Incomplete information, idiosyncratic volatility and stock returns

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Abstract

We develop a $q$–theoretic model of investment under incomplete information that explains the link between idiosyncratic volatility and stock returns. When calibrated to match properties of the US business cycles as well as various firms and industry characteristics, the model generates a negative relation between idiosyncratic volatility and stock returns. We show that conditional on earning surprises, the link is positive after good news and negative after bad news. This result provides new insights on the nature of stock return predictability.

Keywords: Idiosyncratic volatility, incomplete information, cross–section of returns, $q$–theory of investment.

JEL Classification. G12, D83, D92.
According to textbook asset pricing theory, investors are only compensated for bearing aggregate risk and, as a result, idiosyncratic volatility should not be priced. However, numerous recent studies have documented a relation between stock returns and idiosyncratic volatility. In particular, Ang, Hodrick, Xing, and Zhang (2006) and Jiang, Xu, and Yao (2007) provide evidence of a negative relation for the US stock market and Ang, Hodrick, Xing, and Zhang (2008) confirm in a recent study that a similar relation also holds in other markets. There is however no consensus as to the direction of this effect. Indeed, Malkiel and Xu (2001), Spiegel and Wang (2005) and Fu (2005) find positive relations between idiosyncratic volatility and expected returns, while Longstaff (1989) finds a weakly negative relation.

In this paper we propose a model of firm valuation under incomplete information that is able to explain the ambiguous link between idiosyncratic volatility and stock returns. Firms in our model are unable to perfectly anticipate the growth rate of their cash flows, but learn about it by observing a firm specific signal. As a result, the shock perceived by a firm at a given point in time, the so-called innovation process, differs from that which is measured by the econometrician who conducts unconditional tests based on the whole history of stock returns. Indeed, the innovation process is the sum of two terms: the underlying true idiosyncratic shock and the error that the firm makes in estimating the growth rate of its cash flows. In contrast, the econometrician uses the actual underlying distribution to construct his tests and, hence, is able to measure the true idiosyncratic shocks of the firms.

Conditional on the information available to the firm, the estimation error is equal to zero on average and it follows that the expected returns satisfy a conditional version of the CAPM where idiosyncratic volatility plays no role. In contrast, relative to the true underlying distribution, the firm’s estimation error is different from zero and thus appears multiplied by the
idiosyncratic volatility in the expected stock return as measured by the econometrician. This is the mechanism which generates a relation between idiosyncratic volatility and stock returns, and we verify using simulations that our model is able to replicate the empirical regularities documented by Ang et al. (2006; 2008) and Jiang et al. (2007) among others. It is important to observe that the deviation from the CAPM which is implied by our model is not due to a missing factor. Indeed, the additional term in the expression of a firm’s expected stock return is generated by the firm’s own estimation errors and, hence, does not represent remuneration for exposure to a systematic risk factor. The presence of such a component in expected returns is entirely due to learning and could not be generated by introducing additional states variables into an otherwise standard model.

Our aim is to explain properties of the cross-section of stock returns. To this end, we need a valuation model in which heterogeneity among firms arises endogenously through time. Furthermore, we want to be able to calibrate the model to observable firm and industry characteristics. In order to achieve these goals, we focus on a simple version of the $q$-theoretic model of investment with adjustment costs which has been successful in describing many properties of the cross section of stock returns (see Liu, Whited, and Zhang (2007), Li, Livdan, and Zhang (2007) and the references therein). Specifically, we assume that each firm is endowed with a constant returns to scale production function and faces quadratic capital adjustment costs. The growth rate of the firm specific output price follows a two state Markov chain which is common to all firms, and thus proxies for the business cycles.\(^1\) We assume that firms only observe the output prices and must therefore estimate the current value of the growth rate.

This simple specification delivers a closed form expression for the value

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\(^1\)Similar specifications have been used in the asset pricing literature by Veronesi (1999; 2000) and David (1997), in the corporate finance literature by Hackbarth, Miao, and Morellec (2006) and in the investment literature by Guo, Miao, and Morellec (2005) and Eberly, Rebello, and Vincent (2006).
of the firm which allows for a transparent analysis of the relation between idiosyncratic volatility and stock returns. In particular, the model shows that, relative to the true distribution from which the shocks are drawn, the expected excess stock return is the sum of two terms. The first term is the usual remuneration for exposure to aggregate risk, namely the product of the firm’s beta and the market price of risk. The second term is the product of the firm’s idiosyncratic volatility and a normalized earning surprise. This term is unique to our incomplete information setting and is the channel through which idiosyncratic volatility and stock returns are related.

From this specification, we identify two empirically testable implications. First, we show that the response of returns to earning surprises depends on idiosyncratic volatility in an asymmetric way. Specifically, the model implies that, following good news, firms with larger idiosyncratic volatility should produce larger returns. Following bad news, the relation is reversed and firms with larger idiosyncratic volatility should produce lower returns. This implication relates to the vast literature on stock return predictability, see Ball and Brown (1968), Watts (1978), Foster, Olsen, and Shevlin (1984) and Bernard and Thomas (1990) among others. Our contribution is to show, theoretically, why the reaction to news should be stronger among high idiosyncratic volatility firms. Recent results in the empirical literature support this prediction. In particular, Zhang (2006) studies the link between information uncertainty and stock returns. Sorting firms on past return volatility, he shows that firms with high volatility perform relatively better following good news and relatively worse following bad news. Our prediction is consistent with this finding because our model predicts that the firms with large total volatility are also those with large idiosyncratic volatility.

The deviation from the traditional CAPM implied by our setting is driven by earning forecast errors. Therefore the second implication of our model is that, controlling for earning surprises, idiosyncratic volatility should
not have explanatory power for the cross section of stock returns. In other words, introducing this control variable should reduce the volatility anomaly. Here again, there is recent empirical evidence in the literature to support the prediction of our model. Jiang et al. (2007) replicate the findings in Ang et al. (2006) by showing that idiosyncratic volatility has a negative and significant coefficient in the standard Fama-MacBeth regressions. However, when controlling for analyst forecast errors, they obtain a non-significant coefficient for idiosyncratic volatility. Since such forecasting errors are a reasonable proxy for the additional term implied by our model, the results of Jiang et al. (2007) provide strong support for our prediction.

While our model predicts that idiosyncratic volatility and stock returns should be related, it does not say whether this relation should be positive or negative. The empirical evidence documenting the idiosyncratic volatility anomaly in the cross section of stock returns relies on sorting stocks into portfolios on the basis of past idiosyncratic volatilities. The sign of the relation that would be obtained in our model using a similar construction depends on the distribution of the estimation errors among the volatility sorted portfolios. Unfortunately, the complex path dependence of the firms’ estimations prevents us from deriving the properties of this distribution analytically. We therefore resort to simulations to determine whether our model can replicate the empirical evidence.

The starting point of our simulation is a careful choice of the parametric specification. In particular, we calibrate the model to match the dynamics of US business cycles as well as several empirical moments of industry and firms characteristics. We replicate the portfolio construction of Ang et al. (2006; 2008) on simulated panels of data and obtain qualitatively similar results. On average, portfolios of firms with high idiosyncratic volatility generate lower returns and have lower alphas. In our model the sign and magnitude of the idiosyncratic volatility effect is driven by the joint distri-
bution of estimation errors and idiosyncratic volatility. For the calibrated set of parameters, expansions are more frequent than recessions so that the unconditional mean of the growth rate is close to the high state. It follows that the estimation errors are large and negative during recession and small and positive during expansion. While expansions are more frequent, the overall effect is dominated by the negative contribution of recessions. This suggests that the asymmetry in the distribution of the growth rate is the key element needed to generate a negative idiosyncratic volatility effect. We show that this is indeed the case by simulating the model with a symmetric distribution.

The remainder of the paper is organized as follows. In Section I, we formulate our incomplete information model and derive an analytical solution for the value of an individual firm. In Section II, we provide a theoretical analysis of the relation between idiosyncratic volatility and stock returns and we derive testable implications. In Sections III, we detail the simulation methodology and the calibration of the model. We also present the results of regressions performed on artificial panel data and study the determinants of the relation by varying key parameters. Section IV concludes.

I. The Model

In this section we construct a model of capital investment with adjustment costs under incomplete information. Heterogeneity among firms arises endogenously as each firm faces a specific output price that comprises both an idiosyncratic and an aggregate shock.

As in Berk, Green, and Naik (1999), we focus on a partial equilibrium model in the sense that we take the pricing kernel as given. This gives us the tractability we need in order to focus on the relation between idiosyncratic volatility and stock returns.
A. Information Structure

We consider a continuous time model of an economy in which firms sell their output at a firm specific price $X_i$. This firm specific price process has a stochastic growth rate which is common to all firms and is affected by both idiosyncratic and aggregate shocks.

**ASSUMPTION 1:** For each $i$, the firm specific price $X_i$ evolves according to

$$dX_{it} = X_{it}\theta_t dt + X_{it}\sigma\left[\rho dB_{at} + \sqrt{1 - \rho^2}dW_{it}\right]$$  \hspace{1cm} (1)

where $B_a$, $W_i$ are independent Wiener processes and $(\rho, \sigma)$ are constants. The process for the growth rate $\theta$ is described below.

The constant $\rho\sigma$ measures the exposure of the firm specific price process to the aggregate shock $B_a$ which is common to all the firms. The constant $\sigma\sqrt{1 - \rho^2}$ measures the exposure of the price process to the firm specific Wiener process $W_i$. The instantaneous volatility of the price process is identical across firms and equal to $\sigma$. Similarly, the instantaneous correlation between the prices faced by firms $i$ and $j \neq i$ is identical across firms and equal to $\rho$. The firm specific prices grow at the rate $\theta$ which is common to all firms and satisfies the following.

**ASSUMPTION 2:** The growth rate of the price processes follows a two-state, continuous time Markov chain with generator matrix $^2$

$$\Lambda = \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}.$$  \hspace{1cm} (2)

The two states of the Markov chain are denoted by $\theta_h > 0$ and $\theta_l < 0$ and are referred to as the low and the high state of the economy.

The growth rate of the price processes can jump from one state to another, simultaneously for all firms. Furthermore, the above assumption implies

$^2$See Karlin and Taylor (1975, Chapter 4) for a precise definition of the generator matrix. The transition matrix of the chain over a period of length $t$ can be obtained from the generator matrix simply as $\exp(\Lambda t)$. 

that the transition times between the high and the low state on one hand, and between the low and high state on the other hand, are exponentially distributed with parameters $\lambda$ and $\mu$. This is a simple way of introducing business cycles into the model. In the high state, the economy is expanding and all firms benefit from the positive trend in prices. In the low state, the economy is contracting and all firms suffer from the negative trend in prices. Nevertheless, in both states firms may be subject to specific adverse or beneficial market conditions which are modeled through their exposure to the idiosyncratic shocks.

A key feature of our model is that agents have incomplete information about the growth rate of the output price processes. Firms are run by managers who act in the best interest of shareholders. These managers play no role other than processing information and implementing the corporate strategy that maximizes shareholder value. In the model, the managers base their anticipations on the observation of different signals and hence have different perceptions about the current state of the economy. In particular, we make the following assumption.

ASSUMPTION 3: The manager of firm $i$ only observes the realizations of the aggregate shock $B_a$ and the price process $X_i$ faced by his firm.

Since the manager only observes the aggregate shock and the price faced by his firm, his estimation of the current state will depend only the realizations of his firm’s price process. An identical information processing behavior can be obtained by assuming that the managers observe all the price processes but do not recognize the fact that the growth rate is common to all firms. The above assumption is thus behavioral as it implies that the managers are biased in the way they learn about the state of the

\[\text{The interpretation of } \theta \text{ as a model of the business cycle indicates precisely how to calibrate the generator matrix. In the simulations of the model we use the frequencies of the business cycles as reported by the NBER to calibrate the transitions between the two states of the growth rate and investment rates to calibrate its values.}\]
Since the manager of firm $i$ has complete information about the aggregate shocks, he bases his estimation of the growth rate on the observation of a firm specific signal $s_i$ which evolves according to

$$ds_{it} = \frac{dX_{it}}{X_{it}} - \rho \sigma dB_{it} \equiv \theta_{it} dt + (1/\epsilon) dW_{it}. \quad (3)$$

The constant $\epsilon = 1/\sigma \sqrt{1 - \rho^2}$ measures the precision of the signal. When $\epsilon$ is high, the signal has very little dispersion around the true value of the growth rate and firms are able to estimate $\theta$ accurately. On the contrary, when $\epsilon$ is low, the variance of the signal is very large and firms are thus unable to estimate the growth rate accurately.

Let $\mathcal{F}_{it}$ denote the information set which is available at time $t$ to the manager of firm $i$. This information set contains all past realizations of the aggregate shock and the firm specific price process. Further let $E_{it}$ denote the expectation conditional of that information set and the manager’s prior, and

$$m_{it} = E_{it}[\theta_i] \equiv E[\theta_i | \mathcal{F}_{it}] \quad (4)$$

denote the manager’s estimation of the current growth rate of the price process. The following well-known lemma (see for example Liptser and Shiryaev (2001, p.372) or David (1997)) shows that the evolution of the estimated growth rate can be described by a diffusion process.

**LEMMA 1**: Assume that the prior of manager $i$ is represented by $p_i \in [\theta_l, \theta_h]$. Then his estimation of the growth rate evolves according to

$$dm_{it} = (\lambda + \mu)(m_{it} - \bar{m}) dt + \epsilon (m_{it} - \theta_{it})(\theta_h - m_{it}) dB_{it} \quad (5)$$

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4The Brown and Rozeff (1979) model, which is one of the most popular earnings forecast model, relies exclusively on past earnings and does not use any economy wide variables. Since the firm’s output is locally deterministic, forecasting prices or earnings is equivalent in our setting and it follows that Assumption 3 can be understood as an application of the standard forecasting practice.
subject to the initial condition \( m_{i0} = p_i \). In this equation, the constant \( \overline{m} \) is the unconditional mean of the growth rate and

\[
B_{it} = \int_0^t \epsilon \left( ds_{iu} - m_{iu} du \right). \tag{6}
\]

is a standard Wiener process with respect to the information set \( \mathcal{F}_{it} \) which is available to the manager of firm \( i \).

The dynamics of the estimated growth rate given in the above lemma are quite intuitive. The stochastic shock \( dB_i \) is the normalized innovation in the firm’s signal, that is the difference between the observed signal and its expected value divided by the volatility of the signal observed by the firm. The specific form of the volatility of \( m_i \) guarantees that the firm’s estimation takes values in the interval \([\theta_l, \theta_h]\) induced by the support of the growth rate. Finally, the drift is a mean reverting component which pushes back the firm’s estimation towards the unconditional mean

\[
\overline{m} = \theta_l + \frac{\mu}{\mu + \lambda}(\theta_h - \theta_l) \tag{7}
\]

of the growth rate process. The fact that the coefficient of mean reversion increases with \( \lambda + \mu \) is due to the property that the speed of convergence of the Markov chain towards its stationary distribution increases with the frequency of the shifts, see Karlin and Taylor (1975, Chap. 4).

Equipped with the definition of the innovation process, we can rewrite the dynamics of the firm specific price process as

\[
dX_{it} = X_{it} m_{it} dt + X_{it} \left[ \rho \sigma dB_{at} + \left(1/\epsilon \right) dB_{it} \right]. \tag{8}
\]

In conjunction with equation (5), this equation shows that the information set which is available to firm \( i \) coincides with the information set generated by the pair \((m_i, X_i)\). It follows that the relevant state variables for the firm valuation problem are the firm’s price process and its estimation of the current growth rate.
In order to complete the description of the information structure in the model, we need to specify what information is available to the investors in the market. This is the purpose of the following assumption.

**ASSUMPTION 4**: The manager of firm $i$ publicly releases the values of $m_i$ and $X_i$. Investors take these values and the dynamics (5), (8) as given for all firms in the economy.

The first part of the above assumption insures that, for each firm $i$, the manager’s forecast of the firm’s growth rate is readily available to investors. Since investors take the dynamics of $m_i$ and $X_i$ as given, this further implies that all agents in the model agree on the state variables that are relevant to the firm and, hence, also on the value of the firm.

There is empirical evidence, that managers do provide such information to investors, either directly or indirectly through analysts covering the firm. In particular, Ajinkya and Gift (1984) find that managers release earnings forecast in order to move the investors’ earnings expectations towards the management forecast. Similarly, Graham, Harvey, and Rajgopal (2005) find that CFOs provide earnings guidance to analysts if there is a significant gap between analysts’ forecasts and internal projections.

The second part of Assumption 4 is quite natural in the context of our model as it implies that the information available to investors is simply the aggregation of the information available to the managers. However it is important to note that investors are not trying to estimate the true state of the economy from their observation of the processes $m_i$ and $X_i$. They take the dynamics in equations (5) and (8) as given and do not internalize the fact these arise from the filtering of the growth rate by the managers.\(^5\)

\(^5\)An identical information structure can be obtained by assuming that investors do not realize that the growth rate is common to all firms and, hence, estimate the growth rate of firm $i$ by considering $X_i$ and $B_a$ only.
B. Firm Valuation

Each firm uses capital $K$ and labor $L$ to produce output according to the isoelastic Cobb-Douglas production function

$$Y(K, L) = AK^{1-\zeta}L^\zeta$$

where $A$ is a nonnegative constant and $\zeta \in (0, 1)$ represents the constant share of labor. The firm pays a fixed wage $w$ and can costlessly adjust its labor input. As a result, the firm’s operating profit is given by

$$\pi(X_{it}, K_{it}) = \max_x Y(K_{it}, x)X_{it} - wx = DX_{it}^\Phi K_{it}. \quad (9)$$

where $\Phi = 1/(1-\zeta)$ and $D \equiv D(A)$ is a nonnegative constant which we normalize to one by choosing the value of the constant $A$.

The firm undertakes gross investment $I_i$ and incurs depreciation at a constant rate $\delta \geq 0$. Consequently, the dynamics of its capital stock are

$$dK_{it} = (I_{it} - \delta K_{it})dt. \quad (10)$$

Investment is reversible but capital cannot be adjusted costlessly. Following Abel and Eberly (1997) we assume that the instantaneous investment cost function is given by

$$\phi(I_{it}) = bI_{it} + \frac{1}{2\gamma}I_{it}^2 \quad (11)$$

where $b \geq 0$ represents the purchase price of one unit of capital and $\gamma > 0$ is a constant that measures the severity of the adjustment costs. The fact that $\phi$ is convex reflect the fact that the more units of additional capital the firm tries to incorporate into the existing one, the less effective those units are at expanding firm capacity on the margin. The specification of the capital adjustment technology can be generalized to include numerous features such as asymmetric adjustment costs (Zhang (2005), Hall (2001)), fixed costs (Abel and Eberly (1994; 1997), Cooper (2006)) or irreversibility (Abel and Eberly (1994)). We choose to focus on the simple specification in
equation (11) because it allows for an explicit solution to the firm valuation problem.

Following Hall (2001) and Zhang (2005) we assume that the firm can costlessly issue new equity shares if its operating cash flows are not sufficient to finance new investments. On the other hand, when operating cash flows are larger than investment expenses the firm pays dividend to its shareholders. Accordingly, the total cash flow paid to the shareholders of firm $i$ at time $t$ is given by

$$C_{it} = \pi(X_{it}, K_{it}) - \phi(I_{it}) = X_{it}^\Phi K_{it} - bI_{it} - \frac{1}{2\gamma}I_{it}^2. \quad (12)$$

Abstracting from agency issues, we assume that the manager of the firm acts in the best interest of shareholders and hence chooses an investment strategy that maximizes the market value of the firm. To identify the latter, we define a stochastic discount factor.

ASSUMPTION 5: Financial markets are complete. Assets can be valued by discounting future cash flows using the stochastic discount factor

$$\xi_t = \exp \left[ -rt - \kappa B_{at} - \frac{\kappa^2}{2} f \right]. \quad (13)$$

In this equation, the constants $r$ and $\kappa$ represent, respectively, the risk free rate and the market price of aggregate risk.

The specification of the stochastic discount factor is quite natural in the context of our model. Indeed, only the aggregate shocks which are common to all firms carries a risk premium. Furthermore, the fact that all investors observe the aggregate shock $B_a$ implies that they have the same perception of the stochastic discount factor and this property is crucial to guarantee that they agree on the prices of all traded assets.\(^6\)

\(^6\)The specification of the stochastic discount factor could be generalized to allow for a stochastic risk free rate and a stochastic risk premium as in Berk et al. (1999) or Zhang (2005). We focus on a specification where these components are constant for tractability. However, the mechanism which drives our result does not rely on this assumption and hence would still prevail under more general specifications.
Putting together the various pieces of the model, we can now formally define the value of firm $i$ as

$$V_{it} = \max \ E_{it} \int_{t}^{\infty} \xi_{t,s} \left[ \pi(X_{is}, K_{is}) - \phi(I_{is}) \right] ds,$$  

(14)

where $\xi_{t,s} = \xi_t / \xi_s$ is the stochastic discount factor at time $t$ for cash flows which are paid at time $s \geq t$. The following proposition derives an analytical solution for the value of the firm.

PROPOSITION 1: Assume that the parameters of the model satisfy

$$\frac{r}{\Phi} > \max \left[ \theta_h - \rho \sigma \kappa - \frac{1}{2} \sigma^2 (1 - \Phi); 2 \theta_h - 2 \rho \sigma \kappa - \sigma^2 (1 - 2 \Phi) \right].$$  

(15)

Then the value of firm $i$ and its optimal investment policy are given by

$$V_{it} = Q(m_{it}, X_{it})K_{it} + G(m_{it}, X_{it}),$$  

(16)

$$I_{it} = \gamma Q(m_{it}, X_{it}) - \gamma b.$$  

(17)

In the above equations, the marginal value of the firm’s capital, $Q$, and the market value of the firm’s growth options, $G$, are defined by

$$Q(m_{it}, X_{it}) = (q_0 + q_1 m_{it})X_{it}^\Phi,$$  

(18)

$$G(m_{it}, X_{it}) = g_0 + g_1(m_{it})X_{it}^\Phi + g_2(m_{it})X_{it}^{2\Phi},$$  

(19)

where the constants $q_0$, $q_1$ and $g_0$ and the functions $g_1(m)$ and $g_2(m)$ are defined in the appendix.

The above proposition is in line with the neoclassical $q$–theory of investment according to which a firm invests when the value of an additional unit of capital exceeds its purchase price and disinvests otherwise. As in Abel and Eberly (1997), the combination of constant returns to scale and capital independent adjustment costs implies that both the marginal value of capital and the value of the firm’s growth options are independent of firm size as measured by its capital stock.
Specific to our analysis is the fact that there are two state variables influencing the firm’s investment behavior: the specific price \( X_i \) and the expected growth rate \( m_i \). Both of these variables affect the marginal value of capital and hence condition the firm’s investment policy and the evolution of its capital stock. Since the marginal value of capital is a linear function of the estimated growth rate it follows that

\[
Q(m_{it}, X_{it}) = E_{it}[Q(\theta_{it}, X_{it})]
\]

where \( Q(\theta_{it}, X_{it}) \) is the marginal value of capital that would prevail in a full information context.\(^7\) Since investment is linear in the marginal value of capital, this further implies that investment under incomplete information is the expectation of its full information counterpart. Incomplete information reduces the optimal investment level in the high state and increases it in the low state. This effect is illustrated by Figure 1 which plots a simulated path of the firm’s optimal investment policy under both complete and incomplete information.

In the simulated path of Figure 1, the high state is more likely than the low state (\( \mu \gg \lambda \)) and therefore the unconditional mean \( \overline{m} \) of the growth rate is closer to the high value of the growth rate \( \theta_h \). Since the estimated growth rate reverts to its mean \( \overline{m} \), this implies that the firm’s investment adjusts slowly in the low state and rather fast in the high state. In the postwar US economy, business cycles present a similar asymmetry with short recessions and long expansions. We discuss in Section III.C the crucial role of this asymmetry in our explanation of the relation between idiosyncratic volatility and stock returns.

\(^7\)When the growth rate is constant (\( \theta_h = \theta_l = \theta \)) and investors have full information, the expectation becomes irrelevant. In that case, the marginal value of capital is given by \( Q_i = X_i^\theta / B \) for some nonnegative constant \( B \) as in Abel and Eberly (1997).
II. Idiosyncratic Volatility and Stock Returns

This section derives the relation between idiosyncratic volatility and stock returns implied by the model. We first discuss our theoretical findings in Section II.A and then discuss testable implications in Section II.B.

A. Theoretical Findings

Our model implies that the unexpected variations in firm values, as perceived by investors, correspond to the innovation processes induced by the managers’ forecasts. The driving forces remain nonetheless the original aggregate and idiosyncratic shocks $B_a$ and $W_i$, which generate the evolution of the firm specific price processes. When performing unconditional tests on data generated by the model, we do not capture the distributional properties of the returns as perceived by investors. Instead, we measure a combination of perceived returns, which are solely due to exposure to aggregate risk, and forecast errors which are due to incomplete information. In other words, basic regression results provide coefficient estimates which are drawn from the true underlying distribution, and not from the perceived conditional distribution of stock returns.

To clearly identify the respective contributions of exposure to aggregate risk and forecast errors to stock returns, we start by analyzing the dynamics of the firm value. Applying Itô’s lemma to the expression of the firm value given in Proposition 1 we obtain

$$\frac{dV_{it} + C_{it}dt}{V_{it}} = (r + a_{it}\kappa)dt + a_{it}dB_{it} + \iota_{it}dB_{it},$$

(20)

where

$$a_{it} = \rho\sigma X_{it} \frac{V_x(m_{it}, X_{it})}{V(m_{it}, X_{it})},$$

(21)

denotes the firm’s aggregate, or systematic, volatility and

$$\iota_{it} = \frac{1}{\epsilon}X_{it} \frac{V_x(m_{it}, X_{it})}{V(m_{it}, X_{it})} + \epsilon(m_{it} - \theta_l)(\theta_h - m_{it}) \frac{V_m(m_{it}, X_{it})}{V(m_{it}, X_{it})},$$

(22)

15
denotes the firm’s idiosyncratic volatility. Both of these volatilities contain a term which comes from the sensitivity of the firm value to variations in the output price. However, idiosyncratic volatility is also driven by a specific component which comes from incomplete information.

The absence of arbitrage opportunities implies that, conditional on the information available to investors, the expected instantaneous stock return depends only on the firm’s exposure to aggregate risk as measured by the aggregate volatility $a_i$. Therefore, from the point of view of investors, a version of the intertemporal CAPM holds. The econometrician’s perspective is different. Using the link between the innovation process and the original Wiener process, we can write the dynamics of the firm value as

$$\frac{dV_{it}}{V_{it}} = (r + a_{it}\kappa + \iota_{it}\eta_{it})dt + a_{it}dB_{it} + \iota_{it}dW_{it}, \quad (23)$$

where

$$\eta_{it} = \epsilon(\theta_t - m_{it}). \quad (24)$$

Relative to equation (20), the drift now contains an additional component which depends on the idiosyncratic volatility of the firm $\iota_i$ and the manager’s forecast error $\theta_t - m_{it}$. Conditional on the information available to investors this component is null on average since

$$E_{it}[\iota_{it}\eta_{it}] = E_{it}[\iota_{it}\epsilon(\theta_t - m_{it})]$$

$$= \iota_{it}\epsilon(E_{it}[\theta_t] - m_{it}) \equiv 0,$$

by definition of the manager’s forecast $m_i$. However, conditional on the whole information set (i.e. knowing the true state of the economy) this term becomes observable and hence satisfies

$$E_t[\iota_{it}\eta_{it}] = \iota_{it}\eta_{it} \neq 0. \quad (25)$$

This difference in the measurements of the average returns by investors on the one hand and the econometrician on the other is the key mechanism that
allows us to obtain a link between idiosyncratic volatility and stock returns. When $\theta_t > m_{it}$ ($\theta_t < m_{it}$) there is a positive (negative) shock which is interpreted as being part of the innovation and therefore does not contribute to the investors’ perception of the expected return. This shock will however affect a time series estimation of the mean stock returns because these are drawn from the true distribution which includes the additional term $\iota_t\eta_t$ in the drift of the firm value process.

We summarize the previous discussion and present our main result on the expected excess return equation in the following proposition.

**PROPOSITION 2:** The instantaneous expected excess return conditional on the whole information set is given by

$$R_{it} - r = E_t \left[ \frac{dV_{it}}{V_{it} dt} + C_{it} dt \right] - r = a_{it}\kappa + \iota_{it}\eta_{it}. \quad (26)$$

where $a_{i}$ and $\iota_{i}$ denote the firm’s aggregate and idiosyncratic volatility and $\eta_{i}$ is the normalized forecast error.

Equation (26) is not a multi-factor specification in the tradition of Merton (1973) intertemporal CAPM. The first term on the right hand side is a remuneration for the exposure to aggregate risk. The second term, however, comes from the forecast error and is not a remuneration for risk. This term depends on the level of idiosyncratic volatility and on the manager’s forecast, which are both firm specific, but it also depends on the current state of the economy $\theta$. Since the latter is common to all firms, and can only take two values, all forecast errors have the same sign. They do however differ in their magnitude, since a firm’s assessment of its current growth rate depends on the trajectory of its specific price process.

In the model, the dynamics of the cash flows follow from the firms’ investment decisions but this property is not necessary for Proposition 2 to hold for any specification of the common growth rate process.

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8 The two state process underlying this result allows for a simple interpretation of the common growth rate as a proxy for the evolution of the business cycle. It is however not necessary for the validity of our analysis since Proposition 2 holds for any specification of the common growth rate process.
hold. In particular, the return decomposition given in equation (26) holds for any specification of the cash flow as long as the information structure and the state variables are kept the same. The endogenous cash flow specification on which we focus, allows for a straightforward calibration of the model to observed firms and investment characteristics. Furthermore, our model implies that the heterogeneity among firms, and hence among idiosyncratic volatilities, arises endogenously as firms react optimally to changing market conditions. This makes the model more realistic and the level of heterogeneity more plausible than if we had exogenously postulated a cash flow process for each firm.

Proposition 2 describes the risk return relation conditional on the whole information set. In practice, an econometrician performing unconditional tests on data generated by the model would rely on a much smaller set. In particular, portfolio regressions similar to the one we conduct in Section III.C are constructed from realized stock returns which are averaged across time and stocks. Even in such a case, our analysis remains valid. To see this, consider the sample average excess return on an equally weighted portfolio of $n$ stocks over a time period of length $\tau$ starting at time $t$, that is

$$A_t(\tau, n) = \frac{1}{\tau n} \sum_{i=1}^{n} \int_{t}^{t+\tau} \left( \frac{dV_{is} + C_{is} ds}{V_{is}} - r ds \right).$$

To infer the mean excess return on the portfolio, an econometrician computes the time series average of successive realizations of the variable $A$. Using our previous results we may decompose each realization into the sum of three components

$$A_t(\tau, n) = \int_{t}^{t+\tau} \left( \frac{1}{nT} \sum_{i=1}^{n} a_{is} dB_{as} + \frac{1}{nT} \sum_{i=1}^{n} \epsilon_{is} dB_{is} \right)$$

$$+ \int_{t}^{t+\tau} \left( \frac{1}{nT} \sum_{i=1}^{n} a_{is} \kappa \right) ds + \int_{t}^{t+\tau} \left( \frac{1}{nT} \sum_{i=1}^{n} \epsilon_{is} \eta_{is} \right) ds.$$

When averaged across time the three terms in the above decomposition behave differently. The first term averages to zero as it represents the average
shock incurred by the portfolio. The second term is the standard reward for
the exposure of the portfolio to aggregate risk. Even if the firms’ forecasts
are conditionally unbiased, the last term is non zero on average because the
estimation error are multiplied by the firms’ idiosyncratic volatilities.

The sign and magnitude of the effect depend on the joint distribution
of forecast errors and volatilities. Due to the complex path dependence of
these variables, this distribution cannot be computed in closed form. We
therefore rely on simulations to assess the sign of the relation implied by our
model. We show in Section III.C that, when calibrated to match moments of
firms and industry characteristics, the model generates the negative average
relation documented by Ang et al. (2006; 2008) and Jiang et al. (2007).

B. Testable Implications and Empirical Support

The previous results can be used to derive two main implications which are
related to earning forecasts and idiosyncratic volatility. While we do not
perform a formal test of these predictions, we show that they are strongly
supported by recent findings in the empirical asset pricing literature.

Proposition 2 shows that the loading of stock returns on forecast errors
is given by the firm’s idiosyncratic volatility. This suggests that stocks with
larger idiosyncratic volatility should be more responsive to forecast errors.
In particular when the realized growth rate is higher than anticipated, i.e.
when $\theta > m_i$, firms with larger idiosyncratic volatility should have higher
returns than firms with lower idiosyncratic volatility. On the contrary, these
firms should have lower returns when $\theta < m_i$. This leads to the following
testable implication.

IMPLICATION 1: Following good news, firms with larger idiosyncratic volatil-
ity should produce relatively larger returns, and following bad news they
should produce relatively lower returns.

There is a vast literature documenting the predictability of stock returns
following earning announcements, see Ball and Brown (1968), Watts (1978), Foster et al. (1984) and Bernard and Thomas (1990) among others. The fact that good (bad) news are followed by positive (negative) returns is usually referred to in that literature as the post-earning announcement drift. Most of the theories proposed to explain this anomaly are behavioral. In particular, Bernard and Thomas (1990) suggest that investors underreact to news while Barberis, Shleifer, and Vishny (1998) rely on the representative heuristic and conservatism bias. Our model proposes an explanation based on incomplete information and, in addition, predicts that the effect should be stronger among high idiosyncratic volatility firms.\(^9\)

Recent results in the empirical literature provide support to the above implication. In his study of the link between information uncertainty and stock returns, Zhang (2006) provides a detailed analysis of the properties of portfolios sorted on the basis of different proxies for information uncertainty. One of the six proposed proxies is the stock volatility, which is measured by the standard deviation of weekly excess returns.\(^10\) Defining good and bad news according to the direction of the forecast revisions made by analysts, Zhang (2006) finds that firms with high volatility produce relatively lower returns following bad news and relatively higher returns following good news. These results, which are summarized in panel B of Table 1, provide strong support for Implication 1 as long as revisions of analysts’ forecasts qualify as a good proxy for the forecast errors which appear in equation (26).

---

According to Proposition 2, the expected excess return of a firm is the

\(^9\)Since information is revealed continuously through time there is no formal earnings announcement in our model. However, Implication 1 deals with instantaneous returns and thus describes the local relation between news and stock returns.

\(^10\)In general, firms with high total volatility do not necessarily have high idiosyncratic volatility. In our model the two quantities are linked through equations (21)–(22) and we show in Section III.C that sorting stocks on the basis of total or idiosyncratic volatility results in the same portfolios.
sum of two components. The first one is generated by exposure to aggregate risk and can be measured by the firm’s market beta. The second one is an idiosyncratic component, which is the product of the firm’s idiosyncratic volatility and a forecast error. This suggests that if one could control for these firm specific forecast errors, then idiosyncratic volatility should not play any role in explaining the cross section of stock returns. This instantaneous forecast error is difficult to measure empirically but earnings forecast errors should provide a reasonable proxy because, in our model, earnings are linear in the firm specific price process. This naturally leads to the following implication.

**IMPLICATION 2:** *Controlling for earning forecast errors, idiosyncratic volatility should not have explanatory power for the cross-section of stock returns.*

There is recent evidence in the literature that supports this prediction of our model. As part of their study of the information content of idiosyncratic volatility, Jiang et al. (2007) estimate the following linear model

\[
\text{Return}_{t+1} = b_0 + b_1 \text{IVOL} + b_2 \ln (\text{SIZE}) + b_3 \ln (\text{B/M}) + b_4 \text{PrRet}
+ b_5 \text{LEV} + b_6 \text{LIQ} + b_7 \text{SHOCK} + \varepsilon
\]  

(27)

where IVOL is a measure of idiosyncratic volatility and the variable SHOCK stands for six different unexpected earning measures. In particular, one of these measures is an analyst forecast error defined as follows: “realized quarterly earning per share (EPS) in excess of the mean of analysts EPS forecasts at the last month of the portfolio formation quarter (Q0), divided by the previous year’s book value of equity per share”. As explained above, this variable provides a reasonable proxy for the forecast error which appears in the instantaneous return equation (26).
The results of the Fama–MacBeth regression (27) are reported in Table II. First, omitting the SHOCK variables Jiang et al. (2007) obtain a negative and significant relation between idiosyncratic volatility and stock return. When introducing current (FERQ0) and one step ahead (FERQ1) earning forecast errors, idiosyncratic volatility becomes insignificant as predicted by our model. Jiang et al. (2007) provide an explanation which is related to selective corporate information disclosure. Our model does not validate or invalidate their explanation as it addresses the problem in a very different way. However, it is noteworthy that our model explains the direction of the effect following good or bad news, while theirs does not.

III. Implementation of the Model

In this section we use artificial data to show that our model is able to qualitatively replicate the idiosyncratic volatility anomaly. We describe the simulation methodology in Section III.A and discuss the choice of parameters in Section III.B. The results of the tests performed on the simulated data are presented in Section III.C.

A. Simulation Methodology

The model developed in the previous sections is set in continuous time and, hence, needs to be discretized before it can be simulated. To this end, we apply the standard Euler scheme which allows for a transparent discretization of the model’s dynamics.

Let $\Delta$ be a fixed time step. In accordance with the convention taken in Proposition 2, the return of firm $i$ is computed as

$$\text{Return}_{it} = \frac{V_{it} + C_{it}\Delta}{V_{it-\Delta}} - 1$$

where $V_i$ is the firm’s value process as defined in Proposition 1 and

$$C_{it} = X_{it}^\Phi K_{it} - \phi(I_{it})$$
is the cash flow of the firm. To compute beta coefficients, we exogenously define a return process for the market portfolio by letting

\[ dM_t = (r + \kappa)M_t dt + \sigma M_t dB_t. \]

Given this process, the return on the market is computed as

\[ \text{Market Return}_t = \frac{M_t - M_{t-\Delta}}{M_{t-\Delta}} \]

and the beta of firm \( i \) is obtained by regressing \( R_i \) on \( R_M \). In the above specification of the market portfolio, the constant volatility parameter \( \sigma_M \) has no effect on the estimation other than scaling the values of the beta coefficients. We choose its value in such a way that the average beta of the firms across all the simulations is close to one.

We simulate 10,000 artificial panels of data each with 500 firms sampled at a weekly frequency for a period of 20 years. All the statistics which appear in the following are obtained by averaging across simulations.

**B. Calibration**

The model has 12 parameters, which can be separated in three groups. The composition of these groups as well as the chosen parameter values are reported in Table III.

Insert Table III here

The first group contains the parameters whose values are directly measurable from the data or can be obtained from previous studies. The depreciation rate is set equal to \( \delta = 12\% \) following Cooper and Haltiwanger (2006). Following Kydland and Prescott (1982) we set the share of labor in the production function to \( \zeta = 0.7 \) so that the price elasticity of operating profits is given by \( \Phi = 10/3 \). The market price of risk \( \kappa \) is set equal to 0.3 which is close to the average Sharpe ratio in the US over the past 100 years, see Shiller (2005). The last parameter in this group is the risk free interest rate.
which we set equal to 14%. Evidently, this value is largely above the average one year rate of 4.8% measured in the US by Shiller (2005). However, one must recall that in order to guarantee that the firm value is well defined, we need to set the interest rate high enough so that

$$\frac{r}{\Phi} > \max \left[ \theta_h - \rho \sigma \kappa - \frac{1}{2} \sigma^2 (1 - \Phi); 2\theta_h - 2\rho \sigma \kappa - \sigma^2 (1 - 2\Phi) \right].$$

While we could set a more reasonable level of the interest rate by going to a finite horizon model, the assumption of an infinite horizon allows us to obtain closed form solutions which greatly simplify the analysis.\(^{11}\)

The parameters in the second group define the transition matrix of the growth rate process. To match the duration and frequency of business cycles in the postwar US as measured by the NBER (2008), we set the transition intensities to $\lambda = 0.233$ and $\mu = 1.154$. This parametrization implies that, on average, the length of a complete cycle is 61.9 months, with expansion phases of 51.5 months and contractions of 10.4 months.

The parameters of the third group are chosen in such a way that the moments obtained from the simulated data match a number of empirical moments of investment dynamics, stock returns, and book to market ratios. The parameters of the adjustment cost function play an important role in the determination of the size of growth options relative to the total firm value. We choose $\gamma$ and $b$ to match the mean and standard deviation of book-to-market ratios. Following Pontiff and Schall (1998), we construct a book-to-market ratio index by using the Dow Jones Industrial Average Index at a monthly frequency along with the previous year book value obtained from ValueLine (2006). Figure 2 displays the evolution of this index over the period 1920–2008 and shows that the book-to-market ratio significantly decreased in the second part of the century. For the entire sample the average is 69% and the standard deviation is 27%. In contrast, over the period

\(^{11}\)Alternatively, we could assume that firms exit the economy at a constant rate $\lambda_e$ so that the relevant discount rate is given by $\rho = r + \lambda_e$. This would allow us to reduce the risk free rate and still ensure integrability.
1981–2003, for which we also have investment data, the mean and standard deviation of the book-to-market ratio are 42% and 27% respectively. We choose to calibrate the model to match these values, as the idiosyncratic volatility puzzle we are interested in is obtained on recent data. The simulated data generates an average book-to-market ratio of 44% with a monthly standard deviation of 29%.

The states of the growth rate process are set to $\theta_h = 0.707\%$ and $\theta_l = -5.863\%$ to match the mean and standard deviation of investment rate reported in Abel and Eberly (1999a) and Eberly et al. (2006). The model produces a mean investment rate of 10% and a mean disinvestment rate of 5%. The standard deviation of investment rates reported in Eberly et al. (2006) for the period 1981–2003 is 5.5% while our model generates a lower value of 2%. Finally we set the total volatility of the price process to $\sigma = 21\%$ and the correlation between firm specific prices and aggregate shocks to $\rho = 0.1$. This produces an average standard deviation of stock returns of 31% which lies well within the range of empirically reported values (see Campbell, Lettau, Malkiel, and Xu (2001) and Vualteenaho (2001)).

Table IV summarizes the moments obtained in the simulated data along with their empirical counterparts. It shows that our model generates values which are in line with key moments of industry and firms characteristics and provides reasonable level of stock return volatility. The model performs however very poorly regarding the share of adjustment costs as it generates a value twice as high as its empirical counterpart. This is a well-known shortcoming of adjustment cost models, see Chirinko (1993). We could improve the performance of the model along this dimension by modifying the cost function to allow for features such as fixed costs or partial irreversibility. We choose to focus on the simple specification in equation (11) because it allows for an explicit solution to the firm valuation problem.
C. Results

To test the ability of our model to generate the idiosyncratic volatility anomaly we now use the simulated panels of data to construct portfolios sorted on different measures of idiosyncratic volatility. In their original study Ang et al. (2006) use one month of daily observations to construct their measures of idiosyncratic volatility and consider a one month portfolio holding period. Since our data are simulated at a weekly rather than daily frequency, we must consider longer estimation and holding periods to obtain a comparable number of points. We use six months of weekly observations to construct measures of idiosyncratic volatility and hold the portfolios for six months. Ang et al. (2006) use two different measures of idiosyncratic volatility: (i) standard deviation of past returns and (ii) standard deviation of the residuals of a three factor Fama–French model. We focus here on the first measure and replace the second one by the standard deviation of the residuals of a market model since the construction of the Fama–French factors is problematic in our partial equilibrium framework.

Table V displays summary statistic from our simulation for five quintile portfolios ranked by increasing measure of idiosyncratic volatility along with the benchmark results of Ang et al. (2006). As shown by the first and the fourth column of Panel A, the model is able to generate the volatility anomaly. Indeed, the portfolios’ alphas are both decreasing in the measure of idiosyncratic volatility and highly statistically significant. This shows that portfolios which contain high idiosyncratic volatility firms have lower risk adjusted performance. Furthermore, the average portfolio returns are also decreasing in idiosyncratic volatility. The magnitude of the effect generated by our model is however much lower than the empirical results. Ang et al. (2006) obtain an alpha of −16.20% per year for a portfolio long in the fifth
quintile and short in the first quintile, while our model yields an alpha of –1.69% per year. Similarly, Ang et al. (2006) report a negative average return of –11.64% per year for this long-short portfolio while our model generates a negative return of –1.33% per year.\footnote{In their paper Ang et al. (2006) report a monthly alpha of –1.35% and an average monthly return of –0.97% for the long-short portfolio. The numbers reported here are obtained from theirs by multiplying by twelve.} It is important to note that the empirical results reported in Ang et al. (2006) are mainly driven by extremely low returns in the fifth quintile.\footnote{In order to obtain such large effects, we would need our model to generate more heterogeneity among firms. This could potentially be achieved by initially endowing firms with heterogenous cash flow dynamics and production technologies.} Indeed, the difference in average return between the fourth and fifth quintile is ten times larger than between the first on fourth quintile. If we exclude the fifth quintile, our results are much more in line with theirs. In particular, for a portfolio long in the fourth quintile and short in the first, Ang et al. (2006) obtain an alpha of –5.04% and an average return of –0.84% while our model yields an alpha of –1.338% and an average return of –1.014%.

The last three columns of Table V report the average idiosyncratic volatility effect, idiosyncratic volatility and aggregate volatility of the quintiles portfolios. Comparing the two panels of the table shows that sorting on either total volatility or residual volatility correctly identifies the firms with largest idiosyncratic volatility. As expected from the fact that returns are generated according to equation (26), the measured alphas are almost identical to the true idiosyncratic volatility effect reported in the fifth column. In accordance with the prediction of our model the magnitude of the effect is larger for high idiosyncratic volatility firms.

As explained in Section II.A the sign of the idiosyncratic volatility effect depends on the joint distribution of future forecast errors and idiosyncratic volatilities. Figure 3 displays the idiosyncratic volatility effect as a function
of the forecast error and the empirical distribution of the normalized forecast error $\eta$ conditional on the true value of the growth rate. As can be seen from the figure, the distribution is negatively skewed in both states. This is due to the fact that, since expansions are more frequent than recessions for the chosen parameter values, the unconditional mean $\mu$ of the forecast is close to the high value of the growth rate. In the high state, when the forecast error is positive, the effect is small as $m_t$ tends to be close to $\theta_h$ due to mean reversion. However in the low state, the effect is much larger since the distribution of the forecast errors is concentrated on the left.

The idiosyncratic volatility effect which is reported in Table V results from the combination of the effects which occur in the low and high states. While the high state is more frequent, the overall effect is dominated by the negative contribution of the low state. In a model with symmetric regimes and transition probabilities, the idiosyncratic volatility effect should be absent since the contribution of the two states should cancel out. Figure 4 confirms this intuition by showing that in such a model the idiosyncratic volatility effect and the distribution of the forecast errors are both perfectly symmetric. Panel A of Table VI reports the summary statistics obtained from this model and shows that the alphas of the quintile portfolios are not significantly different from zero. However, the long–short portfolio generates a positive average return due to its net exposure to aggregate risk.

The average return generated by the long–short portfolio is not only due to the idiosyncratic volatility effect as it also reflects the different exposure of the quintile portfolios to aggregate risk. In Table V, the firms with highest idiosyncratic volatility also have the highest aggregate volatility. The long–short portfolio therefore exhibits a positive exposure to aggregate risk which
contributes positively to its average return. This exposure could completely offset the idiosyncratic volatility effect. However the alphas of the portfolios should still decrease with idiosyncratic volatility, since larger exposure to aggregate risk increases not only the portfolio returns but also their betas.

This intuition is confirmed by Panel B of Table VI which reports the summary statistics for a model where the correlation coefficient $\rho$ is set equal to 0.5. With this parametrization, aggregate volatilities are much larger and the long–short portfolio generates a positive return even though the idiosyncratic volatility effect is still present. Indeed, the alphas of the portfolios are significantly negative and decrease with idiosyncratic volatility.

IV. Conclusion

In this paper, we propose a rational model of firm valuation under incomplete information that is able to explain the ambiguous link between idiosyncratic volatility and stock returns.

Our theoretical analysis leads to two testable implications. First, firms with larger idiosyncratic volatility should display lower returns following bad news and higher returns following good news. We present results in the recent literature (see Zhang (2006)) which document such an effect. It is important to note that our model generates these properties endogenously through a rational mechanism. Second, we predict that controlling for earning forecast errors should mitigate the idiosyncratic volatility anomaly. This implication is supported by the findings of Jiang et al. (2007) who show that introducing contemporaneous earnings forecasts errors renders the coefficient of idiosyncratic volatility insignificant in Fama–MacBeth regressions.

When calibrated to match important moments of firm and industry characteristics, the model is able to replicate the idiosyncratic volatility anomaly documented by Ang et al. (2006). We show that, in the context of our model, the assymetry of business cycles is crucial to generate the negative relation
between stock returns and idiosyncratic volatility found in the data.
Appendix A: Proofs

Proof of Lemma 1. The proof of this well-known filtering result can be found in numerous places among which Liptser and Shiryaev (2001, p.372). QED.

In order to facilitate the proof of Proposition 1, we start by presenting a useful technical result.

Lemma 2: Let $a$, $b$, $c$ and $\delta$ be arbitrary constants and consider the, possibly infinite, process defined by

$$S_t = E_{it} \left[ \int_t^\infty e^{-a(s-t)} \xi_{t,s} X_{is}^b (c + \delta m_{is}) ds \right].$$

If the parameters of the model are such that

$$\min_{m \in [\theta_l, \theta_h]} \left[ r + a + b \rho \sigma \kappa + \frac{1}{2} b(1-b) \sigma^2 - bm \right] > 0, \quad (28)$$

then the process $S$ is well defined and given by $S_t = (C + Dm_{it}) X_{it}^b$ where the constants $C$ and $D$ are the unique solutions to

$$bC + bD(\theta_1 + \theta_2) = D \left[ r + a + b \rho \sigma \kappa + \lambda + \mu + \frac{1}{2} b(1-b) \sigma^2 \right] - \delta,$$

$$C \left[ r + a + \frac{1}{2} b(1-b) \sigma^2 \right] = D \left[ (\lambda + \mu) m - b \theta \theta_2 \right] + c.$$

Proof. The fact that (28) is sufficient for the finiteness of $S$ follows from the boundedness of $m_i$ and the fact that $\xi X_i^b$ is an exponential process with constant volatility, we omit the details.

In order to establish the second part, let the constants $C$ and $D$ be as in the statement and consider the process defined by

$$M_t = e^{-at} \xi_{it} X_{it}^b (C + Dm_{it}) + \int_0^t e^{-as} \xi_{s} X_{is}^b (c + \delta m_{is}) ds.$$
local martingale. Using the boundedness of \( m_i \) in conjunction with (28) and well-know results on geometric Brownian motion we can show that

\[
E_{i0} \left[ \sup_{t \in [0,T]} |M_t|^2 \right] < \infty,
\]

for any finite time \( T \). This implies that the local martingale \( M \) is a true martingale at least up to time \( T \) and it follows that

\[
e^{-at} X_{it}^b (C + Dm_{it}) = E_{it} \left[ e^{-aT} \xi_{t,T} X_{iT}^b (C + Dm_{iT}) + \int_t^T e^{-as} \xi_{t,s} X_{is}^b (c + \delta m_{is}) ds \right].
\]

Taking the limit as \( T \to \infty \) on both sides of the previous expression and using the dominated convergence theorem we obtain

\[
X_{it}^b (C + Dm_{it}) = S_t + \lim_{T \to \infty} E_{it} \left[ e^{-a(T-t)} \xi_{t,T} X_{iT}^b (C + Dm_{iT}) \right]
\]

and the proof will be complete once we show that the second term on the right is equal to zero. This follows from the boundedness of \( m_i \) and the assumed validity of (28), we omit the details. QED.

**Proof of Proposition 1.** Let \( I \) be an arbitrary investment policy for firm \( i \), denote by \( K_i \) the associated capital stock and let

\[
S_t = E_{it} \int_t^\infty \xi_{t,s} \left[ \pi(X_{is}, K_{is}) - \phi(I_s) \right] ds
\]

denote the corresponding firm value process. Using the dynamics of the capital stock process we obtain

\[
K_{is} = e^{-\delta(s-t)} K_{it} + \int_t^s e^{-\delta(\tau-t)} I_s \, d\tau.
\]

Plugging this back into the expression of the firm value process and using the definition of the firm’s production function in conjunction with the law of iterated expectations we obtain

\[
S_t = Q_{it} K_{it} + E_{it} \int_t^\infty \xi_{t,s} \left[ Q_{is} I_s - \phi(I_s) \right] ds
\]
where the process
\[ Q_{it} = E_{it} \int_t^\infty e^{-\delta(s-t)} \xi_{t,s} X_{t,s}^\Phi ds \]
gives the marginal valuation of the firm’s capital. Assuming that the above expectations are well defined at the optimum (this will be verified below), this shows that the optimal investment policy is given by
\[ I_t^* = \arg\max_{x \in \mathbb{R}} (xQ_{it} - \phi(x)) = \gamma Q_{it} - \gamma b \]
and all that is left to do is to compute the marginal value of capital and the value of the firm’s growth options:
\[ G_t = V_{it} - Q_{it} K_{it} = \frac{\gamma}{2} E_{it} \int_t^\infty \xi_{t,s} (Q_{is} - b)^2 ds. \]
Let us start by computing the firm’s marginal value of capital. Using the result of Lemma 2 we have that if
\[ r + \delta + \Phi \left[ \rho \sigma \kappa - \theta_h + \frac{1}{2} \Phi (1 - \Phi) \sigma^2 \right] > 0, \]
then the firm’s marginal value of capital is well defined and given by equation (18) where the constants \( q_0 \) and \( q_1 \) solve the linear system
\[
q_0 \Phi + q_1 \Phi (\theta_t + \theta_h) = q_1 \left[ r + \delta + \Phi \rho \sigma \kappa + \lambda + \mu + \frac{1}{2} \Phi (1 - \Phi) \sigma^2 \right],
\]
\[
q_0 \left[ r + \delta + \Phi \rho \sigma \kappa + \frac{1}{2} \Phi (1 - \Phi) \sigma^2 \right] + q_1 \Phi \eta \theta_h = 1 + q_1 \eta (\lambda + \mu).
\]
Plugging this into the expression for the value of the firms’ growth options and simplifying the terms we obtain
\[
G_t = g_0 + q_1^2 E_{it} \int_t^\infty \xi_{t,s} m_{t,s}^2 X_{t,s}^\Phi ds
+ E_{it} \int_t^\infty \xi_{t,s} \left[ (q_0^2 + 2q_0 q_1 m_{t,s}) X_{t,s}^\Phi - 2b (g_0 + q_1 m_{it}) X_{t,it}^\Phi \right] ds
\]
where we have set \( g_0 = \gamma b^2/(2r) \). Using the result of Lemma 2 we have that the last expectation on the right hand side is well defined provided that the parameters of the model satisfy
\[ r > \Phi \max \left[ \theta_h - \rho \sigma \kappa - \frac{1}{2} (1 - \Phi) \sigma^2, 2\theta_h - 2\rho \sigma \kappa - (1 - 2\Phi) \sigma^2 \right], \] (29)
and that in this case it is given by
\[(g_{10} + g_{11}m_{it})X^{\Phi}_{it} + (g_{20} + g_{21}m_{it})X^{2\Phi}_{it}\]
where the constants \(g_{10}, g_{11}, g_{20}\) and \(g_{21}\) solve the linear system of four equations with four unknowns given by
\[
\begin{align*}
g_{10}\Phi + g_{11}\Phi(\theta_l + \theta_h) &= g_{11}\left[r + \Phi\rho\sigma\kappa + \lambda + \mu + \frac{1}{2}\Phi(1 - \Phi)\sigma^2\right] + 2bq_1, \\
g_{10}\left[r + \Phi\rho\sigma\kappa\frac{1}{2}\Phi(1 - \Phi)\sigma^2\right] + g_{11}\Phi\theta_l\theta_h &= g_{11}\overline{m}(\lambda + \mu) - 2bq_0, \\
2g_{20}\Phi + 2g_{21}\Phi(\theta_l + \theta_h) &= g_{21}\left[r + 2\Phi\rho\sigma\kappa + \Phi(1 - 2\Phi)\sigma^2 + \lambda + \mu\right] - 2g_0q_1, \\
g_{20}\left[r + 2\Phi\rho\sigma\kappa + \Phi(1 - 2\Phi)\sigma^2\right] + 2g_{21}\Phi\theta_l\theta_h &= g_{21}\overline{m}(\lambda + \mu) + q_0^2.
\end{align*}
\]
In order to obtain the value of the firm’s growth options, it now only remains to compute the first expectation
\[
\Psi(m_{it}, X_{it}) = E_{it}\int_t^\infty \xi_{t,s}m_{it}^2X_{is}^{2\Phi}ds.
\]
Since \(m_i\) is bounded it follows from (29) that the function \(\Phi\) is well defined. On the other hand, using the fact that the dynamics of the price process are linear, it can be shown that the function \(\Psi\) is homogenous of degree 2\(\Phi\) with respect to \(x\) and it follows that
\[
\Psi(m, x) = x^{2\Phi}H(m)
\]
for some bounded function \(H : [\theta_l, \theta_h] \to \mathbb{R}\). Assuming that this function is smooth and applying Itô’s lemma we deduce that it satisfies
\[
m^2 = \left[r + 2\Phi\rho\sigma\kappa + \Phi(1 - 2\Phi)\sigma^2 - 2\Phi m \right]H(m) \\
- \left[(\lambda + \mu)(\overline{m} - m) + 2\Phi(m - \theta_l)(\theta_h - m)\right]H'(m) \\
- \left(1/2\right)\left[\epsilon(m - \theta_l)(\theta_h - m)\right]^2H''(m).
\]
Unfortunately, this ordinary differential equation does not admit a closed form solution because of the quadratic term on the right hand side. As a result we will need to solve it numerically. To this end we have to specify
boundary conditions. Since $\theta_l$ and $\theta_h$ are entrance boundaries for the process $m_i$ (see David (1997)) these boundary conditions take the form

$$\lim_{m \to \theta_l, \theta_h} |H''(m)| < \infty.$$ 

In practice we impose some finite values for the second derivative of the unknown function on the boundaries of the domain and verify that our numerical solution is insensitive to the choice of these values.

Putting everything together we have that the value of an arbitrary firm is given by equations (16), (18) and (19) with

$$g_1(m) = (\gamma/2) [g_{10} + g_{11}m],$$

$$g_2(m) = (\gamma/2) [g_{20} + g_{21}m + H(m)],$$

provided that the parameters of the model are such that inequality (29) holds true. QED.
References


Table I. Properties of portfolios sorted by analyst forecast revision and information uncertainty proxy (reproduced from Zhang (2006, Table III)). This table reports average monthly portfolio returns sorted by analyst forecast revision and information uncertainty proxy. Each month I sort stocks into three categories depending on whether the forecast revision is negative, zero, or positive. The forecast revision is the average of individual revisions by analysts who covered the firm in both months $t-1$ and $t$. For each category, I further sort stocks into ten deciles based on information uncertainty proxy. Firm size (MV) is the market capitalization (in millions of dollars) at the end of month $t$. Firm age (AGE) is the number of years since the firm was first covered by CRSP. Analyst coverage (COV) is the number of analysts following the firm in the previous year. Forecast dispersion (DISP) is the standard deviation of analyst forecasts in month $t$ scaled by the prior year-end stock price. Stock volatility (SIGMA) is the standard deviation of weekly market excess returns over the year ending at the end of month $t$. Cash flow volatility (CVOL) is the standard deviation of cash flow from operations in the past five years (with a minimum of three years), where cash flow from operations is earnings before extraordinary items minus total accruals, scaled by average total assets. $1/MV$, $1/AGE$, and $1/COV$ are the reciprocals of MV, AGE, and COV, respectively. Stocks with a price less than five dollars at the portfolio formation date are excluded from the sample. Stocks are held for one month, and portfolio returns are equally weighted. The sample period is from January 1983 to December 2001; $t$–statistics in parentheses are adjusted for autocorrelation.

<table>
<thead>
<tr>
<th></th>
<th>A. Sorted by DISP</th>
<th>B. Sorted by SIGMA</th>
<th>C. Sorted by CVOL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Rev&lt;0 Rev=0 Rev&gt;0</td>
<td>Rev&lt;0 Rev=0 Rev&gt;0</td>
<td>Rev&lt;0 Rev=0 Rev&gt;0</td>
</tr>
<tr>
<td>D1 (low)</td>
<td>0.71 1.29 1.48</td>
<td>1.25 1.41 1.71</td>
<td>1.01 1.25 1.73</td>
</tr>
<tr>
<td>D2</td>
<td>0.72 1.26 1.70</td>
<td>1.12 1.54 1.73</td>
<td>1.04 1.42 1.64</td>
</tr>
<tr>
<td>D3</td>
<td>0.99 1.34 1.67</td>
<td>1.04 1.34 1.81</td>
<td>1.00 1.19 1.59</td>
</tr>
<tr>
<td>D4</td>
<td>0.83 1.28 1.94</td>
<td>0.91 1.46 1.75</td>
<td>0.82 1.24 1.78</td>
</tr>
<tr>
<td>D5</td>
<td>0.88 1.38 1.97</td>
<td>0.79 1.42 1.77</td>
<td>0.84 1.44 1.74</td>
</tr>
<tr>
<td>D6</td>
<td>0.59 1.44 1.91</td>
<td>0.63 1.42 1.76</td>
<td>0.68 1.18 1.88</td>
</tr>
<tr>
<td>D7</td>
<td>0.82 1.31 1.96</td>
<td>0.54 1.49 1.94</td>
<td>0.66 1.43 1.82</td>
</tr>
<tr>
<td>D8</td>
<td>0.59 1.20 1.76</td>
<td>0.53 1.31 1.90</td>
<td>0.62 1.19 1.99</td>
</tr>
<tr>
<td>D9</td>
<td>0.63 1.28 2.00</td>
<td>0.37 0.88 2.24</td>
<td>0.62 1.00 2.13</td>
</tr>
<tr>
<td>D10 (high)</td>
<td>0.48 1.08 2.04</td>
<td>-0.23 0.51 2.16</td>
<td>0.05 0.79 2.27</td>
</tr>
<tr>
<td>D10-D1</td>
<td>-0.23 -0.21 0.56</td>
<td>-1.47 -0.90 0.44</td>
<td>-0.97 -0.46 0.54</td>
</tr>
<tr>
<td></td>
<td>(-0.77) (-0.95) (1.82)</td>
<td>(-2.23) (-1.36) (0.64)</td>
<td>(-2.10) (-1.04) (1.07)</td>
</tr>
</tbody>
</table>
Table II. Idiosyncratic volatility anomaly: controlling for future earning shocks (reproduced from Jiang et al. (2007, Table IV)). This table reports the results of the Fama-MacBeth regression (27) of future stock returns on idiosyncratic volatility with various control variables. In each quarter, we perform cross-sectional regressions of next quarter returns (RETQ1) on current-quarter IVOL and future earning shocks as measured by standardized unexpected earnings (SUEQ0/SUEQ1/SUE4) and errors of analyst consensus earnings forecast (FERQ0/FERQ1/FER4). Control variables include ln(Size), ln(B/M), PrRet (momentum), LEV (leverage), and LIQ (liquidity measure of Pastor and Stambaugh (2003)). The Newey–West t-statistics are computed with a one quarter lag.

<table>
<thead>
<tr>
<th></th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
<th>Model 4</th>
<th>Model 5</th>
<th>Model 6</th>
<th>Model 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>7.92</td>
<td>7.28</td>
<td>7.07</td>
<td>6.85</td>
<td>5.84</td>
<td>5.74</td>
<td>6.28</td>
</tr>
<tr>
<td></td>
<td>(7.81)</td>
<td>(6.76)</td>
<td>(6.54)</td>
<td>(6.51)</td>
<td>(4.09)</td>
<td>(3.89)</td>
<td>(4.45)</td>
</tr>
<tr>
<td>IVOL</td>
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<td>−0.67</td>
<td>−0.16</td>
<td>0.21</td>
<td>−0.15</td>
<td>0.22</td>
<td>0.90</td>
</tr>
<tr>
<td></td>
<td>(−4.11)</td>
<td>(−3.01)</td>
<td>(−0.69)</td>
<td>−0.85</td>
<td>(−0.42)</td>
<td>(0.54)</td>
<td>(1.93)</td>
</tr>
<tr>
<td>ln(Size)</td>
<td>−0.35</td>
<td>−0.31</td>
<td>−0.43</td>
<td>−0.46</td>
<td>−0.33</td>
<td>−0.39</td>
<td>−0.46</td>
</tr>
<tr>
<td></td>
<td>(−2.90)</td>
<td>(−2.38)</td>
<td>(−3.39)</td>
<td>(−3.95)</td>
<td>(−2.30)</td>
<td>(−2.55)</td>
<td>(−2.83)</td>
</tr>
<tr>
<td>ln(B/M)</td>
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<td>1.14</td>
<td>0.74</td>
<td>−0.20</td>
<td>0.06</td>
<td>0.22</td>
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<tr>
<td></td>
<td>(2.77)</td>
<td>(3.73)</td>
<td>(3.41)</td>
<td>(2.14)</td>
<td>(−0.48)</td>
<td>(0.14)</td>
<td>(0.51)</td>
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<tr>
<td>PrRet</td>
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<td>2.01</td>
<td>1.62</td>
<td>1.74</td>
<td>0.47</td>
<td>1.03</td>
<td>1.43</td>
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<tr>
<td></td>
<td>(8.60)</td>
<td>(4.41)</td>
<td>(3.75)</td>
<td>(3.83)</td>
<td>(0.81)</td>
<td>(1.73)</td>
<td>(2.13)</td>
</tr>
<tr>
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<td>−0.14</td>
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<td>0.60</td>
<td>0.59</td>
<td>0.29</td>
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<tr>
<td></td>
<td>(−0.38)</td>
<td>(−0.50)</td>
<td>(−0.42)</td>
<td>(−0.93)</td>
<td>(1.57)</td>
<td>(1.57)</td>
<td>(0.73)</td>
</tr>
<tr>
<td>LIQ</td>
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<td>−0.04</td>
<td>−0.04</td>
<td>0.00</td>
<td>0.08</td>
<td>0.59</td>
<td>0.19</td>
</tr>
<tr>
<td></td>
<td>(−1.74)</td>
<td>(−1.63)</td>
<td>(−1.01)</td>
<td>(0.02)</td>
<td>(0.17)</td>
<td>(0.81)</td>
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<td>SUEQ0</td>
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<td></td>
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<td>(24.00)</td>
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<td>SUE4</td>
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<td></td>
<td></td>
<td></td>
<td>(15.40)</td>
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<td>1.48</td>
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<td>(10.11)</td>
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<tr>
<td>FERQ1</td>
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<td>1.24</td>
</tr>
<tr>
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<td>(10.52)</td>
</tr>
<tr>
<td>FER4</td>
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<td>0.41</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td>(10.34)</td>
</tr>
<tr>
<td>Adj. $R^2$</td>
<td>0.08</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
<td>0.13</td>
<td>0.12</td>
<td>0.12</td>
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</tbody>
</table>
Table III. Benchmark parameter values. This table lists the parameter values used to simulate the model. We break the parameters into three groups. Group I includes the depreciation rate of capital $\delta$, the market price of risk and the riskfree rate $r$. While the two former are chosen to match prior empirical studies, the latter has to be set high enough to satisfy the condition of Proposition 1. Group II includes the intensity parameters $\lambda$ and $\mu$ which are chosen to match the frequency of the business cycles as reported by the NBER. The parameters in the final group are calibrated in such a way that the model matches key moments of firm and industry characteristics, see Table IV for a list of these moments.

<table>
<thead>
<tr>
<th>Group I</th>
<th>Group II</th>
<th>Group III</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta$</td>
<td>$\delta$</td>
<td>$\kappa$</td>
</tr>
<tr>
<td>0.7</td>
<td>0.12</td>
<td>0.30</td>
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</table>
Table IV. Key moments under the benchmark parameters. This table reports a set of moments generated by the model under the parameter values of Table III. The investment and disinvestment moments are from Abel and Eberly (1999b) and Eberly et al. (2006). The average share of adjustment costs is from Barnett and Sakellaris (1999). The empirical average and standard deviation of book-to-market are computed using the reconstructed times series of book-to-market of the Dow Jones index over the period 1981–2003, see Figure 2. The data sources for the range of volatilities of stock returns are Campbell et al. (2001) and Vualteenaho (2001). All moments are expressed on an annual basis.

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
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<tr>
<td>Average rate of investment</td>
<td>0.15</td>
<td>0.10</td>
</tr>
<tr>
<td>Average rate of disinvestment</td>
<td>0.02</td>
<td>0.05</td>
</tr>
<tr>
<td>Average share of adjustment costs</td>
<td>0.13</td>
<td>0.34</td>
</tr>
<tr>
<td>Standard deviation of investment rate</td>
<td>0.05</td>
<td>0.02</td>
</tr>
<tr>
<td>Average book-to-market ratio</td>
<td>0.42</td>
<td>0.44</td>
</tr>
<tr>
<td>Standard deviation of book-to-market ratio</td>
<td>0.27</td>
<td>0.29</td>
</tr>
<tr>
<td>Average volatility of stock returns</td>
<td>0.25–0.32</td>
<td>0.31</td>
</tr>
</tbody>
</table>
Table V. Properties of portfolios sorted on volatility. The first panel of this table reports the estimation results obtained by Ang et al. (2006) for portfolios sorted on total volatility. Panels A and B report summary statistics obtained in our simulations for portfolios sorted on total return volatility and on idiosyncratic volatility relative to the CAPM. Portfolios are formed every six months, based on volatility computed using weekly observations over the previous six months. Portfolio 1 (5) is the portfolio with the lowest (highest) volatilities. The statistics in the columns labeled $\alpha$, $R$ and $\iota \eta$ are measured in yearly percentage terms. The $\alpha$ and $\beta$ columns report Jensen’s alpha and the portfolio beta with respect to the CAPM. The $R$ column reports the average gross return. The $\iota \eta$ column reports the idiosyncratic volatility effect as defined by equation (24). Finally, the columns labeled $\iota$ and $\alpha$ report the idiosyncratic and aggregate volatilities as defined by equations (22) and (21). All values are computed by taking averages across 10,000 simulations of a panel of 500 firms for 20 years with a weekly time step. The parameters used in the simulations are reported in Table III.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$R$</th>
<th>$\iota \eta$</th>
<th>$\iota$</th>
<th>$\alpha$</th>
</tr>
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<tbody>
<tr>
<td><strong>Benchmark:</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.68</td>
<td></td>
<td>12.72</td>
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<tr>
<td>2</td>
<td>1.56</td>
<td></td>
<td>13.80</td>
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<tr>
<td>3</td>
<td>0.84</td>
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<td>14.64</td>
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</tr>
<tr>
<td>4</td>
<td>-3.36</td>
<td></td>
<td>11.88</td>
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<td></td>
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</tr>
<tr>
<td>5</td>
<td>-14.52</td>
<td></td>
<td>1.08</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5–1</td>
<td>-16.20</td>
<td></td>
<td>-11.64</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>A. Portfolios sorted on total volatility</strong></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>1</td>
<td>0.446</td>
<td>(-4.50)</td>
<td>0.736</td>
<td>15.207</td>
<td>0.465</td>
<td>0.249</td>
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<tr>
<td>2</td>
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<td>1.006</td>
<td>14.736</td>
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<td>0.338</td>
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<td>3</td>
<td>-0.673</td>
<td>(-6.80)</td>
<td>1.082</td>
<td>14.380</td>
<td>-0.729</td>
<td>0.361</td>
</tr>
<tr>
<td>4</td>
<td>-0.892</td>
<td>(-9.01)</td>
<td>1.117</td>
<td>14.193</td>
<td>-0.976</td>
<td>0.371</td>
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<tr>
<td>5</td>
<td>-1.241</td>
<td>(-12.39)</td>
<td>1.153</td>
<td>13.874</td>
<td>-1.244</td>
<td>0.378</td>
</tr>
<tr>
<td>5–1</td>
<td>-1.686</td>
<td>(-17.05)</td>
<td>-1.334</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>B. Portfolios sorted on CAPM residual volatility</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.449</td>
<td>(-4.54)</td>
<td>0.751</td>
<td>15.226</td>
<td>0.451</td>
<td>0.251</td>
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<tr>
<td>3</td>
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<td>1.090</td>
<td>14.388</td>
<td>-0.735</td>
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<tr>
<td>4</td>
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<td>14.188</td>
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<td>0.371</td>
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<tr>
<td>5</td>
<td>-1.225</td>
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<tr>
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<td>(-16.93)</td>
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</table>
Table VI. Properties of portfolios sorted on total volatility under alternative parameter values. This table reports summary statistics for portfolios sorted on total volatility. The parameters used in the simulations are those of Table III except for $\rho = 0.5$ in Panel A and $\mu = \lambda = 1$, $-\theta_l = \theta_h = 4\%$ in Panel B. The statistics in the columns labeled $\alpha$, $R$ and $\iota \eta$ are measured in yearly percentage terms. The $\alpha$ and $\beta$ columns report Jensen’s alpha and the portfolio beta with respect to the CAPM. The $R$ column reports the average gross return. The $\iota \eta$ column reports the idiosyncratic volatility effect as defined by equation (24). Finally, the columns labeled $\iota$ and $a$ report the idiosyncratic and aggregate volatilities as defined by equations (22) and (21). All values are computed by taking averages across 10,000 simulations of a panel of 500 firms for 20 years with a weekly time step.

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$R$</th>
<th>$\iota \eta$</th>
<th>$\iota$</th>
<th>$a$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A. Symmetric Shocks</strong></td>
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Figure 1. Estimation and optimal investment policy. The top panel displays a sample path of the growth rate $\theta_t$ and the corresponding path of the estimation $m_{it}$. The bottom panel displays the associated investment policy under full information (dashed line) and incomplete information (solid line). The parameters used for this figure are given in Table III.
Figure 2. Book-to-market ratio of the Dow Jones index. This figure plots the evolution of the book to market ratio of the Dow Jones industrial average. The book to market ratio is constructed from the publicly available returns on the index and the book value of equity of the companies in the index as reported by Value Line. The sample period runs from January 1920 to February 2008.
Figure 3. **Idiosyncratic component in return and forecast error.** The top panel displays the idiosyncratic component in returns $\epsilon_i \eta_i$ and the empirical distribution of the forecast error $\eta_i$ conditional on being in the low state ($\theta_t = \theta_l$). The bottom panel displays the same quantities conditional on being in the high state ($\theta_t = \theta_h$). In both panels the state variable is set to its average value. The data used for this figure is from a simulated panel of a thousand firms with the parameters of Table III.
Figure 4. Idiosyncratic component in return and forecast error with symmetric shocks. The top panel displays the idiosyncratic component in returns $\xi_t \eta_i$ and the empirical distribution of the forecast error $\eta_i$ conditional on being in the low state ($\theta_t = \theta_l$). The bottom panel displays the same quantities conditional on being in the high state ($\theta_t = \theta_h$). In both panels the state variable is set to its average value. The data used for this figure is from a simulated panel of a thousand firms with the parameters of Table III except for $-\theta_l = \theta_h = 1.2\%$ and $\mu = \lambda = 1$. 