>( \rho(PD, PD_{-1}) )</td>
<td>0.81</td>
<td>0.85</td>
<td>0.78</td>
<td>0.70</td>
</tr>
</tbody>
</table>

4.2.2 Simulation results

In order to compare our model to the data we simulate it for 1000 series of 100 years (400 quarterly draws of the shock) each.\(^{37}\) Table 2 reports basic unconditional statistics. The columns below 'Model' refer to our benchmark calibration, simulated from the indicated initial wealth distributions. As explained in section 4.1, we do not strive to match prices and returns to a high precision because data statistics vary so widely between data samples. Also, as table 2 clearly shows, due to the drift inherent in our model the exact values depend on the choice of starting value. So the way to interpret table 2 is as showing that we are rather close to the data for the means and for the autocorrelation of the price-dividend ratio, but far off in terms of the volatilities. We consider problematic only the low volatilities of excess returns and price-dividend ratio. The historical standard deviation of the risk free rate seems very high anyway and is likely due to \textit{ex post} inflation. Campbell (2003) argues that the volatility of the \textit{ex ante} real risk free rate should be very low. The low volatilities of excess returns and price-dividend ratio are at least partly due to our choice of data for the driving process. We work with consumption data, even though dividends are much more volatile than consumption.\(^{38}\) The fact that excess returns are more volatile than consumption growth and that the price dividend ratio covaries positively with consumption growth (otherwise excess returns would be less volatile than \( \Delta c \)) confirms that our model goes in the right direction of creating "excess volatility". This reiterates what we saw graphically in section 4.2.1.

\(^{36}\)See for example Cogley & Sargent (2005).

\(^{37}\)We use these short run simulations instead of a long run simulation for the same reasons as Adam, Marcet & Nicolini (2006): Heterogeneity and hence predictability is a transitory phenomenon in the model and vanishes in the long run. See Adam et al. (2006) for details on the method.

\(^{38}\)We discuss this choice with its pros and cons in section 4.1.
Table 3 presents long-horizon regressions of log excess stock returns on the log price-dividend ratio in historical and simulated data. We report the slope coefficients and the $R^2$s up to horizons of 10 years. As mentioned in section 4.1, we choose the spread between the two types’ risk aversion coefficients such as to generate predictability, as measured by the $R^2$, of magnitudes similar to those we find in the data. The estimation coefficients on the artificial data tend to be higher than in the historical data. Unsurprisingly, in the light of the foregoing discussion of the properties of our model, there is less predictability at higher initial wealth shares of the less risk averse agents. This is because in these simulations we miss part of the range of wealth distributions for which effective heterogeneity is largest. In order to generate higher $R^2$s also for higher starting values we would have to increase the spread between the two types’ risk aversion coefficients further.

How ‘big’ is heterogeneity in our calibration anyway? The degree of leverage of the less risk averse agents may provide an intuitive measure. We find that on average the L-types hold stocks amounting to 128% of their wealth in the simulation for $w_{L,0} = 0.1$ and less in the other two simulations. The H-types’ portfolio share of stocks is on average 41% in the simulation for $w_{L,0} = 0.1$. This difference may seem rather big, but one has to keep in mind that in this economy the stock is the only asset in positive net supply. Data presented in Vissing-Jorgensen (2002) shows that even the 44% of stockholders among U.S. households hold only about half their financial wealth in stocks, with a standard deviation of 30%. This number translates into a coefficient of variation of the portfolio share of about 0.6, almost equal to the one in our model for the simulation for $w_{L,0} = 0.1$ and bigger than the ones for the other simulations. Taking into account that households owning stock likely own non-financial wealth as well and/or adding the households who do not participate in the stock

<table>
<thead>
<tr>
<th>Horizon (Years)</th>
<th>Data</th>
<th>Model $w_{L,0} = 0.1$</th>
<th>Model $w_{L,0} = 0.2$</th>
<th>Model $w_{L,0} = 0.3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Coefficient</td>
<td>$R^2$</td>
<td>Coefficient</td>
<td>$R^2$</td>
</tr>
<tr>
<td>1</td>
<td>-1.65</td>
<td>.06</td>
<td>-2.77</td>
<td>.11</td>
</tr>
<tr>
<td>2</td>
<td>-3.43</td>
<td>.11</td>
<td>-5.41</td>
<td>.20</td>
</tr>
<tr>
<td>3</td>
<td>-3.92</td>
<td>.13</td>
<td>-7.90</td>
<td>.27</td>
</tr>
<tr>
<td>5</td>
<td>-7.91</td>
<td>.35</td>
<td>-12.40</td>
<td>.38</td>
</tr>
</tbody>
</table>

In line with intuition more heterogeneity leads to more predictability because portfolios become more extreme and the wealth effect thus becomes stronger.  
This may be because the volatility of the price dividend ratio in the simulated data relative to the real data is even lower than the relative volatility of excess returns in simulated and real data. All coefficients are significantly different from zero, both those estimated from artificial and from real data.
Table 4: Volatilities and correlations of individual and aggregate consumption growth \( w_{L,0} = 0.1 \)

<table>
<thead>
<tr>
<th>Volatilities</th>
<th>Correlations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma(\Delta c_L) )</td>
<td>( \sigma(\Delta c_H) )</td>
</tr>
<tr>
<td>6.7%</td>
<td>1.2%</td>
</tr>
</tbody>
</table>

market, the coefficient of variation calculated from the data would actually be higher than in our calibrated model. It therefore seems fair to say that the amount of heterogeneity we impose in our calibration is not excessive.

Another way of looking at heterogeneity in our model is by comparing the volatilities and correlations of individual and aggregate consumption. Table 4 contains the corresponding information. In our benchmark simulation the annual volatility of consumption growth of the less risk averse agents is about twice as big as that of aggregate consumption while the consumption growth volatility of the more risk averse agents is only little more than a third of its aggregate value. Thus quite a bit of consumption insurance is taking place between agents, and from this perspective heterogeneity does seem sizable. For completeness we also report the correlations of aggregate and individual consumption growth rates. They are positive but far from one, in particular where \( c_L \) is involved.

Table 5 presents means and standard deviations as well as predictive regressions for our alternative calibration using dividend data. Unfortunately the results are only partially comparable to those from the consumption calibration because we could not solve the model for comparable degrees of heterogeneity. For the dividend calibration we hence have to use a ratio of risk aversion coefficients of \( \frac{\gamma_H}{\gamma_L} = \frac{2.335}{2.335} \approx 2.43 \), compared to \( \approx 2.71 \) in our benchmark calibration to consumption data. Nevertheless, this alternative calibration is instructive to analyze. Unsurprisingly, we achieve much less predictability than in our benchmark calibration. Nevertheless, the \( R^2 \) rises noticeably with horizon and reaches 5% at a 10 year horizon, which is still more than what Chan & Kogan (2002) achieve in their heterogeneous agent model. The volatilities of excess returns and price-dividend ratio are much higher now than in our benchmark calibration. The big increase in excess return volatility can clearly be traced back to the much higher volatility of dividend growth of 12.8%. Again, the positive correlation of price-dividend ratio and dividend growth adds some more to the volatility of excess returns.

Nevertheless, the volatility of excess returns and price-dividend ratio is still much lower than in the data. It seems that in this model as it stands it is very hard to generate sufficient volatility. One reason is that the price-dividend ratio here is bounded above and below by its respective values in the two polar representative agent models. At 18.86 and 23.97 in our benchmark calibration, these
### Means and Std. dev.’s Long-horizon regressions

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Data</th>
<th>Model</th>
<th>Horizon (Years)</th>
<th>Data</th>
<th>Model</th>
<th>10 x Coeff.</th>
<th>R²</th>
<th>10 x Coeff.</th>
<th>R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>E(r_f)</td>
<td>1.7</td>
<td>1.1</td>
<td>1</td>
<td>-1.65</td>
<td>.06</td>
<td>-1.32</td>
<td>.006</td>
<td></td>
<td></td>
</tr>
<tr>
<td>σ(r_f)</td>
<td>5.1</td>
<td>1.0</td>
<td>2</td>
<td>-3.43</td>
<td>.11</td>
<td>-2.50</td>
<td>.011</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E(r−r_f)</td>
<td>4.6</td>
<td>4.8</td>
<td>3</td>
<td>-3.92</td>
<td>.13</td>
<td>-3.51</td>
<td>.015</td>
<td></td>
<td></td>
</tr>
<tr>
<td>σ(r−r_f)</td>
<td>18.5</td>
<td>13.3</td>
<td>5</td>
<td>-7.91</td>
<td>.35</td>
<td>-5.86</td>
<td>.024</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E(PD)</td>
<td>22.7</td>
<td>23.6</td>
<td>7</td>
<td>-7.92</td>
<td>.20</td>
<td>-8.75</td>
<td>.039</td>
<td></td>
<td></td>
</tr>
<tr>
<td>σ(PD)</td>
<td>6.2</td>
<td>1.8</td>
<td>10</td>
<td>-13.19</td>
<td>.59</td>
<td>-12.04</td>
<td>.052</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: The model is simulated starting at w_{L,0} = 0.1.

Table 5: Means, standard deviations, and long horizon return regressions for the calibration to dividend data

bounds are clearly too tight compared to the data. We could extend them somewhat by choosing a higher value for the elasticity of intertemporal substitution, but this would not help very much[^1].

This lack of volatility is a bit of a concern because it could bias upward our results regarding predictability. For in a way strong predictability and low volatility of price-dividend ratio and excess returns are two sides of the same coin: Strong predictability means that the variability of expected excess returns, i.e. of the equity premium, is high relative to that of ex post excess returns. Ex post excess returns vary for three reasons. The first is variation in expected excess returns as a consequence of wealth shifts between agents. Secondly, dividend shocks directly affect realized excess returns. And so do, finally, movements in the price-dividend ratio. Thus if the volatility of excess returns is too low because the price-dividend ratio is too stable, excess returns may display too little ex post volatility relative to their ex ante volatility, implying too much predictability. An answer to this concern is that the level of heterogeneity in our model is sufficiently moderate as to allow further increases. If we could increase heterogeneity so much as to generate enough volatility we would likely obtain far too much predictability[^2]. In the light of the difficulties of previous contributions to generate sufficient predictability this might actually be a welcome rather than a problematic result. Also, realistic extensions such as modeling equity as a leveraged claim on consumption would help to reduce predictability again by raising volatility for a given amount of heterogeneity.

The upshot of the foregoing discussion is that strong predictability of excess returns is indeed a feature of our model. This is true not only in the sense of generally sizable R²s but also in terms of significant rises in the R² with horizon. To understand this latter property it is best to refer to the discussion of predictability in Cochrane (2005), chapter 20. The point made there is that for the R²s

[^1]: On the downside, the drift in the wealth distribution would become much stronger, which we regard as undesirable.
[^2]: Numerical limitations prevent us from doing so.
of predictive regressions to rise strongly with horizon, the forecasting variable, i.e. the price-dividend ratio for our purposes, has to be very persistent. This is clearly the case in our model. Table 2 reveals that the first-order autocorrelation of the price-dividend ratio in our simulations is of the same order of magnitude (ca. 0.8) as in the data. This persistence of the price-dividend ratio can in turn be traced back to the highly persistent effect of dividend shocks on the wealth distribution. Recall from section 3.3 that if it were not for the drift in the wealth distribution persistence would actually be complete.

The overall magnitude of the $R^2$s, on the other hand, has to do with our choice of recursive utility. This choice allows us to impose heterogeneity only on the coefficient of risk aversion while leaving the elasticity of intertemporal substitution equal across agents. We can thus make the equity premium rather volatile while leaving the real rate fairly constant. It moves only to the extent that changing average risk aversion changes the strength of the precautionary savings motive. As a result, in our model, unlike in Chan & Kogan (2002), return predictability is actually excess return predictability and not gross return predictability driven by predictable movements in the interest rate. In our benchmark calibration the volatility of the risk premium is more than 50% bigger than that of the risk free rate, while in Chan & Kogan (2002) it is almost exactly the other way round.

5 Conclusion

We have analyzed a model of heterogeneous agents whose recursive preferences differ exclusively in the coefficient of risk aversion. Our major finding is that in this world even at moderate degrees of heterogeneity the price-dividend ratio is a significant (negative) predictor of future excess returns. Regressions of excess returns, cumulated over different horizons, on the price-dividend ratio display the empirically observed pattern of $R^2$s rising strongly with horizon. The $R^2$s we find are of roughly the same size as their empirical counterparts.

Heterogeneity in our model also creates some excess volatility of stock prices and excess returns, however, the magnitudes of these volatilities are still far below their empirical counterparts for our calibration. To raise them significantly, we would likely have to raise heterogeneity immensely. In order to improve this aspect of the model it would be interesting to introduce leveraged equity. I.e. bonds would be in positive net supply, and stocks would be a claim on consumption net of payments to bond holders. This would be a way to make dividends more volatile than consumption while retaining the general equilibrium character of the model, which is necessary with heterogeneous agents. We conjecture that this extension would be an alley towards achieving realistic amounts of
volatility and predictability at the same time.

References


A Choice of state space and computational strategy

For all the results presented in section 4.2 we solve the model by approximating four functions on a grid of $w_L$, the wealth share of the less risk-averse agent type. The functions describe $c_L$, the consumption of the L-type as a share of dividends, $\phi_L$, the L-type’s portfolio share of stocks, $PD$, the price-dividend ratio, and $R$, the gross risk free rate. We can use the wealth distribution as the single state variable because dividend growth is i.i.d.

For each wealth distribution we find next period’s wealth distributions on a grid of dividend growth rates as a fixed point of

$$w'_L[PD(w'_L) + 1] \exp(\Delta d') =$$

$$\left(w_L[PD(w_L) + 1] - c_L(w_L)\right) \left(\phi_L(w_L) \frac{PD(w'_L) + 1}{PD(w_L)} \exp(\Delta d') + (1 - \phi_L(w_L)) R(w_L)\right)$$

where we use our approximated functions. The resulting set of $w'_L(w_L, \exp(\Delta d'))$ allows to find the corresponding next period values of the approximated functions and ultimately to state the Euler equations for stocks and bonds for each agent. We iterate on the approximated functions to satisfy the Euler conditions using Broyden’s algorithm.

For the grid of dividend growth rates we use 30 grid points, which are weighted to approximate a log-normal distribution. For the grid of wealth shares 25 Chebychev nodes or less are typically sufficient to achieve a precision of $10^{-5}$ or better off the grid.

B Complete versus incomplete markets

In the model on which the results of our paper are based markets are incomplete because there are only two assets while dividend growth can have many realizations. Also, with heterogeneous agents we are not aware of any spanning properties for this asset structure. Nevertheless, we conjecture that our results do not depend on this market incompleteness. Several arguments suggest so:

1. At a theoretical level, with one source of uncertainty and trade in a stock and a riskless bond markets are incomplete only due to discrete trading intervals. I.e. as we shrink the period length we get closer and closer to complete markets. Numerically, we have made use of this observation and solved the model for ever shorter frequencies in order to check to what extent the solution changes. It turns out that changes are hardly recognizable.

2. We have also programmed and solved the social planner’s problem. (See below for the set-up
and the computational approach.) It is computationally more involved, which makes it less practical for experimenting, and at the point of writing the program is not fully stable. However, for the calibrations tried results are hardly distinguishable from those for the incomplete markets version.

3. We have analyzed the simplified model of section 3.3 for the case in which the dividend shock can take only two values such that markets are complete. The results of lemma 4 uphold and can in fact be strengthened.

B.1 Social planner’s problem

For completeness we explain how we solve the complete markets version of our model. The problem we solve is

$$\max_{\{C_L,t, C_H,t\}} \omega \log U_{L,t} + (1 - \omega) \log U_{H,t}$$

$$C_{L,t} + C_{H,t} = D_t \forall t \geq 1$$

where $U_{i,t} = \left[(1 - \delta)C_i^{1-1/\psi} + \delta[E(U_{i,t+1}^{1-\gamma_i})]^{1-1/\psi}\right]^{1/\psi}$. We use the log-transformation of $U$ in the planner’s problem because in this formulation the welfare weight $\omega$ can be interpreted as the initial wealth share of the L-types, as will be shown below. This property is useful for comparing the complete and incomplete markets versions of our model. Since the log is a positive monotone transformation log $U$ represents the same preferences as $U$.

The first order conditions of this problem are equalization of stochastic discount factors (definition in equation (10)),

$$m_{L,t+1} = m_{H,t+1},$$

for all periods and

$$\omega U_{L,t}^{1/\psi} C_{L,t}^{1/\psi} = (1 - \omega) U_{H,t}^{1/\psi} C_{H,t}^{1/\psi}$$

for period one.

The problem is thus not fully recursive, since we have an extra condition for the initial period. We handle this problem as follows: First we approximate $c_L$, the consumption share of the L-type, as well as $v_i \equiv V_i/D$, the value of each agent type, normalized by dividends, on a grid of dividend growth and last period’s distribution of consumption $c_{L,-1}$. In this we make use of the stochastic discount factor condition (18) as well as the two recursive utility functions. Next we use equation
to find the initial consumption distribution that corresponds to the initial wealth distribution characterized by $\omega$.

The interpretation of the welfare weight $\omega$ as the initial wealth share of the L-types is possible due to the linear homogeneity of $U$. This property implies $U_{i,1}^{1-1/\psi}C_{i,1}^{-1/\psi} = (1 - \delta) \frac{U_{i,1}}{W_{L,1}/w_{L,1}} = (1 - \delta) W_{i,1}^{1/\psi}$ Dividing the corresponding conditions for each type by one another and noticing that $W_{L,1}/W_{H,1} = w_{L,1}/w_{H,1}$ we arrive at equation (19) with $w_{L,1} = \omega$.

\[43\text{See Cochrane (2006) for details on the properties of recursive utility.}\]
Figure 1: The timing of events in the two-period model.
Figure 2: Equity premium (in % p.a.) and price-dividend ratio as functions of the wealth distribution
Figure 3: The share of the stock in each agent type’s portfolio as a function of the wealth distribution.

Figure 4: Stock holdings of the L-type as a function of his wealth share. (The dotted line represents $s_L = w_L$, i.e. the counterfactual of no asset trade.)
Figure 5: Sharpe ratio, equity premium (in % p.a.), and volatility of excess returns as functions of the wealth distribution
Solid lines: heterogeneous agent model
Dashed lines: emulation in the representative agent model
Figure 6: The Role of the intertemporal elasticity of substitution
Figure 7: Impulse responses: Difference between simulation with and without a dividend growth shock of one standard deviation.