Long run forward rates and long yields of bonds and options in heterogeneous equilibria

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Abstract We prove that, in a heterogeneous economy with scale invariant utilities, the yield of a long term bond is determined by the agent with maximal expected marginal utility. We also prove that the same result holds for the long term forward rates.

Furthermore, we apply Cramer’s large deviations theorem to calculate the yield of a long term European call option. It turns out that there is a threshold risk aversion such that the option yield is independent of the risk aversion when risk aversion is above the threshold. Surprisingly, the long term option yield is always greater then or equal to the corresponding equity return. That is, in the long run, it is more profitable to buy a long maturity call option on equity then the equity itself.

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1 Introduction

In this paper we study long run zero coupon bond yields and forward rates in heterogeneous, complete market economies. In equilibrium, long run forward rates and long yields on bonds reflect the price of long run economic risks and, for this reason, help us in understanding the role of these risks. See, for example, Alvarez and Jermann [2] and Dybvig, Ingersoll and Ross [10] (see, also, Hubalek et al. [12] for a general mathematical result).

There is a large literature analyzing asset prices in heterogeneous economies. See, e.g., Dumas [9], Ross [18], Wang [20], Wang [21], Constantinides and Duffie [5], Gollier and Zeckhauser [11]. A direct predecessor of our paper is the paper of Wang [22]. Wang considered an economy populated by two agents with CRRA utility functions, identical discount factors and risk aversions $1$ and $0.5$ and proved that the long run zero coupon bond yield is determined by the agent with the largest expected marginal utility. In this paper, we prove this result for an arbitrary number of agents with arbitrary heterogeneous discount factors and risk aversions. Furthermore, we prove a much stronger statement: an analogous result holds for long run forward rates. Our results can be directly extended to any scale invariant preferences, including heterogeneous beliefs (such as Wang [20], Wang [21]) and state dependent preferences, generated by habit formation (such as, e.g., Constantinides [4] and Abel [1]). Even though we work in discrete time, all our results literally hold for continuous time economies. Note also, that our result extends Lengwiler [14], who considered an economy with heterogeneous discount factors and proved that the long run bond yield is determined by the most patient agent.

It is well known that, in a homogeneous economy with standard, CRRA preferences and geometric random walk (Brownian motion) aggregate endowment, the equity price is proportional to the aggregate endowment and, consequently, options are priced via the standard Black-Scholes formula. Thus, Black-Scholes formula can be considered as a trivial, homogeneous special case of our heterogeneous equilibria. Introducing heterogeneity is a natural way of generating new effects, such as stochastic volatility. For example, Beninga and Mayshar [3] considered a one-period economy with heterogeneous risk aversions and discount factors, and showed how heterogeneity generates correction to the Black-Scholes formula and, in particular, is able to produce smiles and skews, compatible with the real data.

In this paper we, for the first time, study the yield of a long maturity call option. That is, the per-period return on holding the option up to maturity. We explicitly calculate the limit of the option yield as the maturity tends to infinity. Several surprising phenomena arise. First, there is a threshold risk aversion, such that the long run option yield is independent of risk aversion when the latter is above the threshold. Second, the option yield is always greater then or equal to the corresponding equity return. That is, in the long run, it is more profitable to invest in options than in equities. It would be very interesting to compare these theoretical predictions with real data.
It is necessary to point out that the word "agent" is used to denote a set of identical agents, each of measure zero. Of course, an agent of positive measure cannot remain a price taker when he begins to dominate certain asset returns at sufficiently long horizon\textsuperscript{1}. However, we abuse this fine distinction and use the word "agent" alone.

2 The model

We assume a discrete time, pure exchange economy with a single, perishable consumption good (numeraire). The aggregate endowment stream $W_t$, $t \geq 1$, equivalently, the single good stream, is a geometric random walk, normalized by $W_0 = 1$. That is, there exists a sequence of positive, i.i.d. random variables $X_t$ such that

$$W_t = X_1 \cdots X_t$$

for all $t$. The information structure is encoded in the filtration $(\mathcal{F}_t, t \geq 0)$ of the underlying probability space $(\Omega, \mathcal{B}, P)$ generated by the aggregate endowment process $W_t$. We emphasize, that all our results can be directly extended to continuous time, incorporating the model of [22].

We also make the common assumption that there is a "money market" in which a one period risk free bond can be traded at each moment of time. Agents trade competitively in both the equity and money markets and consume the proceeds. Furthermore, we assume that the market is dynamically complete. That is, there are enough risky assets available for trading to make the market dynamically complete. This assumption is naturally fulfilled in standard, Brownian motion driven complete markets (see, e.g., [13]). When time is discrete, things become slightly different, because there might be exceptional situations for a set of parameters of measure zero, for which the dimension of the market subspace falls down. But, for risky assets with generic endowment processes, this does not happen. The standard way to proceed is: first assume that the market is complete and calculate the equilibrium state price densities. Then, find the generic set of (exogeneously specified) risky dividend processes that complete the market. This set will be the complement of a countable set of hyperplanes.

Note, that in discrete time with a finite time horizon we would need infinitely many assets to complete the market if the probability space $\Omega$ were infinite. But, since our proofs and results do not depend on the discreteness of time, we present all the arguments for general probability spaces.

Since the market is dynamically complete, it is well known (see, e.g., [8], [13]) that there exists a unique, positive state price density process $M = (M_t, t \geq 0)$ (normalized by $M_0 = 1$), through which all securities can be priced. In particular, the price of the Lucas tree equity, whose dividend process

\textsuperscript{1} We thank Rajnish Mehra for this important remark
coincides with the aggregate endowment $W_t$ is given by

$$P_t^W = E_t \left[ \sum_{\tau=1}^{\infty} \frac{M_{t+\tau}}{M_t} W_{t+\tau} \right]$$  \hspace{1cm} (2.1)$$

(we assume no bubbles). Abel [1] suggested considering equities with dividend processes $W^\alpha_t$ with $\alpha \in \mathbb{R}$, along with the Lucas tree asset. The parameter $\alpha$ is introduced to account for leverage effects. In general, the price $P^D_t$ at time $t$ of an asset with a dividend process $D = (D_t, t \geq 0)$ is given by

$$P^D_t = E_t \left[ \sum_{\tau=1}^{\infty} \frac{M_{t+\tau}}{M_t} D_{t+\tau} \right]$$

Furthermore, the price at time $t_1$ of a risk free zero coupon bond maturing at time $t_2$ is given by

$$B^F(t_1, t_2) = E_{t_1} \left[ \frac{M_{t_2}}{M_{t_1}} \right]$$  \hspace{1cm} (2.2)$$

That is, specifying asset prices is equivalent to specifying the state price density process. For the Lucas tree equity price to be defined in is necessary that the state price densities lie in the natural price space

$$l_1(W) = \{ M : \sum_{t=0}^{\infty} E[W_t|M_t] < \infty \}$$

The economy is populated by $n$ classes of identical CRRA agents. Since they aggregate (see, [19]), we will be using the name agent $i$ for the representative agent of a class $i$, $i = 1, \cdots, n$. We denote by $N = \{1, \cdots, n\}$ the set of all agents.

Agent $i$ has constant relative risk aversion (CRRA) utility function. He chooses his random consumption $x_{it}$ at each time $t \geq 0$ and each possible state of the world to maximize expected discounted intertemporal utility function

$$E \left[ \sum_{t=0}^{\infty} \delta_t^{1-\gamma_i} \frac{x_{1-\gamma_i}^1 - 1}{1 - \gamma_i} \right]$$

Here, $\gamma_i$ is the relative risk aversion of agent $i$ and $\delta_t$ is his discount factor (patience). Agent $i$ is endowed with $\eta_i$ number of shares of equity and $\sum_{i=1}^{n} \eta_i = 1$ since we normalize the supply to be one. An agent finances his consumption by trading assets. It is easy to show (see, e.g., Wang [22], [8]) that, since the markets are complete, the set of feasible consumption streams (the budget set) of agent $i$ can be easily described in terms of the unique state price densities. Namely, given a state price density process from the price space

\footnote{The reciprocal of relative risk aversion, $b_i = 1/\gamma_i$, is called cautiousness}
\( l_1(W) \), agent \( i \) chooses his optimal consumption stream \((x_{it})_{t \geq 0}\) from the corresponding consumption space

\[ l_1(M) = \left\{ (x_{it})_{t \geq 0} : \sum_{t=0}^{\infty} E[x_{it} M_t] < \infty \right\} \]

satisfying the budget constraint

\[ E \left[ \sum_{t=0}^{\infty} x_{it} M_t \right] = \eta_i E \left[ \sum_{t=0}^{\infty} W_t M_t \right] \]

to maximize his utility. The utility maximization problem can be now easily solved.

**Lemma 2.1** Let \( b_i = \gamma_i^{-1} \). The solution to the utility maximization problem for an agent \( i \)

\[
\max \left\{ E \left[ \sum_{t=0}^{\infty} \delta_t^{1-\gamma_i} - \frac{1}{1-\gamma_i} \right] \left| E \left[ \sum_{t=0}^{\infty} x_{it} M_t \right] = \eta_i E \left[ \sum_{t=0}^{\infty} W_t M_t \right] \right. \right\} \quad (2.3)
\]

is given by \( x_{it} = M_t^{-b_i} \delta_t^{b_i} x_{i0} \) for all \( t \geq 1 \) and

\[ x_{i0} = \eta_i \frac{\sum_{t=0}^{\infty} E[W_t M_t]}{\sum_{t=0}^{\infty} \delta_t^{b_i} E[M_t^{1-b_i}]} \quad (2.4) \]

**3 Market Equilibrium**

If the payoffs of all assets are linearly independent, standard arguments (see, e.g., [8], [13]) imply that the equilibrium market clearing for all assets is equivalent to the market clearing for the consumption good. Thus, our equilibrium can be characterized as an Arrow-Debreu equilibrium.

**Definition 3.1** A positive state price density process \( M := (M_t, t \geq 0) \) is an Arrow-Debreu equilibrium if

\[ \sum_{i \in N} x_{it} = W_t \]

for all \( t \geq 0 \). That is,

\[ \sum_{i \in N} \delta_t^{b_i} M_t^{-b_i} x_{i0} = W_t \quad (3.1) \]

for all \( t \geq 1 \) (recall that \( b_i = \gamma_i^{-1} \)). Market clearing at time-zero follows from the Walras’ law.

Existence of an equilibrium for infinite horizon economies is a nontrivial problem. In [16] we prove the following
Theorem 3.2 Under the assumptions made above, an equilibrium exists if and only if

\[ \delta_i E[X_i^{1-\gamma_i}] < 1 \]  

for all \( i = 1, \ldots, n \). This condition is also necessary and sufficient for the finiteness of the Lucas tree equity price.

Therefore, everywhere in the sequel we make

Assumption 3.3 Inequality (3.2) is fulfilled for any \( i = 1, \ldots, n \).

In general, is the state space and horizon are finite, existence follows from standard results (see, e.g., [6]). Then, we could simply view the infinite horizon yields as limits of finite horizon yields as the horizon goes to infinity. But, viewing the yields directly as prices in infinite horizon economies is, of course, more convenient. For this reason, we present a sketch of proof of Theorem 3.2 in the appendix.

Non-uniqueness of equilibria. Note, that the equilibrium in our economy is not necessarily unique (see, [16] for concrete examples of non-uniqueness). But, these multiple equilibria only differ from each other via the initial consumptions \( x_{i0} \). Our results are universal because they are independent of the initial consumptions \( x_{i0} \) and, consequently, are independent of a particular equilibrium.

4 The aggregator function and its properties

The key ingredient of the proofs is the aggregator function, constructed in

Proposition 4.1 In equilibrium, the \( \tau \)-period stochastic discount factor is given by

\[
\frac{M_{t+\tau}}{M_t} = F_t \left( \delta_t^1 \left( \frac{W_{t+\tau}}{W_t} \right)^{-\gamma_1}, \ldots, \delta_t^n \left( \frac{W_{t+\tau}}{W_t} \right)^{-\gamma_n} \right)
\]

where \( F_t = F_t(y_1, \ldots, y_n) \) is the unique solution to

\[
\sum_{i=1}^n F_t^{-b_i} g_i^b (x_{i1} W_t^{-1}) = 1
\]

Note that the weights \( x_{i1} W_t^{-1} \) sum up to one.
Proof Dividing equilibrium equations (3.1) at time $t+\tau$ by $W_{t+\tau}$, we get

$$1 = \sum_{i=1}^{n} e^{-\rho_i (t+\tau) b_i} M_{t+\tau}^{-b_i} x_{i0} W_{t+\tau}^{-1}$$

$$= \sum_{i=1}^{n} e^{-\rho_i \tau b_i} \left( \frac{M_{t+\tau}}{M_t} \right)^{-b_i} M_t^{-b_i} e^{-\rho_i t b_i} x_{i0} W_t^{-1} W_{t+\tau}^{-1}$$

$$= \sum_{i=1}^{n} e^{-\rho_i \tau b_i} \left( \frac{M_{t+\tau}}{M_t} \right)^{-b_i} (x_{i\tau} W_t^{-1}) \left( \frac{W_{t+\tau}}{W_t} \right)^{-\gamma_i} b_i$$

$$= \sum_{i=1}^{n} \left( \frac{M_{t+\tau}}{M_t} \right)^{-b_i} \left( e^{-\rho_i \tau} \left( \frac{W_{t+\tau}}{W_t} \right)^{-\gamma_i} b_i \right) (x_{i\tau} W_t^{-1})$$  (4.1)

and the claim immediately follows. \qed

Proposition 4.1 allows us to formulate most important properties of the state price densities in terms of the aggregator function. In Malamud [15], we use the aggregator function to prove sharp estimates for asset prices.

We will need the following important

Lemma 4.2 The function $F_t$ satisfies

$$\max_i \left( y_i x_{i\tau} W_t^{-1} \right) \leq F_t(y_1, \ldots, y_n) \leq \left( \sum_{i \in N} y_i^{-1} (x_{i\tau} W_t^{-1})^{-1} \gamma_i \right) \gamma \leq n \gamma - 1 \sum_{i \in N} y_i \gamma$$  (4.2)

where $\gamma$ is any number satisfying the inequality $\gamma^{-1} \leq \min \{ \min_i b_i, 1 \}$.

Proof The first inequality follows from

$$F_t^{-b_j} y_j b_j x_{j\tau} W_t^{-1} \leq \sum_{i=1}^{n} F_t^{-b_i} y_i b_i (x_{i\tau} W_t^{-1}) = 1$$

for any $j \in N$. Suppose now that

$$F_t(y_1, \ldots, y_n) > \left( \sum_{i \in N} y_i^{-1} (x_{i\tau} W_t^{-1})^{-1} \gamma_i \right) \gamma$$

Then,

$$1 = \sum_{i=1}^{n} F_t^{-b_i} y_i b_i (x_{i\tau} W_t^{-1}) \leq \sum_{i=1}^{n} \left( \frac{y_i^{-1} (x_{i\tau} W_t^{-1})^{-1} \gamma_i}{\sum_{i \in N} y_i^{-1} (x_{i\tau} W_t^{-1})^{-1} \gamma_i} \right) b_i \gamma$$

$$\leq \sum_{i=1}^{n} \frac{y_i^{-1} (x_{i\tau} W_t^{-1})^{-1} \gamma_i}{\sum_{i \in N} y_i^{-1} (x_{i\tau} W_t^{-1})^{-1} \gamma_i} = 1$$  (4.3)

Contradiction.

The last inequality follows from Jensen’s inequality and convexity of $x^\gamma$. \qed
The long run yield of a zero coupon bond and long run forward rates

Recall that the aggregate endowment $W_t$ is assumed to be a geometric random walk,

$$W_t = X_1 \cdots X_t$$

and $X_t$ are i.i.d. In a homogeneous economy populated by identical agents with risk aversion $\gamma$ and discount factors $\delta$, bond prices are constant and are given by

$$B^F(t, t+\tau)(\delta, \gamma) = \delta^\tau E[X_t^{-\gamma}]$$

and the yield of the zero coupon bond is, by definition,

$$-\tau^{-1} \log B^F(t, t+\tau)(\delta, \gamma) = -\log \left( \delta E[X_t^{-\gamma}] \right)$$

When agents are heterogeneous, bond prices can not any more be calculated explicitly, but their asymptotic behavior can be studied in detail. The following result is a substantial extension of Theorem 4 in Wang [22] for the general class of heterogeneous CRRA economies.

**Theorem 5.1** We have

$$\lim_{\tau \to \infty} \tau^{-1} \log B^F(t, t+\tau) = \max_i \log (\delta_i E[X_i^{-\gamma_i}]) \quad (5.1)$$

**Remark 5.2** Wang [22] considered an economy with two agents having identical discount factors and risk aversion 1 and 1/2, and proved (5.1) in this very special case.

**Proof** The proof of this theorem is based on the following simple

**Lemma 5.3** Let $\rho_i \in \mathbb{R}$, $i = 1, \cdots, m$. If $a_i(\tau) = e^{\tau \rho_i + O(1)}$ for all $\tau > 0$ then

$$\lim_{\tau \to \infty} \tau^{-1} \log \sum_{i=1}^m a_i(\tau) = \max_i \rho_i$$

By Lemma 4.2 and Proposition 4.1,

$$\delta_j^\tau (E[X_1^{-\gamma_j}])^\tau (x_j, W_t^{-\gamma_j}) \leq E_t \left[ \frac{M_{t+\tau}}{M_t} \right] = B^F(t, t+\tau) \leq n^{\gamma-1} \sum_{i \in N} \delta_i^\tau E_t[(W_{t+\tau}/W_t)^{-\gamma_i}]$$

$$= n^{\gamma-1} \sum_{i \in N} \delta_i^\tau (E[X_1^{-\gamma_i}])^\tau \quad (5.2)$$

for any $j \in N$. The required assertion immediately follows. $\Box$
A much more subtle question is to understand the behavior of the forward rates
\[ f_t(\tau) = \frac{B^F(t, t + \tau)}{B^F(t, t + \tau + 1)} \]

Note, that the bond yields satisfy
\[ -\tau^{-1} \log B^F(t, t + \tau) = \tau^{-1} \sum_{\theta=1}^{\tau} \log f_t(\theta) \]

and, consequently, if the forward rates converge to a limit, so do the bond yields (see, also, [10] and [12]), but the converse is generally not true.

Surprisingly, this is true in our class of economies under a simple genericity assumption.

**Assumption 5.4** There exists a unique agent \( b \in N \) such that
\[ \delta_b E[X_1^{-\gamma_1}] = \max_i \delta_i E[X_1^{-\gamma_1}] \]

Obviously, Assumption 5.4 holds for generic economies (the set of such economies is a complement of a finite number of smooth hypersurfaces).

**Theorem 5.5** Let Assumption 5.4 be fulfilled. Then,
\[ \lim_{\tau \to \infty} f_t(\tau) = \left( \delta_b E[X_1^{-\gamma_1}] \right)^{-1} \]

We will need the following auxiliary

**Lemma 5.6** Let \( \gamma \geq 1 \). Then the function \( f : \mathbb{R}^n_+ \to \mathbb{R} \) defined by
\[ f(x_1, \cdots, x_n) = \left( \sum_{i=1}^{n} x_i^{1/\gamma} \right)^\gamma \]
is concave.

**Proof** The Hessian \( H(f) \) of the function \( f \) is given by
\[
H(f) = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{i,j=1}^{n} \\
= (1 - \gamma^{-1}) \left( \sum_{i=1}^{n} x_i^{1/\gamma} \right)^{\gamma-2} \left( x_i^{1/\gamma-1} x_j^{1/\gamma-1} \right)_{i,j=1}^{n} \\
- \left( \sum_{i=1}^{n} x_i^{1/\gamma} \right) \text{diag}[x_i^{1/\gamma-2}]_{i=1}^{n} \\
= (1 - \gamma^{-1}) f^{1-2/\gamma} \text{diag}(x_i^{\gamma-1/2})_{i=1}^{n} \left( x_i^{\gamma-1/2} x_j^{\gamma-1/2} \right)_{i,j=1}^{n} \\
- f^{1/\gamma} I \right) \text{diag}(x_i^{\gamma-1/2})_{i=1}^{n}
Here, $I$ is the identity matrix. The matrix
\[ A = \left( x_i^{\gamma^{-1}/2} x_j^{\gamma^{-1}/2} \right)_{i,j=1}^n \]
equals $f_1^{1/\gamma}$ times the orthogonal projection onto the vector $(x_i^{\gamma^{-1}/2})_{i=1}^n$. Thus, $\|A\| = f_1^{1/\gamma}$ and the matrix
\[ \left( x_i^{\gamma^{-1}/2} x_j^{\gamma^{-1}/2} \right)_{i,j=1}^n - f_1^{1/\gamma} I \]
is negative definite. Therefore, $H(f)$ is also negative definite. $\Box$

The Jensen inequality immediately yields the following

**Lemma 5.7** Let $\gamma \in \mathbb{N}$ be a natural number and $X_1, \ldots, X_n$ nonnegative random variables. Then,
\[ E \left[ \left( X_1^{1/\gamma} + \cdots + X_n^{1/\gamma} \right)^\gamma \right] \leq \left( E[X_1]^{1/\gamma} + \cdots + E[X_n]^{1/\gamma} \right)^\gamma \]

We are now ready to prove the theorem.

**Proof (of Theorem 5.5)** Let $b$ be the (by assumption, unique) agent from Assumption 5.4 and 
\[ \alpha_j = x_j \tau \]
for all $j \in \mathbb{N}$. Then, applying Lemma 5.7 to the random variables
\[ X_j = \delta_j \left( W_t + \tau \right)^{-\gamma} \alpha_j^{\gamma} \]
and using Proposition 4.1 and Lemma 4.2, we arrive at the inequality
\[ \frac{E_t[M_{t+\tau}]}{M_t} \leq \left( \sum_{j=1}^n \left( \delta_j \left( W_t + \tau \right)^{-\gamma} \alpha_j^{\gamma} \right)^{1/\gamma} \right)^\gamma \tag{5.3} \]
for some $\gamma > 1$. Denote
\[ \kappa := \max_{i \neq b} \left( \frac{\delta_i E[X_i^{-\gamma}]}{\delta_b E[X_b^{-\gamma}]} \right) \]
By Assumption 5.4, $\kappa < 1$. Applying Lemma 4.2 and using (5.3), we get
\[ \left( \delta_b E[X_b^{-\gamma}] \right)^{\gamma} \alpha_b \leq B^F(t, t + \tau) \leq \left( \delta_b E[X_b^{-\gamma}] \right)^{\gamma} \alpha_b (1 + K \kappa^{\gamma^{-1}}) \]
where
\[ K = \sum_{j \neq b} \left( \alpha_j^{\gamma} \alpha_b^{-\gamma} \right)^{1/\gamma} \]
Consequently,
\[ B^F(t, t + \tau) = \left( \delta_b E[X_b^{-\gamma}] \right)^{\gamma} \alpha_b (1 + o(1)) \]
and the forward rates satisfy
\[ \frac{B^F(t, t + \tau)}{B^F(t, t + \tau + 1)} = \left( \delta_b E[X_b^{-\gamma}] \right)^{-1} (1 + o(1)) \]
which is what had to be proved. $\Box$
6 The yield of a long maturity European call option

Recall that the Lucas tree equity is the asset whose dividend process coincides with the aggregate endowment and the price $P_t^W$ is given by

$$P_t^W = E_t \left[ \sum_{\tau = 1}^{\infty} \frac{M_{t+\tau}}{M_t} W_{t+\tau} \right]$$

Note that, in a homogeneous economy with parameters $(\delta, \gamma)$, the equity price is proportional to the dividend and is given by

$$P_t^W(\delta, \gamma) = W_t \frac{\delta E[X_1^{1-\gamma}]}{1-\delta E[X_1^{1-\gamma}]}$$

In general, Proposition 4.1 and Lemma 4.2 together yield

**Lemma 6.1**

$$C_1 W_t \leq P_t^W \leq C_2 W_t \quad (6.1)$$

for some constants $C_2 > C_1 > 0$

**Proof** It follows directly from Proposition 4.1 and Lemma 4.2 that the constants

$$C_1 = \frac{\delta_1 E[X_1^{-\gamma_1}]}{1 - \delta_1 E[X_1^{-\gamma_1}]}$$

and

$$C_2 = n^{\gamma-1} \sum_{j \in N} \frac{\delta_j E[X_1^{-\gamma_j}]}{1 - \delta_{ij} E[X_1^{-\gamma_j}]}$$

satisfy the required estimates. See, [15] for sharp bounds for the equity price.

By no arbitrage, the price $\text{Call}_t(K, t+\tau)$ at time $t$ of a European call option with strike $K$ and maturity $t + \tau$ is given by

$$\text{Call}_t(K, t+\tau) = E_t \left[ \frac{M_{t+\tau}}{M_t} (P_t^W - K)^+ \right]$$

The payoff of the option at maturity is, by definition, $(P_{t+\tau}^W - K)^+$ and therefore, the log expected per-period return on holding the option up to maturity (i.e., the option yield) is given by

$$\tau^{-1} \log \frac{E_t \left[ (P_{t+\tau}^W - K)^+ \right]}{E_t \left[ \frac{M_{t+\tau}}{M_t} (P_{t+\tau}^W - K)^+ \right]}$$

In this section we study the asymptotic behavior of the option yield as its maturity tends to infinity. We will need several definitions.
Assumption 6.2 We assume that the jump $X_1$ of the aggregate endowment satisfies the Donsker-Varadhan condition
\[ E[X_1^\alpha] < \infty \]
for all $\alpha \in \mathbb{R}$. This condition guarantees that the Cramer’s large deviation result holds. See, e.g., [7], p.6.

Definition 6.3 Let
\[ Z(\beta, X) = E[X^\beta] \]
and
\[ S(x, X) = \sup_{\beta \in \mathbb{R}} (x \beta - \ln Z(\beta, X)) \]
The function $S(x, X)$ is the Legendre transform of the moment generating function $Z$.

Lemma 6.4 $S(x, X)$ is a strictly convex function of
\[ x \in (\text{essinf log } X, \text{esssup log } X) \]
with
\[ \min_x S(x, X) = S(E[\log X], X) = 0 \]
Consequently, $S(x, X) > 0$ for all $x \neq E[\log X]$.

Lemma 6.4 means that $S(x, X)$ measures the deviation of $x$ from the mean $E[\log X]$.

If $H = \log X$ is a binomial variable taking values $h_1 < h_2$ with probabilities $p_1$ and $p_2 = p$, then
\[ S(x, X) = -I_y(p) \]
where
\[ y = p + \frac{x - E[H]}{h_2 - h_1} \]
and $I_y(p)$ is the relative entropy function, given by
\[ I_y(p) = y \log \left( \frac{p}{y} \right) + (1 - y) \log \left( \frac{1 - p}{1 - y} \right) . \]

Cramer’s large deviations theorem states that $S(x, X)$ is the exact rate at which $\log X$ deviates from its mean. Namely, the following is true.

Theorem 6.5 (Cramer’s large deviations theorem) We have
\[ \Pr [W_t \geq e^{\tau x}] = e^{-\tau \left( S(x, X_1) + o(1) \right) } \]
for any $x$, satisfying $\text{esssup } X_1 > x \geq E[\log X_1]$. In particular, if $E[\log X_1] < 0 < \text{esssup } X_1$, then, for any positive constant $K > 0$,
\[ \Pr [W_t \geq K] = e^{-\tau \left( S(0, X_1) + o(1) \right) } \]
See, e.g., [7], p.6 for a proof. We will need modifications of the large deviations theorem under an equivalent change of measure.

**Definition 6.6** Define on each sigma-algebra $\mathcal{F}_t$ an equivalent probability measure

$$dP^\gamma_t = \frac{W_t^{-\gamma}}{E[W_t^{-\gamma}]} dP$$

It is easy to see that the family of measures $dP^\gamma_t$ is consistent (because $W_t$ is a geometric random walk) and, therefore, by the Kolmogorov extension theorem, there exists a measure $dP^\gamma$ whose restrictions on finite horizon algebras $\mathcal{F}_t$ coincide with $dP^\gamma_t$. The mathematical expectation with respect to $dP^\gamma$ is denoted by $E^\gamma$.

It is easy to see that $W = (W_t)$ is also a random walk under this modified probability measure $dP^\gamma$, the Donsker-Varadhan condition is also fulfilled and therefore the large deviations theorem also holds. We denote by

$$Z^\gamma(\beta, X) = E^\gamma[X^\beta]$$

and

$$S^\gamma(x, X) = \sup_{\beta \in \mathbb{R}} (x \beta - \ln Z^\gamma(\beta, X))$$

the corresponding Cramer function. Then, the large deviations theorem takes the form

$$\text{Prob}_t^\gamma[W_t \geq e^{\tau x}] = e^{-\tau \left(S^\gamma(x, X_1) + o(1)\right)}$$

We will also make the following economically natural assumption

**Assumption 6.7** $	ext{Prob}[X_1 > 1] \cdot \text{Prob}[X_1 < 1] > 0$. That is,

$$0 \in (\text{essinf } \log X_1, \text{esssup } \log X_1)$$

Assumption 6.7 means that both booms (growth) and recessions happen with positive probability.

**Lemma 6.8** The function $f(\delta, \gamma) = \delta E[X_1^{-\gamma}]$ is strictly convex in $\gamma$. Under the Assumption 6.7,

$$\lim_{\gamma \to +\infty} f(\gamma) = \lim_{\gamma \to -\infty} f(\gamma) = +\infty$$

and it has a unique global minimum $\Theta$ satisfying

$$E[X_1^{-\Theta} \log X_1] = 0$$

Furthermore, $\Theta > 0$ if and only if

$$E[\log X_1] > 0$$

**Proof** All claims are immediate consequences of the definitions. The positivity of $\Theta$ follows because $f'(0) = -E[\log X_1]$. □
Lemma 6.9 The function
\[ g(\gamma) = -(\log f)'_\gamma = E^\gamma[\log X_1] = \frac{E[X_1^{-\gamma} \log X_1]}{E[X_1^{-\gamma}]} \]
is monotone decreasing in \( \gamma \) and satisfies

\[ g(\mathcal{G}) = 0 \]

Consequently,

\[ S^\gamma(0, X_1) = -\log \frac{E[X_1^{-\mathcal{G}}]}{E[X_1^{-\gamma}]} \]

We will need the following well known correlation inequality.

Lemma 6.10 If \( h_1(x) \) and \( h_2(x) \) are monotone increasing, then

\[ E[h_1(X) h_2(X)] \geq E[h_1(X)] E[h_2(X)] \]

for any random variable \( X \).

Proof (of Lemma 6.9) Using Lemma 6.10, we get

\[ E^{\gamma_1}[\log X_1] = \frac{E^{\gamma_2}[X_1^{\gamma_2-\gamma_1} \log X_1]}{E^{\gamma_2}[X_1^{\gamma_2-\gamma_1}]} \geq E^{\gamma_2}[\log X_1] \]
for any \( \gamma_2 \geq \gamma_1 \) and the required monotonicity follows. Now,

\[ S^\gamma(0, X_1) = -\log \frac{E[X_1^{\beta-\gamma}]}{E[X_1^{-\gamma}]} \]

where \( \beta \) satisfies the first order condition

\[ E[X_1^{\beta-\gamma} \log X_1] = 0 \]
That is, \( \beta = \gamma - \mathcal{G} \) and the claim follows. \( \square \)

Theorem 6.11 Let

\[ \mathcal{O}(\delta, \gamma) = \begin{cases} \log (\delta E[X_1^{\gamma_1-\gamma}]) & , \quad \mathcal{G} \geq \gamma - 1 \\ \log (\delta E[X_1^{-\mathcal{G}}]) & , \quad \mathcal{G} \leq \gamma - 1 \end{cases} \]

Then,

\[ \lim_{\tau \to \infty} \tau^{-1} \log \frac{E_t[(P_t^W - K)^+]}{E_t[M_t((P_t^W - K)^+)]} = \mathcal{O}(1, 0) - \max_{i} \mathcal{O}(\delta_i, \gamma_i) \]
Proof We consider only the case \( t = 0 \). The case \( t > 0 \) is completely analogous. We will first show that

\[
\lim_{\tau \to \infty} \tau^{-1} \log E \left[ M_\tau (P^W_\tau - K)^+ \right] = \max_i O(\delta_i, \gamma_i)
\]

In fact, we will prove that

\[
\lim_{\tau \to \infty} \tau^{-1} \log E \left[ M_\tau (C W_\tau - K)^+ \right] = \max_i O(\delta_i, \gamma_i)
\]

for any constant \( C > 0 \). Then, the required assertion will follow from Lemma 6.1. Proposition 4.1 and Lemma 4.2 together yield that

\[
n^{-1} \left( \min_j (x_{j0})^{\gamma_j} \right) \sum_{i \in N} \delta_i^\gamma W_\tau^{-\gamma_i} \leq M_\tau \leq n^{\gamma-1} \sum_{i \in N} \delta_i^\gamma W_\tau^{-\gamma_i}
\]

Consequently, it suffices to prove that

\[
\lim_{\tau \to \infty} \tau^{-1} \log E \left[ \sum_{i \in N} \delta_i^\gamma (W_\tau - K)^+ \right] = \max_i O(\delta_i, \gamma_i) \quad (6.4)
\]

since the limit is independent of \( C \) and \( K \). Denote

\[
\pi(K, \tau, \gamma) = \text{Prob}^\gamma [W_\tau > K] = (E[X_1^{-\gamma}])^{-\tau} E \left[ W_\tau^{-\gamma}, W_\tau > K \right]
\]

We have

\[
E \left[ \delta_i^\gamma (W_\tau - K)^+ \right] = E \left[ \delta_i^\gamma W_\tau^{1-\gamma_i} - K \delta_i^\gamma W_\tau^{-\gamma_i}, W_\tau > K \right]
\]

\[
= f(\delta_i, \gamma_i - 1)^\tau \pi(K, \tau, \gamma_i - 1) - K f(\delta_i, \gamma_i)^\tau \pi(K, \tau, \gamma_i)
\]

\[
\leq f(\delta_i, \gamma_i - 1)^\tau \pi(K, \tau, \gamma_i - 1) \quad (6.5)
\]

and

\[
E \left[ \delta_i^\gamma (W_\tau - K)^+ \right] = E \left[ \delta_i^\gamma W_\tau^{1-\gamma_i} - K \delta_i^\gamma W_\tau^{-\gamma_i}, W_\tau > K \right]
\]

\[
\geq E \left[ \delta_i^\gamma W_\tau^{1-\gamma_i} - K \delta_i^\gamma W_\tau^{-\gamma_i}, W_\tau > 2K \right]
\]

\[
\geq \frac{1}{2} E \left[ \delta_i^\gamma W_\tau^{1-\gamma_i}, W_\tau > 2K \right] = \frac{1}{2} f(\delta_i, \gamma_i - 1)^\tau \pi(2K, \tau, \gamma_i - 1)
\]

(6.6)

By Cramer's large deviations theorem, the asymptotic behavior of \( \pi(K, \tau, \gamma) \) is independent of \( K \) and therefore it suffices to understand the asymptotic behavior of the right hand side of (6.5). We will consider two cases.

1. \( \gamma_i \geq \mathcal{G} + 1 \). In this case, Lemma 6.9 implies that

\[
E^{\gamma_i - 1} [\log X_1] \leq 0
\]
Therefore, Cramer’s large deviations Theorem and Lemma 6.9 together yield that

$$\pi(K, \tau, \gamma_i - 1) = e^{-\tau(S^{\gamma_i-1}(0, X_1)+o(1))} = \left( \frac{E[X_1^{-\gamma_i}]}{E[X_1^{-\gamma_i-1}]} \right)^\tau e^{-\tau o(1)}$$

and, consequently,

$$f(\delta_i, \gamma_i-1)^\tau \pi(K, \tau, \gamma_i - 1) - K f(\delta_i, \gamma_i)^\tau \pi(K, \tau, \gamma_i) = f(\delta_i, \Theta)^\tau e^{\tau o(1)}$$

(2) \( \gamma_i < \Theta + 1 \). Then,

$$E^{\gamma_i-1}[\log X_1] > 0$$

and therefore

$$\pi(K, \tau, \gamma_i) = 1 - e^{-\tau(a + o(1))}$$

for a positive constant \( a > 0 \).

Thus, we have proved that

$$E \left[ \delta_i^\tau W_{\tau-\gamma_i} (W_{\tau} - K)^+ \right] = O(\delta_i, \gamma_i)^\tau e^{\tau o(1)}$$

as \( \tau \to \infty \) and (6.4) immediately follows. Finally, by the same arguments,

$$\lim_{\tau \to \infty} \tau^{-1} \log E\left[\left(P_W^{\tau} - K\right)^+\right] = \log O(1, 0) = \begin{cases} \log E[X_1], & \Theta \geq -1 \\ \log E[X_1^{-\Theta}], & \Theta \leq -1 \end{cases}$$

\( \square \)

**Remark 6.12** Theorem 6.11 generates many theoretical predictions that should not be difficult to test with real data.

The first amazing consequence of Theorem 6.11 is that the option yield is independent of risk aversions \( \gamma_i \) if \( \gamma_i \geq \Theta + 1 \).

The second, even more interesting consequence is that the option yield is always greater than or equal to the corresponding equity return. Namely, in a homogeneous economy with parameters \((\delta, \gamma)\), the equity return is given by

$$r_{\tau+1}^W = \frac{E_{\tau+1}(\delta, \gamma) + W_{\tau+1}}{P_{\tau}^W(\delta, \gamma)} = \frac{W_{\tau+1}}{W_{\tau}} \cdot \frac{1}{f(\delta, \gamma - 1)}$$

and, consequently, the expected cumulative return

$$E[r_1^W r_2^W \cdots r_t^W]$$

on constantly reinvesting all the money in equity is given by

$$E[r_1^W r_2^W \cdots r_t^W] = \left( \frac{E[X_1]}{f(\delta, \gamma - 1)} \right)^\tau$$

Thus, the per period return on holding equity is given by

$$\frac{E[X_1]}{f(\delta, \gamma - 1)}$$
Now, if the dividends $W$ are growing on average, i.e., $E[\log X_1] > 0$, then the per period return on holding a long maturity call option is, by Theorem 6.11, given by
\[
\frac{E[X_1]}{f(\delta, \min\{\gamma - 1, \mathcal{G}\})} \geq \frac{E[X_1]}{f(\delta, \gamma - 1)}
\]
where the last inequality follows from Lemma 6.8. That is, in the long run, it is more profitable to hold a long maturity option than the equity itself.

This is a very important theoretical prediction. It would be very interesting to compare yields on long maturity call options with long run cumulative equity returns and check the above theoretical prediction. In particular, it would allow us to get some idea about the size of risk aversion $\gamma$ and to find out whether $\gamma \geq (\leq) \mathcal{G}$.

7 Appendix: A sketch of proof of Theorem 3.2

To prove existence, we introduce the excess utility map (see, e.g., [6]). We introduce new parameters (social utility weights) $\lambda_i := x_i^{\gamma_i}$ (note, that we do not normalize $\lambda_i$’s to sum up to 1). Let $\mathbf{T} = \{0, \cdots, T\}$ and $e: \mathbb{R}^n_+ \to \mathbb{R}^n$ be defined by
\[
e_i(\lambda_1, \cdots, \lambda_n) := \lambda_i^{-1} \left( \lambda_i^{b_i} \left( \sum_{t \in \mathbf{T}} \delta_t^{b_i} E[G^{1-b_i}_t] \right) - \sum_{t \in \mathbf{T}} E[w_t G_t] \right)
\]
where $G_t = G_t(\lambda_1, \cdots, \lambda_n, s)$ solves
\[
\sum_{i \in \mathbb{N}} G_t^{-b_i} \delta_t^{b_i} \lambda_i^{b_i} = W_t(s).
\]
(We do not use here the normalization $G_0 = 1$ (i.e. $M_0 = 1$) but $G_0$ fulfills the same kind of equation as $G_t$ for $t \geq 1$).

**Proposition 7.1** State price densities $M_t$, $t \geq 1$ solve equilibrium equations if and only if
\[
e(\lambda_1, \cdots, \lambda_n) = 0
\]
Here, $\lambda_i = x_i^{\gamma_i}$.

We prove that $e$ has all properties of an excess demand and then the standard existence result from [17], p.585 implies existence.

**Lemma 7.2** The excess utility map $e$ satisfies
1. $e$ is homogeneous of degree zero;
2. $\sum_{i \in \mathbb{N}} \lambda_i e_i = 0$;
3. $e$ is continuous in $\mathbb{R}^n_+$;
4. $e_i$ is bounded from above for all $i$ and if $\lambda_i \to 0$ then $e_i \to -\infty$. 

Properties 1. and 2. follow from the definition of $e$.

Property 3. We treat the two terms in (7.1) separately.

We now apply the trick of viewing a sequence of random variables as one variable, but on a larger probability space. Let $\Omega_\infty = \Omega \times T$ be the union of an infinite number identical copies of $\Omega$ and let $\nu_\infty$ be the measure on $\Omega_\infty$, coinciding with $\mu$ (the original probability measure on $\Omega$) when restricted to any copy of $\Omega$. Let $A := (\lambda_1, \ldots, \lambda_n)$ and $G(A)$ be the random variable on $\Omega_\infty$, equal to $G_t$ on the $t$-th copy. In just the same way we define $W = (W_t)_{t \geq 0}$ and $w_i = (w_{it})$. Let also $D_i := (\delta^i_t)_{t \in T}$.

Then we can rewrite (7.1) in the form

$$e_i(A) = \lambda_i^{-1} \left( \lambda_i^b \int_{\Omega_\infty} D_i^b G^{1-b} d\nu_\infty - \int_{\Omega_\infty} w_i G d\nu_\infty \right) \quad (7.2)$$

Lemma 4.2 yields that

$$G \leq K \sum_j \lambda_j D_j W^{-\gamma_j} \quad (7.3)$$

and therefore

$$w_i \cdot G \leq \sum_j \lambda_j D_j w_i W^{-\gamma_j} \leq \sum_{j \in N} \lambda_j D_j W^{1-\gamma_j}. \quad (7.4)$$

In terms of $\Omega_\infty$, Assumption 3.2 means that the variable $D_j W^{1-\gamma_j}$ is $d\nu_\infty$-integrable for any $j \in N$. It follows from the definition of $G_t$ that $G_t(A)$ is continuous in $A \in \mathbb{R}_++^n$ for any $t, s$. Estimate (7.4) and the Lebesgue dominated convergence theorem imply that we can pass to the limit under the integral $\int_{\Omega_T} w_i G d\nu_\infty$ and hence $\int_{\Omega_T} w_i G d\nu_\infty$ is continuous in $A$. To control the first term in (7.2), we note that for any compact set $X \subset \mathbb{R}_++^n$ there exists a constant $K$ (depending on $x$), such that

$$\delta^b_t G_t^{1-b}(A, s) \leq K \sum_{j \in N} \delta^j_t W_t^{1-\gamma_j}(s),$$

that is

$$D_i^b G_t^{1-b} \leq K \sum_{j \in N} D_j W^{1-\gamma_j}. \quad (7.5)$$

Now, the Lebesgue dominated convergence theorem yields the required continuity.

Properties 4 and 5. The case $b_i \geq 1$. In this case

$$\lambda_i^b \sum_{t \in T} \delta^b_t E[G_t^{1-b}] \leq \lambda_i \sum_{t \in T} \delta^b_t E[W_t^{1-\gamma}] = \lambda_i K_1.$$

It follows from Lemma 4.2 that

$$\sum_{t \in T} E[w_t G_t] \geq K \sum_{j \in N} \lambda_j \sum_{t \in T} \delta^j_t E[w_j W_t^{-\gamma_j}] \geq K_1 \sum_{j \in N} \lambda_j. \quad (7.6)$$
remains strictly positive. Thus we have
\[ e_i \leq K_1 - K_2 \lambda_i^{-1} \sum_{j=1}^{n} \lambda_j. \]
It follows that \( e_i \) is uniformly bounded from above and goes to \(-\infty\) as soon as not all \( \lambda_j \)'s go to zero.

If \( b_i < 1 \), then
\[ \lambda_i^{b_i} \sum_{i \in T} \delta_i^{b_i} E[G_i^{1-b_i}] \leq K \lambda_i^{b_i} \sum_{j \in N} \lambda_j^{1-b_i}. \quad (7.7) \]
If \( \lambda_i \) goes to zero and \( \lambda_j \) stay bounded for \( j \neq i \) and not all of them go to zero, we have by (7.6), (7.7) that
\[ e_i \leq \lambda_i^{-1} \left( K_1 \lambda_i^{b_i} \sum_{j \in N} \lambda_j^{1-b_i} - K_2 \sum_{j \in N} \lambda_j \right). \quad (7.8) \]
If \( \lambda_j \)'s stay in a bounded region, \( \lambda_i \to 0 \) and \( \lambda_j \not\to 0 \) at least for one \( j \), then (7.8) implies
\[ e_i \leq -K_3 \lambda_i^{-1} \to -\infty. \]
Finally, it is not difficult to see that
\[ e_i \leq \lambda_i^{-1} K_1 \left( \lambda_i^{b_i} \sum_{j \in N} \lambda_j^{1-b_i} - (K_2 K_1^{-1}) \sum_{j \in N} \lambda_j \right) \leq K_1 \frac{L(K_2 K_1^{-1}) \lambda_i}{\lambda_i} = K_1 L(K_2 K_1^{-1}). \quad (7.9) \]