Multivariate Pricing of Capital Structure Derivatives
with Stochastic Smiles and Skews

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Abstract

We develop a convenient structural framework for the joint modeling of credit spreads, stock prices, stock options and basket credit derivatives, using a multivariate structural firm value model with skewed asset returns. We show that our setting successfully addresses several empirical facts, which are not easily accounted for in Merton’s (1974) structural model. First, stochastic volatility and left-skewed asset returns increase corporate credit spreads to more realistic levels, at short maturities and for low firm leverage. Second, our multifactor volatility structure is consistent with no-arbitrage violations relative to Merton’s model, such as positive comovement between credit spreads and stock returns. Third, our model generates endogenously a stochastic skewness of stock returns, which is a key feature for explaining the dynamics of individual option smiles. Fourth, individual implied volatility skews can be both positive and negative, depending on firm leverage and asset return skewness. Finally, the model yields realistic patterns also for implied correlation smiles of multi-issuer credit derivatives, as observed in the market for CDOs, index tranches or CDS baskets. These features are obtained in a tractable matrix-valued affine diffusion setting that yields semi closed-form solutions for credit spreads, stock and credit derivative prices.

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1 Introduction

In this paper, we study a multivariate extension of the Merton [M74] structural firm value model with skewed asset returns and stochastic variance-covariance matrix. We demonstrate that these features imply a joint behaviour of credit spreads, stock prices, stock volatilities and multi-issuer credit derivative prices which is more consistent with the empirical evidence. In contrast to the Merton approach, in which the firm value process is modeled as a single-factor geometric Brownian Motion, we assume asset returns featuring stochastic volatilities and correlations. By introducing an instantaneous dependence between asset returns and their conditional second moments, we also introduce a skewed asset returns distribution. We model the stochastic covariance matrix of asset returns using a matrix valued diffusion process with Wishart transition densities. This process is affine in the sense of [DFS03]. Therefore, it yields explicit expressions for the Laplace transform of asset returns, which are convenient to price capital structure derivatives in our model, such as equity, corporate bonds, credit default swaps and credit default swap baskets, using Fourier methods. The Wishart diffusion process was first studied by [B91] and has been introduced to the finance context by [GS04] and [GJS04], showing how it can be conveniently used to model multivariate volatility and dependencies for several financial problems.

We combine the stochastic covariance matrix of asset returns with a flexible parametrization of asset returns skewness to obtain realistic patterns for credit spreads, stock returns, stock volatility, individual option-implied volatilities and correlation smiles on multi-asset credit derivatives. In this way, we obtain a unified framework for pricing consistently single and multi-name financial derivatives on firms capital structures.

First, we demonstrate that for low levels of leverage and small maturities our model can generate higher credit spreads than predicted by a Merton model with the same unconditional asset variance covariance matrix. This is an interesting finding, since Merton [M74] structural approach, and structural models in general, have often been criticized for generating too low credit spreads, a property that is linked to the so-called 'credit spread puzzle' in the literature.\footnote{Several approaches have been proposed to deal with the credit spread puzzle. E.g., [G77], [LT96], [LS95] and [CDG01] introduce additional exogenous risk factors to explain the variation of credit spreads. A comparative analysis of these approaches is given in [EHH04]. Some more recent attempts to explain the credit spread puzzle in a general equilibrium model introduce, e.g., belief heterogeneity (see [BTV07]) or macroeconomic uncertainty (see [D06]) in otherwise standard Lucas-type economies.} Two main features of our model imply the higher credit spreads. First, stochastic volatilities and correlations of asset returns lead
to a leptokurtic distribution and to a higher probability of large shortfalls in asset values. Second, credit spreads are sensitive to the skewness of asset returns. A negative correlation between asset returns and their volatilities introduces a negative skewness in our model. This feature can further increase corporate credit spreads. These findings show that volatility and correlation risk may be important ingredients for understanding the unexplained variation of credit spreads in the data.

Second, our approach is consistent with common no-arbitrage violations relative to the Merton model. In such a single-factor setting, equity and debt prices co-move positively, which induces the negative co-movement between equity prices and credit spreads. As documented by [BTV07], this relationship is violated in the data for a good fraction of the cases. Due to its multi-factor structure of asset returns and volatilities, our model can generate both a positive and a negative co-movement between equity prices and debt spreads. The strength of this co-movement depends on firm leverage and the skewness of asset returns.

Third, in our model the correlation between stock returns and their volatilities is random, which is linked to a stochastic skewness of stock returns. As noted in [CHJ07], among others, this property is key for explaining the variations in the option implied volatility shape over time. Many popular stochastic volatility models, such as [H93] or [HW87], allow for a non zero, but constant, skewness of stock returns.

Fourth, our model can generate both positive and negative slopes for the implied volatility smile of stock options, depending on firm leverage and asset return skewness. This is an important departure from the Merton [M74] setting, which instead can only generate negatively sloped smiles.2 [BKM03] and [BTV07] find that for the 30 largest stocks in the S&P100 index and for a panel of 337 US stocks there is a significant fraction of cases with option implied volatility smiles that are positively sloped. Our model can produce positively sloped option-implied volatility smiles, since the risk neutral skewness of stock returns is endogenously determined by firm leverage. For most levels of leverage, the risk neutral skewness of stock returns is negative, leading to a negatively sloped smile. However, there are low to moderate levels of leverage for which skewness can become positive and can cause the slope of the option implied volatility smile to change sign.

Finally, our approach can yield realistic implied correlation smiles for multi-issuer credit derivatives, a feature that we illustrate by considering first-, second- and third-to-default tranches of a three-asset credit default swap basket. This prototypical example is representative of more compli-

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2See [HNW04] for an analysis of option implied volatility smiles in the Merton setting.
cated structures encountered in market practice, such as Collateralized Debt Obligations (CDOs) or index tranches. Previous multi-asset extensions of the Merton [M74] model, such as [LI99] or [LI00], assume that asset returns have constant correlations. In empirical data, however, correlations across tranches of multi-issuer basket structures are not constant, which generates implied correlation smiles. Based on this observation, a vast literature employing fat-tailed distributions for asset returns has emerged, see e.g. [KSW05], [HW04] or [LG02]. While being able to produce realistic shapes of the implied correlation smile at a given point in time, these last approaches are mostly static and therefore not suited to the pricing of derivatives on basket tranches. Further, the joint distribution of asset values is typically not available in closed form, which limits the possibly to study also other derivatives on the firm capital structure, like stocks or stock options.

Our approach is related to several strands of the literature. The intuition for part of our results can be supported by the general equilibrium economy with disagreeing investors in [BTV07]. In contrast to that paper, we assume an exogenous dynamics for asset values and price all derivatives on the firm capital structure in a setting that is more analytically tractable and convenient for applications in the multi-assets setting. [GS04] specify asset returns and volatilities using a Wishart diffusion process and study the pricing of a single defaultable asset having a stochastic default boundary. Since they choose a framework in which asset returns are not skewed by construction, the properties of our setting cannot be mapped into their model. [BPT06] use the Wishart process to model stochastic correlations and skewed returns in intertemporal optimal portfolio choice, and show that it gives rise to a separate hedging demand for correlation risk. [BCT07] apply the Wishart process to term structure modeling in a completely affine setting and show that it can account for many empirical term structure regularities. [DGT06] consider a multivariate Heston [H93]-type option pricing model with Wishart state dynamics, in which the correlation between stock returns and their volatilities, i.e. the skewness of individual stock returns, is constant. [LT08] propose a new class of matrix affine jump diffusions and develop their analytical transform analysis. These processes include the Wishart diffusion as a special case and are convenient to model together stochastic second moments and jumps in the multivariate context.

The structure of the paper is as follows. We present the basic model assumptions in Section 2. The pricing of capital structure derivatives is

3 The interpretation behind the models in [LI99] and [LI00] is slightly different from our one, since they model the joint distribution of default times for multiple firms using a single-factor model with a Gaussian copula. As shown by [LI00], this approach can be related to a structural firm value model by interpreting changes in factors as asset returns.
introduced in Section 3. In Section 4, we study the main implications of the model and examine the pricing of credit default swaps, bond spreads, equity prices, equity options and credit default swap baskets in concrete examples. Section 5 concludes.

2 Model Setup

The Merton [M74] approach uses the cash flows generated by the firm’s assets as its main state variable and models equity and debt as contingent claims on firms’ assets. It has been widely used as a model of corporate default in academic literature and has found great interest in the financial industry. Following this approach, we assume that firms have a very simple capital structure consisting of equity and a fixed-maturity zero-coupon bond. Since the values of the firms assets are random, there are two possible scenarios at maturity of the debt. If a firm’s assets are worth more than the face value of the bond, debtholders are fully repaid and equityholders receive the residual. In this case, they may either liquidate the firm or possibly re-leverage and continue its operations. However, if the firm’s asset value is smaller than the face value of debt at maturity, bondholders immediately force it into default. They liquidate its assets at the current market value and equityholders are left with zero wealth.

We consider an economy with a financial market in which uncertainty is modeled on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) equipped with the filtration \((\mathcal{F}_t)_{t \geq 0}\), where \(\mathbb{P}\) denotes the actual measure. There are \(n\) firms in the economy, whose asset values vector \(V_t = (V_{1t}, \ldots, V_{nt})'\) evolves according to the following stochastic differential equation under probability \(\mathbb{P}\):

\[
dV_t = diag(V_t) \left( (r \mathbf{1} + \Lambda_V(\Sigma_t, t)) dt + \sqrt{\Sigma_t} dZ_t \right)
\]  

where \(r\) is the constant riskless interest rate, \(\mathbf{1}\) is a \((n \times 1)\)-vector of ones, \(\Lambda_V(\Sigma_t, t)\) is a \((n \times 1)\)-vector of risk premia and \(Z\) is a \((n \times 1)\)-vector Brownian motion.

In contrast to multivariate versions of the Merton model assuming a constant covariance matrix, we model the covariance matrix \(\Sigma_t\) of asset returns by a matrix diffusion process given by

\[
d\Sigma_t = (\Omega \Omega' + C \Sigma_t + \Sigma_t C') dt + \sqrt{\Sigma_t} dW_t Q + Q' dW_t' \sqrt{\Sigma_t}.
\]

In equation (2), \(\Omega, C, Q\) are \((n \times n)\)-matrices, with \(\Omega\) invertible and satisfying \(\Omega \Omega' = \beta Q' Q\), and \(W\) is a \((n \times n)\)-matrix standard Brownian motion. The matrix \(C\) controls the mean-reversion of the covariance matrix process and
is assumed negative semidefinite. Matrix $Q$ determines the co-volatilities of the elements of $\Sigma$. From equation (2), the long-run covariance matrix $\Sigma_\infty$ solves the algebraic Lyapunov equation:

$$-\Omega\Omega' = C\Sigma_\infty + \Sigma_\infty C'.$$

This feature is helpful to compare the features of our model with those of multivariate Merton models with constant covariance matrix $\Sigma_\infty$.

The continuous-time process (2) is a matrix-valued extension of the well-known square-root diffusion process, which was introduced in the term-structure literature by [CIR85] and in the option pricing literature by [H93]. The Wishart diffusion is convenient to model stochastic volatilities and correlations due to the following properties. First, if $\Omega\Omega' \geq Q'Q$, then $\Sigma$ is a well-defined covariance matrix process with correlations bounded between $-1$ and $1$. Second, as shown by [B91], if $\Omega\Omega' = \beta Q'Q$ for $\beta > n - 1$ then $\Sigma(t)$ has a Wishart distribution.\(^{5}\) Third, the specification of the covariance matrix process along with the return dynamics in equation (1) implies that the model is affine in the sense of [DFS03]. This feature yields closed form expressions for the conditional Laplace transform of the process and is useful in order to compute efficiently asset prices by Fourier inversion methods.

We introduce leverage effects in asset returns by allowing for a nonzero correlation with their stochastic covariance matrix. This can be achieved by letting the Brownian Motion $Z$ and $B$ to be correlated in the following way:

$$dZ_t = \sqrt{1 - \rho'\rho} dB_t + dW_t \rho.$$  \hspace{1cm} (4)

where $\rho$ is an $(n \times 1)$ - vector, such that $\| \rho \| \leq 1$ and $B$ is a $(n \times 1)$-vector Brownian Motion independent of $W$. In this way, we can model both positive or negative skewness in the distribution of asset returns. More precisely, the instantaneous correlation between the return of asset $j$ and its volatility is given by:

$$\text{corr} \left( \frac{dV^j_t}{V^j_t}, d\Sigma^{ij}_t \right) = \frac{\rho'Q^j_j}{\| Q^j_j \|}$$  \hspace{1cm} (5)

where $Q^j_j$ is the $j^{th}$ column of the matrix $Q$. It follows that the correlation between asset returns and volatilities in the model is constant and completely determined by the parameters $\rho$ and $Q$: When $\rho'Q^j_j \leq 0$ ($\rho'Q^j_j \geq 0$) the $j-$th asset return features a negative (positive) skewness. In particular, a diagonal

\(^{4}\)See [J68] for ways of solving algebraic Lyapunov equations.

\(^{5}\)In the special case where $\beta$ is integer-valued, the distribution of $\Sigma$ has $\beta$ degrees of freedom and equals the distribution of the sum of $\beta$ outer products of iid $n$-dimensional Ornstein-Uhlenbeck processes; see [GJS04].
choice of the matrix $Q$ implies a Heston [H93]-like correlation structure of the form
\[ \text{corr}\left(\frac{dV_j}{V_j}, d\Sigma_{ij}\right) = \rho_j. \] (6)

The model also trivially nests the multivariate version of the structural Merton [M74] model when $Q$ and $M$ are both matrices of zeros.

### 3 Pricing of Capital Structure Derivatives

Pricing capital structure derivatives requires specifying the risk-neutral distribution of asset returns and the covariance matrix. In order to determine this distribution, we have to specify the risk premia associated with asset returns and the covariance matrix. With respect to asset returns, we follow [BPT06] and assume that the risk premium is an affine function of variances and covariances given by $\Lambda_V(\Sigma_t, t) = \Sigma_t \lambda_V$ where $\lambda_V$ is a $(n \times 1)$-vector. Further, extending the univariate specification of the risk premium for volatility taken in [H93] to the multivariate case, we assume that the risk premium of the covariance matrix is given by $\Sigma_t \Lambda' Q + Q' \Lambda \Sigma_t$ where $\Lambda \Sigma$ is a $(n \times n)$-matrix.

In the following, we denote the market prices of risk associated with the matrix brownian motion $\tilde{W}_t$ by the $(n \times n)$-matrix $\Theta_t$ and the market price of risk associated with $B_t$ by the $(n \times 1)$-vector $\theta_t$. Standard theory implies that the market prices of risk satisfy the equations
\[ \Sigma_t \Lambda' Q + Q' \Lambda \Sigma_t = \sqrt{\Sigma_t} \Theta_t + \Theta'_t \sqrt{\Sigma_t} \]
\[ \Sigma_t \lambda_V = \sqrt{1 - \rho' \rho} \cdot \sqrt{\Sigma_t} \theta_t + \sqrt{\Sigma_t} \rho \]

Solving yields
\[ \Theta_t = \sqrt{\Sigma_t} \Lambda'_\Sigma Q \]
\[ \theta_t = \frac{1}{\sqrt{1 - \rho' \rho}} \cdot \sqrt{\Sigma_t} (\lambda_V - \Lambda'_\Sigma Q \rho) \]

We define the risk-neutral measure $Q$ such that $\tilde{W}_t = W_t + \Theta_t$ is a $(n \times n)$-matrix brownian motion and $\tilde{B}_t = B_t + \theta_t$ is a $(n \times 1)$-vector brownian motion under $Q$. We summarize the change of measure and the risk-neutral dynamics of asset values and their covariance matrix in the following Lemma.

**Lemma 1.** Define $X_t = [W_t, B_t]$ and $\Xi_t = [\Theta_t, \theta_t]$. The Radon-Nykodym derivative of $\mathbb{P}$ with respect to $Q$ satisfies
\[ \frac{d\mathbb{P}}{dQ} = \exp \left( - \int_0^T tr(\Xi'_t dX_t) - \frac{1}{2} \int_0^T tr(\Xi'_t \Xi_t) \right) \] (7)
The risk-neutral dynamics of asset values and their covariance matrix are given by
\[dV_t = \text{diag}(V_t) \left( r1dt + \sqrt{\Sigma_t d\tilde{Z}_t} \right)\]
\[\Sigma_t = (\Omega\Omega' + M \Sigma_t + \Sigma_t M')dt + \sqrt{\Sigma_t d\tilde{W}_t} + Q'\tilde{W}_t'\sqrt{\Sigma_t}\]

where \(\tilde{Z} = \sqrt{1-\rho'\rho B} + \tilde{W}\rho\) and \(M = C - \sqrt{\Sigma_t} A\Sigma_t Q\).

The structure of our model does not permit closed-form solutions for derivatives on the firm value. However, the conditional Laplace transform of log asset values is available in closed form. Therefore, the prices of many contingent claims are computable in semi-closed form using Fourier inversion methods.\(^6\)

Let \(v \equiv \log(V)\) denote the logarithm of the firms’ asset values and consider a European-style derivative contract with maturity \(T\) and payoff \(U(v_T)\). By risk-neutral valuation, the value of this contract at any time \(t \leq T\) is given by
\[P_t^U = e^{-r(T-t)} \mathbb{E}^Q[U(v_T)|\mathcal{F}_t]\] (8)

By Fourier inversion, we have:
\[U(v_T) = \frac{1}{(2\pi)^n} \int_Z e^{-iz'v_T} \int_{\mathbb{R}^n} e^{iz'v_T} U(v_T) dv_T dz\] (9)

for a (possibly complex) \((n \times 1)\)-vector \(z\) and a suitable domain of integration \(Z\), which yields:\(^7\)
\[P_t^U = e^{-r(T-t)} \frac{1}{(2\pi)^n} \int_Z \Psi_v(-iz) \hat{F}_U(z) dz .\] (10)

where \(\Psi_v(-iz) = \mathbb{E}^Q[\exp{-iz'v_T}|\mathcal{F}_t]\) is the conditional Laplace transform transform of \(v_T\) evaluated at the complex vector \(-iz\), and \(\hat{F}_U(z)\) is the Fourier

\(^6\)See, e.g., [L00] for a comprehensive treatment of Transform Analysis applied to option pricing under stochastic volatility.

\(^7\)We have:
\[P_t^U = e^{-r(T-t)} \mathbb{E}^Q \left[ \frac{1}{(2\pi)^n} \int_Z e^{-iz'v_T} \int_{\mathbb{R}^n} e^{iz'v_T} U(v_T) dv_T dz | \mathcal{F}_t \right] \]
\[= e^{-r(T-t)} \frac{1}{(2\pi)^n} \int_Z \mathbb{E}^Q \left[ e^{-iz'v_T} | \mathcal{F}_t \right] \hat{F}_U(z) dz \]
\[= e^{-r(T-t)} \frac{1}{(2\pi)^n} \int_Z \Psi_v(-iz) \hat{F}_U(z) dz .\]
transform of the payoff function $U(v_T)$. To compute asset prices, we need just to compute the Fourier transform of the derivative payoff and the conditional Laplace transform of the asset returns, as well as to specify restrictions on the domain of integration $Z$ ensuring existence of the integral (10). Since the model is affine, the conditional Laplace transform of $v_T$ satisfies a closed-form exponentially affine formula that is stated for completeness in Section 6 of the Appendix.

3.1 Pricing Equity and Corporate Debt

Denoting by $(T_1, ..., T_n)$ the maturities of the bonds issued by the $n$ firms and by $(D_1, ..., D_n)$ the corresponding face values, the payoff of the $j^{th}$ corporate bond at maturity is the sum of its face value and the payoff of a short put position on the firm’s assets, with strike $D_j$:

$$D_j - (D_j - V_{T_j}^j)^+. \quad (11)$$

The time-$T_j$ value of the firm to equityholders is the payoff of a call option struck at $D_j$:

$$(V_{T_j}^j - D_j)^+. \quad (12)$$

Denoting by $S_t^j$ the price of firm $j$’s equity at time $t \leq T_j$, it thus follows:

$$S_t^j = \exp(-r(T_j - t)) \cdot \mathbb{E}^Q [(V_{T_j}^j - D_j)^+ | \mathcal{F}_t] \quad (13)$$

The corporate bond price $CB_t^j$ is the difference between the firm value and the price of shareholders’ claim:

$$CB_t^j = V_t^j - S_t^j. \quad (14)$$

Finally, the corporate bond spread is given by

$$s_t^j = -\frac{1}{T_j - t} \left[ \log(CB_t^j) - d_j \right] - r \quad (15)$$

where $d_j \equiv \log(D_j)$.

**Proposition 1.** The price of equity for firm $j \in 1, ..., n$ at time $t \leq T_j$ is given by

$$S_t^j = e^{-r(T_j - t)} \frac{1}{2\pi} \int_{Z_j} \Psi_{\nu}(-iz_j) \hat{F}_C(z_j) dz_j \quad (16)$$
where $z_j = (0, ..., 0, z_j, 0, ... 0)'$ is a $(n \times 1)$ vector having all of its elements equal to zero except for the $j^{th}$ one which is equal to $z_j$,

$$
\hat{F}_C(z_j) = \frac{e^{iz_j d_j + d_j}}{(iz_j + 1)iz_j}
$$

is the Fourier transform of a Call option payoff with strike equal to $D_j$ and the domain of integration is given by the set $Z_j = \{z_j \in \mathbb{C} \mid \text{Im}(z_j) > 1\}$. The prices of corporate bonds and the bond spread follow by substituting (16) into equations (14) and (15).

Proof. See section 7.1 in the Appendix.

From the above expression for the price of equity, the volatility of stock returns follows from Itô’s Lemma.

**Proposition 2.** The conditional variance of instantaneous changes in equity value is given by

$$
\frac{1}{dt} \text{var}_t(dS^j_t) = 4Tr \left[ (DS^j_t) \Sigma_t (DS^j_t)^Q'Q \right] + \left( \frac{\partial S^j_t}{\partial v^j} \right)^2 \Sigma_{ij}^j
$$

$$
+ 4Tr \left[ \Sigma_t R_j Q (DS^j_t) \right] \frac{\partial S^j_t}{\partial v^j}
$$

where $R_j$ is a $(n \times n)$-matrix whose elements are all equal to zero except for its $j^{th}$ row, which is equal to $\rho^j$.

Proof. See section 7.2 of the Appendix.

From Proposition 2, the volatility of stock returns is stochastic and depends on both the firm value $V_t$ and the asset returns covariance matrix $\Sigma_t$. Therefore, the model can feature option implied volatility smiles and skews.

### 3.2 Pricing Stock Options

Since equity is a call option on the firm’s assets, stock options are compound options on firm value in our setting. In the single-factor Merton [M74] model, the price of compound options are available in closed form; see [G79]. However, in models with stochastic volatility closed-form pricing formulas are not generally not available.\(^8\) Therefore, we resort to Monte Carlo methods.

\(^8\)See [FH05] on pricing compound options in stochastic volatility models.
to compute the price of an option on firm \( j \)'s equity with maturity \( \tau_j \leq T_j \) as the risk neutral expectation of its payoff:

\[
e^{-r(\tau_j-t)E(Q)^{[h(S_j^{\tau_j}, K)|F_t]}(19)}
\]

where \( h(s, k) \) is the option’s payoff function at maturity.

### 3.3 Pricing Credit Derivatives

Credit Default Swaps (CDS) are by far the most popular credit derivatives. In short, a CDS is an insurance contract against the risk of default by a particular company. The seller of the CDS usually pays a contractually fixed amount (called the principal of the contract) to the buyer in case of default, and in turn receives a periodic premium (called the CDS spread).

In our model, a company may default only at the maturity date of its debt. Therefore, we set the settlement date of the CDS contract equal to this date. This leads to the following payoff for a CDS on firm \( j \):

\[
P_{T_j} = \mathbb{1}_{\{V_{T_j}^j < D_j\}}.
\] (20)

Thus the CDS payoff is equal to the payoff of a digital put option on firm’s \( j \) assets with strike at \( D_j \), and its price can be easily computed with Fourier methods.

**Proposition 3.** The price of the CDS contract at time \( t \leq T_j \) is given by:

\[
P_t^j = e^{-r(T_j-t)} \frac{1}{2\pi} \int_{Z_j} \Psi(t)(-iz_j) \hat{F}^j_{CDS}(z_j) dz_j
\] (21)

where \( z_j = (0, ..., 0, z_j, 0, ...0)' \) is a \((n \times 1)\) vector,

\[
\hat{F}_{CDS}(z_j) = \frac{1}{iz_j} e^{iz_j}
\] (22)

and the domain of integration is given by \( Z_j = \{z_j \in \mathbb{C} \mid Im(z_j) < 0\} \).

**Proof.** See section 7.3 in the Appendix.

Since the payoff function of the CDS contract is the indicator of the default set for firm \( j \), the CDS price equals the risk-neutral probability of default.\(^\text{10}\)

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\(^{9}\) See, e.g., [HW00] for a detailed discussion on various types of CDS and for a description of the structure of payments and obligations between seller and buyer of these contracts.

\(^{10}\) This finding holds because we have assumed a zero recovery rate. Adding a nonzero recovery rate makes the CDS price proportional to the risk-neutral default probability. See, e.g., [HW00] on modeling CDS contracts with a nonzero recovery rate.
We can also price credit derivatives with payoff depending on a basket of reference entities. In this way, we can study the link between the (stochastic) correlations structure of asset returns and the prices of such credit derivatives. To this end, we focus on a stylized three-assets setting in which all debt maturities are equal across firms: $T_1 = T_2 = T_3 = T$. This is the simplest multi-asset credit derivative that allows us to get nontrivial results for the implied correlations.\footnote{At least three assets are necessary to obtain nontrivial predictions with regard to implied correlations of $n^{th}$-to-default structures, because in a two-asset setting there is only one implied correlation depending on the joint probability of default of both firms. See also the derivations below and the Appendix on implied correlations.}

Similar to CDO contracts, we assume a basket payoff consisting of an 'equity' tranche, which pays off if at least one firm defaults, a 'mezzanine' tranche, which pays in case of two or more defaults, and a 'senior' tranche, which pays off only if all three reference entities default. In other words, the payoff from the 'equity' tranche corresponds to a first-to-default, the one from the 'mezzanine' to a second-to-default, and the payoff from the 'senior' tranche to a third-to-default credit default swap basket. The payoff of the 'equity' tranche is:

$$p_e^T = \mathbb{I}_{\{V_1^T < D_1\}} + \mathbb{I}_{\{V_2^T < D_1\}} + \mathbb{I}_{\{V_3^T < D_1\}} - \mathbb{I}_{\{V_1^T < D_1\}} \cdot \mathbb{I}_{\{V_2^T < D_2\}} - \mathbb{I}_{\{V_1^T < D_1\}} \cdot \mathbb{I}_{\{V_3^T < D_3\}} - \mathbb{I}_{\{V_2^T < D_2\}} \cdot \mathbb{I}_{\{V_3^T < D_3\}}$$ (23)

The one of the 'mezzanine' tranche is:

$$p_m^T = \mathbb{I}_{\{V_1^T < D_1\}} \cdot \mathbb{I}_{\{V_2^T < D_2\}} + \mathbb{I}_{\{V_1^T < D_1\}} \cdot \mathbb{I}_{\{V_3^T < D_3\}} + \mathbb{I}_{\{V_2^T < D_2\}} \cdot \mathbb{I}_{\{V_3^T < D_3\}} - 2 \mathbb{I}_{\{V_1^T < D_1\}} \cdot \mathbb{I}_{\{V_2^T < D_2\}} \cdot \mathbb{I}_{\{V_3^T < D_3\}}$$ (24)

Finally, the payoff of the 'senior' tranche is:

$$p_s^T = \mathbb{I}_{\{V_1^T < D_1\}} \cdot \mathbb{I}_{\{V_2^T < D_2\}} \cdot \mathbb{I}_{\{V_3^T < D_3\}}$$ (25)

The prices of these payoffs are again computed by Fourier methods.

**Proposition 4.** The prices of the tranches of the three assets basket are given by:

$$P_e^T = P_1^T + P_2^T + P_3^T - P_{12}^T - P_{13}^T - P_{23}^T + P_{123}^T$$

$$P_m^T = P_{12}^T + P_{13}^T + P_{23}^T - 2 \cdot P_{123}^T$$

$$P_s^T = P_{123}^T$$

The prices of these payoffs are again computed by Fourier methods.
where $P^j_i$, $j = 1, 2, 3$ is given in proposition 3,

$$P^i_j = \frac{1}{(2\pi)^2} \int_{Z_{ij}} \Psi_{\nu}(-iz)\hat{F}^{ij}_{CDS}(z_{ij})dz_idz_j$$  \hspace{1cm} (26)

where $Z_{ij} = \{(z_i, z_j) \in \mathbb{C}^2 \mid \text{Im}(z_i) < 0 \text{ and } \text{Im}(z_j) < 0\}$, $z_{ij}$ is a vector having zeros everywhere except on its $i^{th}$ and $j^{th}$ positions and

$$\hat{F}^{ij}_{CDS}(z_{ij}) = -\frac{1}{z_iz_j}e^{i(z_id_i+zjd_j)}.$$  \hspace{1cm} (27)

Similarly,

$$P^{123} = \frac{1}{(2\pi)^3} \int_{Z_{123}} \Psi_{\nu}(-iz)\hat{F}^{123}_{CDS}(z)dz_1dz_2dz_3$$  \hspace{1cm} (28)

where $z = (z_1, z_2, z_3)'$, $Z_{123} = \{z \in \mathbb{C}^3 \mid \text{Im}(z_1) < 0, \text{Im}(z_2) < 0 \text{ and } \text{Im}(z_3) < 0\}$ and

$$\hat{F}^{123}_{CDS}(z) = -\frac{1}{iz_1z_2z_3}e^{i(z_1d_1+z_2d_2+z_3d_3)}.$$  \hspace{1cm} (29)

Proof. See section 7.4 in the Appendix. \hfill \square

# 4 Model Analysis

We study the behaviour of corporate credit spreads, equity prices, the prices of stock options and credit derivatives in our model. We first focus on the setting with two firms, using the following parameter choice:

$$M = \begin{pmatrix} -0.5 & -0.2 \\ -0.2 & -0.5 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.25 & 0.1 \\ 0.1 & 0.25 \end{pmatrix}$$  \hspace{1cm} (30)

and $\beta = 2$. From equation (3), these parameters imply an unconditional covariance matrix equal to:\textsuperscript{12}

$$\Sigma_\infty = \begin{pmatrix} 0.125 & 0.05 \\ 0.05 & 0.125 \end{pmatrix}$$

This unconditional covariance matrix is useful to assess the effects of stochastic volatility and correlations on derivative prices, and to compare them with the results of a multivariate Merton model with constant covariance matrix $\Sigma_\infty$. The remaining model parameters are: $V_t = (20, 20)'$, $r = 0$, $t = 0$ and $T_1 = T_2 = 1$.

\textsuperscript{12}As shown by [J68], the solution to the equation $MS_\infty + S_\infty M' = -\Omega\Omega'$ in the two-asset case is given by

$$\Sigma_\infty = \frac{1}{2\text{tr}(M)}\left[ -\Omega\Omega' - |M|^{-1}\Omega\Omega' (M^{-1})' \right].$$
4.1 Asset Returns and CDS Spreads

Given our parameter choice, both firms behave identically and it is sufficient to focus on firm 1. For brevity, we also restrict $\rho$ to be of the form $\rho = (\rho^*, \rho^*)'$. All elements of $Q$ are positive. Therefore, if $\rho^* < 0$ ($\rho^* > 0$) equation (5) implies a left-skewed (right-skewed) asset return distribution. The implied CDS spreads of a firm financed by 25%, 50%, 75% and 95% of debt are presented in Figure 1.

![CDS Spread Graphs](image)

Figure 1: CDS Spread versus $\rho^*$ and leverage. The solid line denotes the CDS spread implied by a Merton model with constant covariance matrix $\Sigma_\infty$. The dash-dotted line represents the spreads generated by our model.

In the upper panels, the CDS spread decreases with $\rho^*$ and it can be significantly higher than in the Merton model for negative values of $\rho^*$. The lower panels of Figure 1, reveal an opposite relationship for high leverage firms, since CDS spreads increase in $\rho^*$. To get an intuition for these effects, note that a CDS contract can be considered as a digital put option on the firm’s asset value, with strike at the debt level. For low leverage firms, this option is far out of the money. Therefore, its value largely depends on the
left tail of the asset return distribution. Introducing stochastic volatility yet keeping $\rho^*$, and thereby skewness, equal to zero increases the kurtosis of the asset return distribution, by shifting probability mass from the center to the tails. This feature increases the risk-neutral probability that the put will end up in the money, and implies a higher put price than in the Merton case. In the upper panel of Figure 1, this effect is amplified by introducing a negative skewness in the asset return distribution. In the opposite case, as the distribution of asset returns becomes more right-skewed, the probability of exercising the digital put declines, leading to a lower price of the CDS contract, which can happen to be below the one implied by the Merton setting. For high leverage firms, the price of the CDS is less sensitive to the tails of the asset return distribution. The lower panels of Figure 1 show that the relation between CDS spreads and firm value skewness may be positive in this case.

4.2 Term Structure of Corporate Bond Spreads

Next, we focus on the term structure of bond spreads, and plot it for maturities between 1 and 10 years in Figures 2–5, as a function of various degrees of leverage.

First, we see that our model can produce a wide variety of shapes in the term structure of debt spreads. Credit spreads increase with maturity for low levels of leverage, but we find an opposite relation for high leverage firms. This finding is consistent with the empirical observation that firms with low credit ratings tend to have inverted spread term structures. Second, at short maturities and low levels of leverage our model can produce higher bond spreads than in Merton’s model for the same long-run covariance matrix. For example, the 1-year spread of the firm financed by 50% of debt is approximately 46 bps in the Merton case. In our model, the bond spread increases to 74 bps for $\rho^* = 0$ and 103 bps for $\rho^* = -0.2$. As leverage is lowered to 25% debt financing, these effects are even more pronounced.

The intuition for these findings can be understood from the fact that the corporate bond price equals the price of a default-free zero-coupon bond and a short put option with strike at the debt level; see again equation (11). Stochastic volatility and negative skewness increase the value of the put option, especially for low leverage firms, by increasing the probability that the option will end up in the money. This leads to the higher credit spreads. A common criticism of the [M74] approach is linked to the fact that it produces on average too low credit spreads, especially for firms with high credit ratings.
Figure 2: Term structure of bond spreads for firm financed with 25% of debt. Dotted line: Spreads for $\rho^* = -0.2$. Dashed line: Spreads for $\rho^* = 0$. Dash-dotted line: Spreads for $\rho^* = 0.2$. Solid line: Merton case.

Figure 3: Term structure of bond spreads for firm financed with 50% of debt. Dotted line: Spreads for $\rho^* = -0.2$. Dashed line: Spreads for $\rho^* = 0$. Dash-dotted line: Spreads for $\rho^* = 0.2$. Solid line: Merton case.
Figure 4: Term structure of bond spreads for firm financed with 75% of debt. Dotted line: Spreads for $\rho^* = -0.2$. Dashed line: Spreads for $\rho^* = 0$. Dash-dotted line: Spreads for $\rho^* = 0.2$. Solid line: Merton case.

Figure 5: Term structure of bond spreads for firm financed with 95% of debt. Dotted line: Spreads for $\rho^* = -0.2$. Dashed line: Spreads for $\rho^* = 0$. Dash-dotted line: Spreads for $\rho^* = 0.2$. Solid line: Merton case.
and for short corporate debt maturities. In other words, individual firm characteristics implied by Merton’s setting are unable to explain empirically corporate credit spreads, which is linked to the so-called ‘credit spread puzzle’. Our findings show that stochastic volatility and stochastic correlations of asset returns may potentially contribute to raise corporate credit spreads to more realistic levels.

### 4.3 Stock Prices and Firm Value Skewness

In structural credit risk models, equity is a call option on firm value. In our setting, stochastic volatility and asset return skewness strongly impact on the price of this option. Moreover, the sign of the dependence of stock prices on firm value skewness depends on firm leverage. We summarize these findings in Figure 6.

First, for low leverage firms the price of equity in our model is higher than in the Merton setting when there is zero asset return skewness ($\rho^* = 0$). However, as leverage increases the stock price in our model can also be lower than in the Merton setting. For example, the stock price of the firm financed by 50% of debt decreases from 10.074 to 10.046 when moving from the stochastic volatility case to a Merton model having the same covariance matrix, but it increases from 3.157 to 3.265 for the firm with 95% of debt financing.

Second, the equity price is decreasing in the correlation between asset returns and their volatilities for firms with low to moderate leverage. For instance, the stock price of the firm financed by 50% of debt is worth 10.16 units when $\rho^* = -0.6$. Increasing $\rho^*$ to 0.6 lowers the price of equity to 10.00. However, in the case of a firm financed by 95% of debt, we can also observe a non-monotonic relation between $\rho^*$ and the price of equity price.

The intuition for these effects is best motivated by invoking put-call parity to rewrite the equity as a portfolio that is long the firm’s assets and a put option on firm value, with strike at the debt level, and which is short $D$ riskless zero coupon bonds:

$$S_t = V_t + P(D, t, T) - DB(t, T)$$

where $P(D, t, T)$ and $B(t, T)$ are the prices of the put option and the zero-coupon bond, respectively.

Since $V_t$ and $B(t, T)$ are known, the stochastic behaviour of asset returns influences the price of equity price through its impact on the price of the put option.

---

13See, e.g., [EHH04] for a comparative analysis of several structural models, as well as [G04] and the references therein.

14See, e.g., [D06] for a detailed discussion on the credit spread puzzle.
Figure 6: Equity Price versus $\rho$ and leverage. For simplicity we take $\rho_1 = \rho_2 = \rho^*$. The solid line denotes the Equity price implied by our model and the dash-dotted line the equity price resulting from a Merton approach with the same long-run covariance matrix.

option, and similar intuitions as in the previous sections apply: When firm leverage is small the put option is far out-of-the-money and its price is highly sensitive to increases in the kurtosis and the left skewness of the asset returns distribution. If firm leverage is high, the put option is close to at-the-money and an opposite situation arises. While it gives rise to a higher probability of large negative asset returns, negative skewness also shifts probability mass away from small decreases in asset value, which would cause the put option to have positive contingent payoffs at expiry. Figure 6 (bottom right panel) shows that the latter effect dominates, which yields and equity price that decreases as asset returns becomes more left-skewed. A similar argument obtains in the case of strongly right-skewed returns, which causes the price of equity to decline as the correlation between asset returns and volatility
becomes increasingly positive.$^{15}$

### 4.4 Co-movement of Equity Prices, Credit Spreads and Stock Return Volatility

In the Merton [M74] setting, the price of equity and credit spreads always co-move negatively. This property follows directly from the single-factor structure of the model when asset values are modeled with a geometric Brownian Motion. While being consistent with numerous observations in market practice, there are several cases in which this property is violated. A prominent example is the downgrade of General Motors in May 2005, which was followed by an increase in both credit spreads and the price of GM stocks.$^{16}$

In our multifactor stochastic volatility model, stock prices depend on asset values, and their volatilities and co-volatilities. Changes in asset values and different asset return correlations can impact differently on the prices of equity and debt prices, giving rise to situations in which both a positive and a negative co-movement between equity prices and debt spreads can be observed. To illustrate, consider the firms financed by 50% and 95% of debt when $\rho^t = -0.6$. Increasing $V_t$ by 1 to 21 and $\Sigma_{11}^{11}$ from 0.125 to 0.15, while keeping all other elements of the covariance matrix unchanged, yields an increase in the price of equity from 10.16 (3.13) to 11.16 (4.01). At the same time, credit spreads rise (decrease) from 162 bps (1190 bps) to 166 bps (1120 bps) for the low leverage firm (high leverage firm). A more direct measure of the tendency of equity prices and debt spreads to co-move is the instantaneous correlation between stock and corporate bond returns.

**Proposition 5.** The instantaneous correlation between changes in the value of equity and the corporate bond is given by

$$
corr_t(dS^j_t, dCB^j_t) = \frac{< S^j_t, V^j_t >_t - < S^j_t, S^j_t >_t}{\sqrt{< V^j_t, V^j_t >_t} \sqrt{< S^j_t, S^j_t >_t}}
$$

---

$^{15}$These results are in line with the analysis of [BTV07], who find a positive response of equity value when moving from the Merton setting to a left-skewed asset return distribution in the low leverage case, and a negative response in the high leverage case. In their model, skewness of the asset return distribution arises endogenously in an economy with beliefs heterogeneity. In our model, it is implied by the choice of the correlation vector between asset returns and volatilities.

$^{16}$See also [BTV07] for a more detailed discussion.
where

\[ < S^j_t, V^j_t >_t = dS^j_t \cdot dV^j_t \]
\[ = V^j_t \cdot \left[ \frac{\partial S^j_t}{\partial v^j} \Sigma^j_t + 2Tr \left[ \Sigma_t R_j QDS^j_t \right] \right] dt, \quad (32) \]
\[ < V^j_t, V^j_t >_t = (V^j_t)^2 \Sigma^j_t dt. \quad (33) \]

and \( < S^j_t, S^j_t >_t = \text{var}_t(dS^j_t) \) is given in Proposition 2.

\textbf{Proof.} See section 7.5 in the Appendix.

We use these results to plot this instantaneous correlation for different values of leverage and asset return skewness in figure 7. At our parameter values, equity and debt prices tend to co-move positively. Moreover, leverage and asset returns skewness strongly affect the size of this correlation. For firms with high leverage, the correlation is close to 1, but for low leverage firms it may be rather small, increasing the chances to observe a positive co-movement between equity prices and debt spreads, especially when the asset return distribution is right-skewed.
The co-movement between stock returns and their volatilities is characterized next.

**Proposition 6.** Define $\sigma^j_t$ as the instantaneous volatility of equity. The instantaneous correlation between changes in equity prices and equity volatility is given by

$$
\text{corr}_t(d\sigma^j_t, dS^j_t) = \frac{< \sigma^j_t, S^j_t >_t}{\sqrt{< \sigma^j_t, \sigma^j_t >_t} \sqrt{< S^j_t, S^j_t >_t}}
$$

The explicit expressions for $< \sigma^j_t, S^j_t >_t$, $< \sigma^j_t, \sigma^j_t >_t$ and $< S^j_t, S^j_t >_t$ are given in the Appendix.

**Proof.** See section 7.6 in the Appendix.

The instantaneous correlation between stock returns and their volatility is stochastic, because it depends on $V^j_t$ and $\Sigma_t$, which yields a stochastic skewness for stock returns. This finding is interesting in light of the recent literature on stochastic skewness in option pricing models (see, e.g., [CHJ07]) emphasizing the failure of stochastic volatility models with deterministic skewness, such as those in [H93] or [HW87], to capture the dynamics of option-implied volatility smiles.

### 4.5 Option-Implied Volatility Smiles

In structural firm value models, stock options are compound options on asset values. [G79] shows how to compute the price of stock options in the single-factor, constant volatility, Merton [M74] model. In this model, stock returns volatility is stochastic, which can feature implied volatility smiles. However, as [HNW04] note, only negatively sloped smiles for put options can be generated. This is an undesirable restriction, as empirical evidence shows that some stocks can exhibit positively sloped implied volatility smiles over time. [BKM03] find that positive smiles can arise in up to 20% of the cases for some stocks in the S&P 100 index. [BTV07] find similar results using a different dataset.

An important feature of our model is that the skewness of stock returns is endogenous and dependent on leverage. To illustrate the main implications of this feature, we compute the implied volatility smiles for put options across several debt levels, strike prices and values of $\rho^*$. We consider European put options with 30 days maturity for $\rho^* = -0.3, 0, 0.3$. Stock option prices are computed by Monte Carlo simulation and the results are summarized in Figures 8–10.
Figure 8: Implied volatility smiles versus leverage for Put Options and $\rho = (-0.3, -0.3)'$. 

Figure 9: Implied volatility smiles versus leverage for Put Options and $\rho = (0, 0)'$. 

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Figure 10: Implied volatility smiles versus leverage for Put Options and $\rho = (0.3, 0.3)'$.

The skewness of equity returns implied by the different parameter choices is computed in table 1.

<table>
<thead>
<tr>
<th>$\rho^*$</th>
<th>25% debt</th>
<th>50% debt</th>
<th>75% debt</th>
<th>95% debt</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.3</td>
<td>-0.3694</td>
<td>-0.5427</td>
<td>-0.7247</td>
<td>-0.7804</td>
</tr>
<tr>
<td>0</td>
<td>-0.1276</td>
<td>-0.3126</td>
<td>-0.4810</td>
<td>-0.4858</td>
</tr>
<tr>
<td>0.3</td>
<td>0.1171</td>
<td>-0.0714</td>
<td>-0.1918</td>
<td>-0.1574</td>
</tr>
</tbody>
</table>

Table 1: Skewness of equity returns.

First, we see that the level of option-implied volatilities increases with leverage. This is consistent with the intuition that higher leverage increases the risk of negative returns and at the money put option prices. Second, even if implied volatility smiles are negatively sloped in most cases, they feature a positive slope for right-skewed asset returns and low levels of leverage. To understand these effects, note that the skewness of equity returns in Table 1 is increasing in $\rho^*$. Therefore it is increasing in the skewness of asset returns. For a firm financed by 25% of debt and $\rho^* = 0.3$, the skewness of stock returns

\footnote{See also [GZ07] for a discussion of equity option pricing in the [M74] model.}
is positive, which implies the positively sloped implied volatility smile. The skewness of stock returns is decreasing in leverage for \( \rho^* = 0, -0.3 \), but it first decreases and then increases again for \( \rho^* = 0.3 \). From Figures 8 and 9, as we move to higher levels of firm leverage and the skewness of stock returns becomes increasingly negative, the negative slope of the implied volatility smile also increases. This property is consistent with the empirical evidence in [BKM03] who document a positive relation between stock returns skewness and the slope of the smile of single stock options.

### 4.6 Joint Default Probabilities

The multivariate Merton model implies joint default probabilities that are functions of firm leverage, the volatility of asset returns and their correlation. In our setting, additional degrees of freedom arise through the stochastic covariance matrix and skewness properties of the asset return distribution. This is illustrated by Figure 11.

Consider the joint probability of default for \( \rho^* = 0 \) first. The joint PDs are significantly higher than in the multivariate Merton case for firms with low and moderate levels of debt financing. The intuition is straightforward: Allowing for stochastic volatility leads to heavier tails of asset return distributions and thereby increases not only the marginal, but also the joint probability of negative shortfalls. Figure 11 reveals that the effects of skewness in the marginal distributions of asset returns persists in the multi-firm case: For increasingly negative values of \( \rho^* \), the joint probability of default increases, but when the distribution becomes more right-skewed joint default probabilities decrease. Interestingly, for higher levels of leverage, this behavior might be completely different, as shown in the bottom right panel of Figure 11, where the joint PD is increasing in \( \rho^* \) and it reaches a highest levels for positively skewed asset returns. The model implied default correlations are presented in figure 12, where default correlation is defined as the linear correlation coefficient between the indicator variables of default events. Figure 12 reveals that also default correlations strongly depend on firm leverage and the skewness of asset returns. For instance, default correlation is about 0.18 for low leverage firms when \( \rho^* = -0.6 \), but it is almost zero when the skewness of asset returns is positive. On the other hand, high leverage firms exhibit significant default correlations across a wide set of parameter values.
Figure 11: Joint Default Probability versus $\rho^*$. Solid line: Joint PDs with stochastic correlations. Dash-dotted line: Joint PDs implied by Merton model.
Figure 12: Default correlation versus $\rho^*$. 
4.7 The Three-Asset Basket

We analyze the effects of a stochastic covariance matrix of asset returns for pricing the tranches of a simple three-asset basket under the parameter choices:

\[ M = \begin{pmatrix} -0.5 & -0.2 & -0.2 \\ -0.2 & -0.5 & -0.2 \\ -0.2 & -0.2 & -0.5 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.25 & 0.1 & 0.1 \\ 0.1 & 0.25 & 0.1 \\ 0.1 & 0.1 & 0.25 \end{pmatrix}, \]

for an initial covariance matrix

\[ \Sigma(t) = \begin{pmatrix} 0.12 & -0.03 & -0.03 \\ -0.03 & 0.12 & -0.03 \\ -0.03 & -0.03 & 0.12 \end{pmatrix}, \]

and for \( \beta = 3, V(t) = [20, 20, 20]^{T}, r = 0 \) and \( t = 1. \)

We investigate the model’s ability to reproduce implied correlation smiles consistent with those observed in the market for basket credit derivatives, such as Collateralized Debt Obligations (CDOs) or credit default swap baskets. The notion of implied correlation is linked to a Gaussian one-factor model used to represent inter-asset correlation structures. This model has become standard in the pricing of CDOs and index tranches.\(^{18}\) It assumes that log-asset returns are jointly normal with all correlation parameters collapsed into one. Similar to the implied volatility for equity options, implied correlation for credit derivatives is computed by matching observed market quotes to model prices and by solving for the implied correlation parameter. Note however that market quotes across tranches of correlation-dependent credit derivatives are usually not consistent with a single solution for the correlation parameter. This feature generates the implied correlation smiles.\(^{19}\)

There are two common types of implied correlations for credit derivatives: Compound correlation and base correlation.\(^{20}\) Compound correlation is determined by matching the price of each tranche with the corresponding model price. In our case,

\[ P^{k}_{\text{market}} = P^{k}_{\text{Gaussian}}(PD^{1}, ..., PD^{n}, \rho^{\text{compound}}) \]

\(^{18}\)See, e.g., [LI99] and [LI00] for early applications of the Gaussian Copula Model to credit risk modeling.

\(^{19}\)See, e.g., [F05].

\(^{20}\)For a discussion of the differences between the two approaches, see, e.g., [F05] or [TBP06].
the compound correlation parameter and $P^k_{\text{Gaussian}}$ is the tranche price using the Gaussian single-factor model.\(^\text{21}\) In practice, compound correlation frequently suffers from the problem that the parameter $\rho^{\text{compound}}$ cannot be determined uniquely, or does not exist at all. This problem is especially pronounced for mezzanine tranches.\(^\text{22}\) Base correlation is a partial remedy to the disadvantages of compound correlation, in that it may be computed for a larger set of parameter choices.\(^\text{23}\) The idea underlying base correlations is to replace the actual tranches by a sequence of equity tranches with the same upper attachment point. Base correlations are then synthesized from market-observed tranche prices using a bootstrap approach.\(^\text{24}\)

In Figures 13 and 14, we plot compound correlations for each tranche of the three-asset basket and for firms with 50% and 75% of debt financing. For notational simplicity, we maintain the assumption that all the elements of $\rho$ are equal: $\rho = (\rho^*, \rho^*, \rho^*)'$.

In the multivariate Merton model, asset return correlations are deterministic and the implied correlation is constant, which yields a a flat implied correlation surface across tranches and firm leverage. As Figures 13 and 14 highlight, our model gives rise to correlation smiles of different level and shape. First, even for a zero skewness of asset returns ($\rho^* = 0$) we obtain a non-constant implied correlation across basket tranches, which features (possibly inverted) correlation smiles. This feature follows from the fact that stochastic correlations imply fatter tails of the multivariate asset return distribution than in the Gaussian case. Second, asset return skewness has a strong effect on the level and shape of the correlation smile. When the asset return distribution is left-skewed, the smile is inverted for firms financed by 50% and 75% of debt. However, as the correlation between asset returns and volatilities increases, the implied correlation smile flattens and - depending on leverage - it eventually might even turn into a true smile.

The base correlations for the equity and mezzanine tranches are presented in Tables 2 and 3.\(^\text{25}\) Consistent with empirical observations, base correlations have an opposite behavior than compound correlations. For low leverage and negative asset return correlation, base correlations for the mezz-

\(^{\text{21}}\)An explicit formula for $P^k_{\text{Gaussian}}$ is given in the Appendix.

\(^{\text{22}}\)For problems relating to the computation of compound correlations, see e.g. [DL06] or [F05].

\(^{\text{23}}\)See, e.g., TBP06.

\(^{\text{24}}\)The concept of base correlations goes back to [ABMW04], [DL06] and [R04] discuss alternative approaches for computing base correlations. Section 8.2 in the Appendix describes the calculation of base correlations for the three-asset basket considered here.

\(^{\text{25}}\)For the technical reasons described in the appendix, we compute base correlations only for the equity and mezzanine tranches of the three-asset basket.
Figure 13: Implied correlation smiles for a firm financed by 50% of debt.
<table>
<thead>
<tr>
<th>Equity</th>
<th>Mezzanine</th>
<th>Senior</th>
</tr>
</thead>
<tbody>
<tr>
<td>−0.25</td>
<td>−0.15</td>
<td>−0.05</td>
</tr>
<tr>
<td>0.05</td>
<td>0.15</td>
<td>0.25</td>
</tr>
<tr>
<td>0.35</td>
<td>0.4</td>
<td>0.45</td>
</tr>
<tr>
<td>0.5</td>
<td>0.55</td>
<td>0.6</td>
</tr>
<tr>
<td>0.6</td>
<td>0.65</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Figure 14: Implied correlation smiles for a firm financed by 75% of debt.
zanne are lower than for the equity tranche. This effect fades and eventually changes sign for a high leverage and a positive skewness of the asset return distribution.

5 Conclusion

We develop a convenient multivariate firm value model for pricing capital structure derivatives. Our setting features asset returns driven by a general stochastic variance-covariance matrix process and a flexible leverage structure. These features generate in a single coherent setting many realistic pricing patterns for the implied volatility smile of equity options, the (stochastic) skewness of stock returns, the term structure of credit default swaps and the implied correlation smile of basket credit derivatives. At the same time, due to the affine structure of the state process, our model is tractable enough to compute the prices of many capital structure derivatives by means of semi-closed form formulas.
6 Appendix A: Conditional Laplace transform of the asset return process

The conditional Laplace transform of the logarithmic asset values is given in the following Lemma.

Lemma 2. The conditional Laplace transform of the logarithm of asset values $v_T$ conditional on $v_t$ for $t \leq T$ is given by

$$
\Psi_{v(t)}(\gamma) = \mathbb{E}^Q \left[ e^{\gamma' v(T)} | \mathcal{F}_t \right] = \exp \{ \text{Tr}[A(\tau)\Sigma(t)] + \gamma' v(t) + c(\tau) \} \tag{36}
$$

with $\tau = T - t$ and where $A(\tau) = A_2^2(\tau)^{-1}A_1^2(\tau)$ with

$$
\begin{pmatrix}
A_1^1(\tau) & A_1^3(\tau) \\
A_1^2(\tau) & A_2^2(\tau)
\end{pmatrix} = \exp \tau \begin{pmatrix}
M + Q' \rho \gamma' & -2Q'Q \\
\frac{1}{2}(\gamma \gamma' - \text{diag}(\gamma)) & -(M' + \gamma \rho' Q)
\end{pmatrix} \tag{37}
$$

and

$$
c(\tau) = -\frac{\beta}{2} \text{Tr} \left[ \log(A_2^2(\tau)) + \tau M' + \tau \gamma (\rho' Q) \right] + \tau \text{Tr} [r \gamma'] \tag{38}
$$

7 Appendix B: Proofs of Propositions

7.1 Proof of Proposition 1

Given equations (10) and (36) all that needs to be proved is the form of the Fourier transform of the call option price and a suitable domain of integration.
We have
\[
\tilde{F}_C(z_j) = \int_{\mathbb{R}} e^{iz_j v^j_T} \left[ e^{v^j_T} - e^{d_j} \right]^+ dv^j_T
\]
\[
= \int_{d_j}^\infty \left[ \frac{1}{iz_j} e^{iz_j v^j_T} - \frac{1}{iz_j} e^{iz_j d_j} \right] dv^j_T
\]
\[
= e^{iz_j d_j + d_j} \frac{1}{(iz_j + 1)iz_j}
\]
where the last step follows by assuming that \( \text{Im}(z_j) > 1 \).

### 7.2 Proof of Proposition 2

Note that the value of equity price of firm \( j \) is a function of \( t \) and the state variables \( v^j \) and \( \Sigma \). It follows from Ito’s Lemma that
\[
dS^j_t = \left( \frac{\partial S^j_t}{\partial t} + \mathcal{L}_{(v^j, \Sigma)} S^j_t \right) dt
\]
\[
+ \text{Tr} \left[ (\sqrt{\Sigma_t} d\tilde{W}_t Q + Q'(d\tilde{W}_t)' \sqrt{\Sigma_t}) DS^j_t \right] + e_j \sqrt{\Sigma_t} d\tilde{Z}_t \frac{\partial S^j_t}{\partial v^j} \tag{39}
\]
where \( e_j \) is a \((1 \times n)\)-vector having all of its elements equal to zero except for its \( j \)th one which is equal to 1, \( D_{ik} = \frac{\partial}{\partial v^k_i} \) is the matrix differential operator and \( \mathcal{L}_{(v^j, \Sigma)} \) is the generator of the joint process \((v^j, \Sigma)\) given by
\[
\mathcal{L} = \text{Tr} \left[ (\Omega \Omega' + M \Sigma_t + \Sigma_t M') D + 2\Sigma_t DQ'QD \right]
\]
\[
+ \left( r - \frac{1}{2} \Sigma_t \right) \frac{\partial}{\partial v^j} + \frac{1}{2} \Sigma_t^{-1} \frac{\partial^2}{(\partial v^j)^2}
\]
\[
+ 2\text{Tr} [\Sigma_t R_j Q D] \frac{\partial}{\partial v^j}
\]
From equation (39), the conditional variance of the instantaneous change in stock value is
\[
\text{var}_t(dS^j_t) = \text{var}_t \left( \text{Tr} \left[ (\sqrt{\Sigma_t} d\tilde{W}_t Q + Q'(d\tilde{W}_t)' \sqrt{\Sigma_t}) DS^j_t \right] \right)
\]
\[
+ \text{var}_t \left( e_j \sqrt{\Sigma_t} d\tilde{Z}_t \frac{\partial S^j_t}{\partial v^j} \right)
\]
\[
+ 2\text{cov}_t \left( \text{Tr} \left[ (\sqrt{\Sigma_t} d\tilde{W}_t Q + Q'(d\tilde{W}_t)' \sqrt{\Sigma_t}) DS^j_t \right], e_j \sqrt{\Sigma_t} d\tilde{Z}_t \frac{\partial S^j_t}{\partial v^j} \right)
\]
\[34\]
First, note that
\[
\sqrt{\Sigma_t d \hat{W}_t Q} + Q'(d \hat{W}_t)\sqrt{\Sigma_t} = d \Sigma_t - \mathbb{E}^Q [d \Sigma_t] 
\] (40)
and therefore
\[
var_t \left( Tr \left[ (\sqrt{\Sigma_t d \hat{W}_t Q} + Q'(d \hat{W}_t)\sqrt{\Sigma_t}) DS_t^j \right] \right) = var_t \left( Tr [d \Sigma_t DS_t^j] \right) 
\] (41)
and
\[
cov_t \left( Tr \left[ (\sqrt{\Sigma_t d \hat{W}_t Q} + Q'(d \hat{W}_t)\sqrt{\Sigma_t}) DS_t^j \right], e_j \sqrt{\Sigma_t d \hat{Z}_t \frac{\partial S_t^j}{\partial v_j}} \right) 
\] = \[
cov_t \left( Tr [d \Sigma_t DS_t^j], e_j \sqrt{\Sigma_t d \hat{Z}_t \frac{\partial S_t^j}{\partial v_j}} \right) 
\] (42)
Next, since \( DS_t^j \) is a real symmetric matrix, it may be decomposed as
\[
DS_t^j = \sum_{i=1}^{n} \lambda_i a_i a_i' 
\] (43)
where \( \lambda_i \) and \( a_i, i = 1, ..., n \) are the eigenvalues and eigenvectors of \( DS_t^j \), respectively. Using equation (43), one gets
\[
var_t \left( Tr [d \Sigma_t DS_t^j] \right) = \left[ var_t \left( Tr \left( \sum_{i=1}^{n} \lambda_i a_i a_i' d \Sigma_t \right) \right) \right] 
\]
\[
= \left[ var_t \left( Tr \left( \sum_{i=1}^{n} \lambda_i a_i' d \Sigma_t a_i \right) \right) \right] 
\]
\[
= \sum_{i=1}^{n} cov_t (\lambda_i a_i' d \Sigma_t a_i) 
\]
\[
= \sum_{i=1}^{n} \left[ cov_t (\lambda_i a_i' d \Sigma_t a_i, \lambda_i a_i' d \Sigma_t a_i) \right] 
\]
\[
= 4 \sum_{i=1}^{n} \sum_{k=1}^{n} \lambda_i a_i' \Sigma_t a_k \lambda_k a_k' Q' Q a_i d \Sigma_t d t 
\]
\[
= 4 \sum_{i=1}^{n} \sum_{k=1}^{n} \lambda_i a_i' \Sigma_t \lambda_k a_k a_k' Q' Q a_i d \Sigma_t d t 
\]
\[
= 4 \sum_{i=1}^{n} \lambda_i a_i' \Sigma_t (DS_t^j) Q' Q a_i d \Sigma_t d t 
\]
\[
= 4Tr \left[ (DS_t^j) \Sigma_t (DS_t^j) Q' Q \right] d \Sigma_t d t 
\] (44)
Moreover,
\[
\text{var}_t \left( e_j \sqrt{\Sigma_t d\tilde{Z}_t} \frac{\partial S_t}{\partial v_j} \right) = \left( \frac{\partial S_t}{\partial v_j} \right)^2 e_j \sqrt{\Sigma_t d\tilde{Z}_t} \sqrt{\Sigma_t e'_j} \\
= \left( \frac{\partial S_t}{\partial v_j} \right)^2 e_j \sqrt{\Sigma_t} \sqrt{\Sigma_t e'_j} dt \\
= \left( \frac{\partial S_t}{\partial v_j} \right)^2 \Sigma_t^{1j} dt
\] (45)

Finally, we obtain:
\[
\text{cov}_t \left( \text{Tr} \left[ d\Sigma_t (DS'_t) \right], e_j \sqrt{\Sigma_t d\tilde{Z}_t} \frac{\partial S'_t}{\partial v_j} \right) = 2 \text{Tr} \left[ \sqrt{\Sigma_t d\tilde{W}_t} Q(DS'_t) \right] e_j \sqrt{\Sigma_t d\tilde{Z}_t} \frac{\partial S'_t}{\partial v_j} \\
= 2 \text{Tr} \left[ \sqrt{\Sigma_t d\tilde{W}_t} Q(DS'_t) \right] e_j \sqrt{\Sigma_t} \sqrt{1 - \rho' \rho dt} \frac{\partial S'_t}{\partial v_j} \\
= 2 \text{Tr} \left[ \sqrt{\Sigma_t d\tilde{W}_t} Q(DS'_t) \right] \cdot \text{Tr} \left[ \sqrt{\Sigma_t e'_j \rho' d\tilde{W}'_t} \frac{\partial S'_t}{\partial v_j} \right] \\
= 2 \text{Tr} \left[ \sqrt{\Sigma_t d\tilde{W}_t} Q(DS'_t) \right] \cdot \text{Tr} \left[ \sqrt{\Sigma_t R'_j d\tilde{W}_t} \frac{\partial S'_t}{\partial v_j} \right] \\
= 2 \text{Tr} \left[ \sqrt{\Sigma_t Q(DS'_t) R'_j} \frac{\partial S'_t}{\partial v_j} \right] dt \\
= 2 \text{Tr} \left[ \Sigma_t Q(DS'_t) R'_j \frac{\partial S'_t}{\partial v_j} \right] dt
\] (46)

where \( R'_j = e'_j \rho' \).

7.3 Proof of Proposition 3

The Fourier transform of the payoff function of the CDS is given by
\[
\hat{F}^{i_j}_{CDS}(z_j) = \int_{\mathbb{R}} e^{iz_j v'_j} \mathbb{I}_{\{v'_j < d_j\}} dv'_j \]
\[
= \left[ \frac{1}{iz_j} e^{iz_j d'_j} \right]_{-\infty}^d \\
= \frac{1}{iz_j} e^{iz_j d'_j}
\] (47)
provided that \( \text{Im}(z_j) < 0 \).

### 7.4 Proof of Proposition 4

The Fourier transform of the payoff function \( p_{ij}^T \) is given by

\[
\hat{F}_{CDS}^{ij}(z_{ij}) = \int_{\mathbb{R}^2} e^{i(z_{ij}v_i^j + z_jv_j^j)} 1\{v_i^j < d_i\} \cdot 1\{v_j^j < d_j\} dv_i^j dv_j^j \\
= \int_{-\infty}^{d_i} \int_{-\infty}^{d_j} e^{i(z_{ij}v_i^j + z_jv_j^j)} dv_i^j dv_j^j \\
= -\frac{1}{z_i z_j} e^{i(z_i d_i + z_j d_j)} \tag{48}
\]

provided that \( \text{Im}(z_i) < 0 \) and \( \text{Im}(z_j) < 0 \).

Similarly, the Fourier transform of the payoff function \( p_{123}^{ij} \) is given by

\[
\hat{F}_{CDS}^{123}(z) = \int_{\mathbb{R}^3} e^{i(z_{123}v_1^1 + z_{23}v_2^2 + z_{3}v_3^3)} 1\{v_1^1 < d_1\} \cdot 1\{v_2^2 < d_2\} \cdot 1\{v_3^3 < d_3\} dv_1^1 dv_2^2 dv_3^3 \\
= \int_{-\infty}^{d_1} \int_{-\infty}^{d_2} \int_{-\infty}^{d_3} e^{i(z_{123}v_1^1 + z_{23}v_2^2 + z_3 v_3^3)} dv_1^1 dv_2^2 dv_3^3 \\
= -\frac{1}{z_1 z_2 z_3} e^{i(z_1 d_1 + z_2 d_2 + z_3 d_3)} \tag{49}
\]

provided that \( \text{Im}(z_1) < 0 \), \( \text{Im}(z_2) < 0 \) and \( \text{Im}(z_3) < 0 \).

### 7.5 Proof of Proposition 5

Since \( CB_t^l = V_t^l - S_t^l \) it follows that the instantaneous covariation between the stock and bond prices is given by

\[
< CB_t^l, S_t^l >_t = < V_t^l, S_t^l >_t - < S_t^l, S_t^l >_t = dV_t^l \cdot dS_t^l - dS_t^l \cdot dS_t^l \tag{50}
\]

Further, recall from Proposition 2 that

\[
dS_t^j = \left( \frac{\partial S_t^j}{\partial t} + \mathcal{L}_{\{v^j, \Sigma\}} S_t^j \right) dt \\
+ Tr \left[ (\sqrt{\Sigma_t dW_t Q + Q'(dW_t)'\sqrt{\Sigma_t}}) DS_t^j \right] + e_j \sqrt{\Sigma_t} d\tilde{Z}_t \frac{\partial S_t^j}{\partial v^j} \tag{51}
\]

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It follows that
\[
<V_t^j, S_t^j> = \left( \text{Tr} \left[ (\sqrt{\Sigma_t} d\tilde{W}_t Q + Q'(d\bar{W}_t)' \sqrt{\Sigma_t}) DS_t^j \right] \right.
\]
\[
\left. + e_j \Sigma_t d\tilde{Z}_t \frac{\partial S_t^j}{\partial v^j} \right) \cdot V_t^j e_j \sqrt{\Sigma_t} d\tilde{Z}_t
\]
\[
= 2\text{Tr} \left[ \sqrt{\Sigma_t} dW_t Q (DS_t^j) \right] \cdot V_t^j e_j \sqrt{\Sigma_t} d\tilde{Z}_t + V_t^j \Sigma_t^j \frac{\partial S_t^j}{\partial v^j} dt
\]
\[
= 2\text{Tr} \left[ \Sigma_t Q (DS_t^j) R_j \right] \cdot V_t^j + V_t^j \Sigma_t^j \frac{\partial S_t^j}{\partial v^j} dt
\]

Further, the instantaneous co-variation of $V_t^j$ is given by
\[
<V_t^j, V_t^j> = dV_t^j \cdot dV_t^j
\]
\[
= e_j \sqrt{\Sigma_t} d\tilde{Z}_t V_t^j \cdot e_j \sqrt{\Sigma_t} d\tilde{Z}_t V_t^j
\]
\[
= (V_t^j)^2 e_j \sqrt{\Sigma_t} d\tilde{Z}_t d\tilde{Z}_t' \sqrt{\Sigma_t} e_j'
\]
\[
= (V_t^j)^2 \Sigma_t^j dt
\]

Taken together, these results prove Proposition 5.

### 7.6 Proof of Proposition 6

In order to derive the conditional correlation between changes in equity value and equity volatility, we have to compute their instantaneous co-variation. As a preparatory step, recall from Proposition 2 that the instantaneous variance of changes in the equity price is given by
\[
\sigma_t^j = 4 \cdot \text{Tr} \left[ (DS_t^j) \Sigma_t (DS_t^j) Q'Q \right] + \left( \frac{\partial S_t^j}{\partial v^j} \right)^2 \Sigma_t^j
\]
\[
+ 4 \cdot \text{Tr} \left[ \Sigma_t R_j Q (DS_t^j) \right] \frac{\partial S_t^j}{\partial v^j}
\]
\[
(52)
\]

By Ito’s Lemma, we get
\[
d\sigma_t^j = \left( \frac{\partial \sigma_t^j}{\partial t} + \mathcal{L}_{\{v^j, \Sigma\}} \sigma_t^j \right) dt
\]
\[
+ \text{Tr} \left[ (\sqrt{\Sigma_t} d\tilde{W}_t Q + Q'(d\bar{W}_t)' \sqrt{\Sigma_t}) D\sigma_t^j \right]
\]
\[
+ e_j \sqrt{\Sigma_t} d\tilde{Z}_t \frac{\partial \sigma_t^j}{\partial v^j}
\]
\[
(53)
\]
where \( \mathcal{L}_{(v, \Sigma)} \) is the generator of the joint process \((v^j, \Sigma)^{26} \).

The explicit form of the partial derivatives \( \frac{\partial \sigma^j}{\partial v^j} \) and \( D\sigma^j \) are described in the following. Consider \( \frac{\partial \sigma^j}{\partial v^j} \) first. In order to facilitate the calculation, note that the first term of \( \sigma^j \) can be re-written as follows.

\[
\text{Tr} \left[ DS^j_i \Sigma_i DS^j_i Q'Q \right] = \text{Tr} \left[ Q'Q \Sigma_i (DS^j_i)^2 \right]
\]

Taking the derivative of this expression yields

\[
\frac{\partial}{\partial v^j} \text{Tr} \left[ DS^j_i \Sigma_i DS^j_i Q'Q \right] = \text{Tr} \left[ Q'Q \Sigma_i \left( \frac{\partial DS^j_i}{\partial v^j} DS^j_i + DS^j_i \frac{\partial DS^j_i}{\partial v^j} \right) \right] = 2 \cdot \text{Tr} \left[ DS^j_i \Sigma_i \frac{\partial DS^j_i}{\partial v^j} Q'Q \right] \tag{54}
\]

The derivation of the other terms in \( \frac{\partial \sigma^j}{\partial v^j} \) is straightforward and we get

\[
\frac{\partial \sigma^j}{\partial v^j} = 8 \cdot \text{Tr} \left[ (DS^j_i) \Sigma_i \frac{\partial DS^j_i}{\partial v^j} Q'Q \right] + 2 \cdot \frac{\partial S^j_i}{\partial v^j} \cdot \frac{\partial^2 S^j_i}{\partial (v^j)^2} \Sigma_i^{ij} + 4 \cdot \text{Tr} \left[ \Sigma_i R_j Q \frac{\partial DS^j_i}{\partial v^j} \right] \cdot \frac{\partial S^j_i}{\partial v^j} + 4 \cdot \text{Tr} \left[ \Sigma_i R_j Q (DS^j_i)^2 \right] \cdot \frac{\partial^2 S^j_i}{\partial (v^j)^2} \tag{55}
\]

Consider \( D\sigma^j \) next. We show how to compute the \((k, l)^{th}\) element of this matrix derivative. Consider the derivative of the first term in \( \sigma^j \) and recall from above that

\[
\text{Tr} \left[ (DS^j_i) \Sigma_i (DS^j_i) Q'Q \right] = \text{Tr} \left[ Q'Q (DS^j_i)^2 \Sigma_i \right]
\]

\(^{26}\)The explicit form of the generator of the joint process \((v^j, \Sigma)\) is stated explicitly in the proof of proposition 2.
Taking the derivative of this term with respect to $\Sigma^{kl}$ yields

\[
\frac{\partial}{\partial \Sigma^{kl}} \text{Tr} \left[ Q'(DS^j_i)^2 \Sigma_t \right] = \text{Tr} \left[ Q'Q \left( \frac{\partial (DS^j_i)^2}{\partial \Sigma^{kl}} \Sigma_t + (DS^j_i)^2 \frac{\partial \Sigma_t}{\partial \Sigma^{kl}} \right) \right]
\]

\[
= \text{Tr} \left[ Q'Q \Sigma_t \left( \frac{\partial DS^j_i}{\partial \Sigma^{kl}} \cdot DS^j_i + DS^j_i \cdot \frac{\partial DS^j_i}{\partial \Sigma^{kl}} \right) \right]
\]

\[
+ \text{Tr} \left[ Q'Q(\partial DS^j_i)^2 \frac{\partial \Sigma_t}{\partial \Sigma^{kl}} \right] = 2 \cdot \text{Tr} \left[ Q'Q \Sigma_t \frac{\partial DS^j_i}{\partial \Sigma^{kl}} \cdot DS^j_i \right]
\]

\[
= 2 \cdot \text{Tr} \left[ Q'Q \Sigma_t \frac{\partial DS^j_i}{\partial \Sigma^{kl}} \cdot DS^j_i \right] + \text{Tr} \left[ Q'Q(\partial DS^j_i)^2 \frac{\partial \Sigma_t}{\partial \Sigma^{kl}} \right], \quad (56)
\]

The derivative of the second term of $\sigma^j_i$ is given by

\[
\frac{\partial}{\partial \Sigma^{kl}} \left( \left( \frac{\partial S^j_i}{\partial v^j} \right)^2 \Sigma^{jj}_t \right) = 2 \cdot \frac{\partial S^j_i}{\partial v^j} \cdot \frac{\partial^2 S^j_i}{\partial v^j \partial \Sigma^{kl}} \cdot \Sigma^{jj}_t
\]

\[
+ \left( \frac{\partial S^j_i}{\partial v^j} \right)^2 \cdot I \{k=j;l=j\} \quad (57)
\]

which follows from straightforward calculations and by noting that the derivative of $\Sigma^{jj}$ is zero except if $k, l = j$. Finally, note that the third term of $\sigma^j_i$ is given by

\[
\text{Tr} \left[ \Sigma_t R_j Q(DS^j_i) \right] = \text{Tr} \left[ R_j Q(DS^j_i) \Sigma_t \right] \quad (58)
\]

Differentiation of this expression yields

\[
\frac{\partial}{\partial \Sigma^{kl}} \text{Tr} \left[ R_j QDS^j_i \Sigma_t \right] = \text{Tr} \left[ R_j Q \left( \frac{\partial DS^j_i}{\partial \Sigma^{kl}} \Sigma_t + DS^j_i \frac{\partial \Sigma_t}{\partial \Sigma^{kl}} \right) \right] \quad (59)
\]
Taken together, we have
\[
\frac{\partial \sigma_j^t}{\partial \Sigma_{kl}} = 8 \cdot Tr \left[ Q' \Sigma_t \frac{\partial DS_j^t}{\partial \Sigma_{kl}} DS_j^t \right] + 4 \cdot Tr \left[ Q'(DS_j^t)^2 \frac{\partial \Sigma_t}{\partial \Sigma_{kl}} \right] 
\]
\[
+ 2 \cdot \frac{\partial S_j^t}{\partial v^j} \cdot \frac{\partial^2 S_j^t}{\partial v^j \partial \Sigma_{kl}} \cdot \Sigma_{ij} + \left( \frac{\partial S_j^t}{\partial v^j} \right)^2 \cdot \mathbb{1}_{\{k=j, l=j\}} 
\]
\[
+ 4 \cdot Tr \left[ \Sigma_t R_j Q DS_j^t \right] \cdot \frac{\partial S_j^t}{\partial v^j} 
\]
\[
+ 4 \cdot Tr \left[ R_j Q \left( \frac{\partial DS_j^t}{\partial \Sigma_{kl}} \Sigma_t + DS_j^t \frac{\partial \Sigma_t}{\partial \Sigma_{kl}} \right) \right] \cdot \frac{\partial S_j^t}{\partial v^j} \] (60)

Now, consider the instantaneous co-variation between the equity price and equity volatility. Applying an argument similar to Proposition 5, we get
\[
< \sigma_j^t, S_j^t >_t = d\sigma_j^t \cdot dS_j^t 
\]
\[
= \left( Tr \left[ d\Sigma_t (D\sigma_j^t) \right] + e_j \sqrt{\Sigma_t} d\tilde{Z}_t \frac{\partial \sigma_j^t}{\partial v^j} \right) \cdot \left( Tr \left[ d\Sigma_t (DS_j^t) \right] + e_j \sqrt{\Sigma_t} d\tilde{Z}_t \frac{\partial S_j^t}{\partial v^j} \right) \] (61)

Further computations give
\[
Tr \left[ d\Sigma_t D\sigma_j^t \right] \cdot Tr \left[ d\Sigma_t (DS_j^t) \right] = 4Tr \left[ D\sigma_j^t \Sigma_t DS_j^t Q'Q \right] dt \] (62)

Moreover,
\[
e_j \sqrt{\Sigma_t} d\tilde{Z}_t \frac{\partial \sigma_j^t}{\partial v^j} \cdot e_j \sqrt{\Sigma_t} d\tilde{Z}_t \frac{\partial S_j^t}{\partial v^j} = \Sigma_{ij} \frac{\partial \sigma_j^t}{\partial v^j} \frac{\partial S_j^t}{\partial v^j} dt. \] (63)

Finally, we compute
\[
Tr \left[ d\Sigma_t (DS_j^t) \right] \cdot e_j \sqrt{\Sigma_t} d\tilde{Z}_t \frac{\partial \sigma_j^t}{\partial v^j} = 2Tr \left[ \Sigma_t R_j Q (DS_j^t) \right] \frac{\partial \sigma_j^t}{\partial v^j} dt \] (64)

and
\[
Tr \left[ d\Sigma_t (DS_j^t) \right] \cdot e_j \sqrt{\Sigma_t} d\tilde{Z}_t \frac{\partial S_j^t}{\partial v^j} = 2Tr \left[ \Sigma_t R_j Q (D\sigma_j^t) \right] \frac{\partial S_j^t}{\partial v^j} dt. \] (65)

Taken together, the co-variation of changes in equity value and equity volatility is given by
\[
< \sigma_j^t, S_j^t >_t = 4Tr \left[ D\sigma_j^t \Sigma_t (DS_j^t) Q'Q \right] dt + \Sigma_{ij} \frac{\partial \sigma_j^t}{\partial v^j} \frac{\partial S_j^t}{\partial v^j} dt 
\]
\[
+ 2Tr \left[ \Sigma_t R_j Q (DS_j^t) \right] \frac{\partial \sigma_j^t}{\partial v^j} dt + 2Tr \left[ \Sigma_t R_j Q (D\sigma_j^t) \right] \frac{\partial S_j^t}{\partial v^j} dt. \]
The quadratic variation of changes in equity value is stated in the proof of Proposition 2. Analogously, we can get the instantaneous quadratic variation of the change in equity volatility as

\[
< \sigma^j_t, \sigma^j_t >_t = 4T \text{tr} \left[ D \sigma^j_t \Sigma_t D \sigma^j_t Q'Q \right] dt + \frac{\partial \sigma^j_t}{\partial v} \Sigma_{ij} dt \\
+ 4T \text{tr} \left[ \Sigma_t R_j Q D \sigma^j_t \right] \frac{\partial \sigma^j_t}{\partial v} dt
\]

(66)

It is easily seen that all terms of the instantaneous correlation between changes in the equity price and equity volatility are dependent on the levels of \( v^j \) and \( \Sigma \) and that these terms do not cancel out making the correlation stochastic thus proving the statement in the proposition.

8 Appendix C: Additional Material

8.1 The Gaussian Model for multi-issuer credit derivatives

Before proceeding to analyze the three-asset basket, we will briefly review the concept of implied correlation and its significance in the present setting. The notion of implied correlation departs from so-called one-factor models of corporate default and assumes that asset returns are driven by two sources of uncertainty:

- a firm-specific factor proprietary to each single firm and independent of the firm-specific factors of all other firms
- a common factor influencing the asset returns of all firms

This definition implies that asset returns (and thereby defaults) may be correlated only through the common factor, while conditional on the common factor asset returns are independent.

To focus ideas, consider the model

\[
y_{j,t,T} = \sqrt{\psi} m + \sqrt{1 - \psi} \epsilon_j
\]

(67)

where \( y_{j,t,T} \) denotes the standardized log asset return within the time interval \([t, T]\), \( \epsilon_j \) is an iid standard normal random variable denoting the firm-specific factor of firm \( j \) for \( j = 1, ..., n \), \( m \) is a standard normal random variable
with density $\phi$, representing the common factor in asset returns, and $\psi$ is a correlation parameter.\(^{27}\)

In order to use (67) for the modeling of corporate defaults, assume that the marginal risk-neutral default probability of firm $j$ within the time interval $[t, T]$ is given by $p_j$. In practice, $p_i$ would in general be derived from quoted single name CDS spreads. There exists a critical value $y_{j,t,T}^c$ of $y_{j,t,T}$ such that

$$p_j = \Phi(y_{j,t,T}^c)$$

(68)

where $\Phi(.)$ denotes the cumulative distribution function of the standard normal distribution. Solving this equation for $y_{j,t,T}^c$ we get

$$y_{j,t,T}^c = \Phi^{-1}(p_j)$$

(69)

Given a realization of the common factor $m$ and the threshold $y_{j,t,T}^c$ we can now calculate the default probability of firm $j$ conditional on $m$ as

$$\mathbb{P}(y_{j,t,T} \leq y_{j,t,T}^c | m) = \Phi \left( \frac{y_{j,t,T}^c - \sqrt{\psi} m}{\sqrt{1 - \psi}} \right) = \Phi \left( \frac{\Phi^{-1}(p_j) - \sqrt{\psi} m}{\sqrt{1 - \psi}} \right)$$

(70)

The unconditional default probability $p_j$ is recovered by calculating the expectation of (70) with respect to $m$.

Having defined the main ingredients of a simple Gaussian one-factor model, we can now easily derive the counterparts of expressions (26) and (28) in the present setup. As noted above, the marginal default probabilities resp. CDS for each firm spreads are not affected by by the choice of the correlation model. However, we get for the joint probability of default of $k$ assets

$$\mathbb{P}(y_{i,t,T} < y_{i,t,T}^c, \ldots, y_{k,t,T} < y_{k,t,T}^c) = \int_{-\infty}^{\infty} \prod_{i=1}^{k} \Phi \left( \frac{\Phi^{-1}(p_i) - \sqrt{\psi} m}{\sqrt{1 - \psi}} \right) \phi(m) dm$$

(71)

Just as implied volatilities in the case of equity options, implied correlation is defined as the level of the correlation parameter $\psi$ that sets the difference between the observed market price of a multi-asset credit derivative and its theoretical price using the one-factor model described above equal to zero.

\(^{27}\)For calculating implied correlations, we follow the standard practice of assuming constant inter-asset correlations across firms. It is easily seen from equation (67) that the pairwise correlation between any two different asset returns $i$ and $j$ is $\rho_{ij} = \psi$. 43
8.2 Computation of base correlation for the three-asset basket

Base correlation is defined as the implied correlation of some fictitious equity tranches. These fictitious equity tranches are constructed by replacing with zero the lower attachment point of each tranche and by leaving unchanged its upper attachment point. By definition, the base and compound correlations of equity tranches are identical.

The fictitious equity tranches can be synthesized using combinations of existing tranches. Consider, for instance, the mezzanine tranche of the three-asset basket. We can define a corresponding fictitious equity tranche, having identical upper attachment point, by adding to the payoff of the mezzanine tranche the one of the equity tranche:

\[
p_{\text{em}}^T = p_e^T + p_m^T = I\{V_1^T < D_1\} + I\{V_2^T < D_2\} + I\{V_3^T < D_3\} - I\{V_1^T < D_1\}I\{V_2^T < D_2\}I\{V_3^T < D_3\}.
\]

The base correlation is the correlation parameter that equates the price of the payoff (72) implied by the Gaussian single-factor model to the sum of the observed market prices of the equity and mezzanine tranches. Similarly, for the senior tranche, we can define a corresponding fictitious equity tranche by adding the payoffs of the equity, the mezzanine and the senior tranches in our basket structure. This payoff is given by:

\[
p_{\text{es}}^T = p_e^T + p_m^T + p_s^T = I\{V_1^T < D_1\} + I\{V_2^T < D_2\} + I\{V_3^T < D_3\}.
\]

Therefore, the price of this fictitious tranche is just the sum of the marginal default probabilities of the three firms. Since these marginal probabilities are used as input parameters for the computation of the implied correlation, we cannot solve in this case for a correlation parameter that matches the market price of the payoff \(p_{\text{es}}^T\) to the price implies by the single-factor Gaussian model.
References


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[LI99] Li, D., 1999, The Valuation of Basket Credit Derivatives, CreditMetrics Monitor, April, p. 3450.


