Bubbles and multiplicity of equilibria under portfolio constraints

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Abstract

This article shows that, as long as agents are required to maintain positive wealth, the presence of portfolio constraints may give rise to asset pricing bubbles in equilibrium even if there are unconstrained agents in the economy who can benefit from the induced arbitrage opportunity. Furthermore, it is shown that the presence of bubbles in the aggregate price system can lead to both multiplicity and real indeterminacy of equilibrium. The general results are illustrated by two explicitly solved examples where seemingly innocuous portfolio constraints make bubbles a necessary condition for the existence of an equilibrium.

Keywords: asset pricing bubbles, general equilibrium, portfolio constraints, limited participation, real indeterminacy.

JEL Classification. D51, D52, D53, G11, G12.
1 Introduction

The absence of exploitable arbitrage opportunities, defined as the possibility of simultaneously buying and selling the same asset at two different prices, is the cornerstone of modern finance theory. Yet, violations of this basic paradigm are frequently observed. In particular, over the past twenty years numerous deviations from the fundamental value implied by no-arbitrage restrictions, so-called asset pricing bubbles, have been detected in various financial markets across the world.¹

Despite this mounting evidence, neoclassical financial economics has little to say about such phenomena mainly because the presence of bubbles on positive supply securities is inconsistent with the existence of equilibrium in the frictionless, representative agent framework which is the workhorse of modern asset pricing theory.² As a result, most of the theoretical studies of asset pricing bubbles fall into the so-called behavioral strand of the finance literature, see Harrison and Kreps (1978); Scheinkman and Xiong (2003); DeLong, Schleifer, Summers, and Waldman (1990) and Abreu and Brunnermeier (2003) among others. The common feature of these articles is that they study the occurrence of bubbles in partial equilibrium settings that allow irrational agents to have a significant impact on prices.³ There is some literature where arbitrage opportunities arise endogenously in equilibrium due the presence of portfolio constraints, see Gromb and Vayanos (2002) and Basak and Croitoru (2000; 2006), but in these models the ability of arbitrageurs to benefit from, and thereby eliminate, price discrepancies is limited by exogenous portfolio and/or solvency constraints. If these constraints are lifted, then the arbitrageurs can scale their position to an arbitrary size and,

¹Famous examples include the internet bubble (see Ofek and Richardson (2003) and Lamont and Thaler (2003b)) and the simultaneous trading of shares from Royal Dutch and Shell (see Rosenthal and Young (1990) and Lamont and Thaler (2003a)).
²In the representative agent model, the existence of an equilibrium, with or without complete markets, does not rule out the possibility of bubbles on zero net supply securities such as options or futures contracts, see Loewenstein and Willard (2000a).
³Loewenstein and Willard (2006) show that the conclusion of these models generally do not persist once one imposes market clearing on all traded assets.
as a result, the presence of mispricing becomes again inconsistent with the existence of equilibrium. The main contribution of this paper is to show that this need not be the case in general: the presence of portfolio constraints can generate non trivial equilibrium pricing bubbles even if the economy includes unconstrained arbitrageurs.

I consider a continuous-time, pure exchange economy with a single consumption good, heterogenous agents and portfolio constraints. To keep things as simple as possible, I assume that there are only two classes of agents: unconstrained agents who are free to choose the composition of their portfolio as they see fit; and constrained agents who have logarithmic utility and are subject to convex portfolio constraints. In this setting, I show that the presence of bubbles in equilibrium prices can be assessed by studying the properties of a single economic state variable which is the ratio of the agents’ marginal utility of consumption. This state variable can be seen as the weight of the constrained agent in the utility function of the representative agent and is thus referred to as the weighting process. The optimality of the agents’ decisions imply that this process has no drift and, hence, behaves like a martingale on time intervals of infinitesimal length. This does not mean, however, that the process is a true martingale over the horizon of the economy because additional integrability conditions are needed for a driftless process to be a true martingale. In fact, the main result of this paper shows that the weighting process is a true martingale.

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4The assumption of a single consumption good is meant to simplify the exposition but the results of this paper still hold if there is more than one consumption good. In particular, they can be applied to assess the presence of bubbles in the models of Pavlova and Rigobon (2007), Soumare and Wang (2006) and Schornick (2007).

5As shown by Cvitanić and Karatzas (1992), the assumption of logarithmic utility is necessary to obtain a simple characterization of optimality under portfolio constraints. A similar assumption is used in Detemple and Murthy (1997); Basak and Cuoco (1998); Basak and Croitoru (2000; 2006); Gallmeyer and Hollifield (2008); Wu (2006); Pavlova and Rigobon (2007) and Schornick (2007) among others.

6Cuoco and He (1994) were the first to use this state variable in order to characterize equilibria in dynamic economies with incomplete markets. Since then this construction has been used by numerous authors including Basak and Cuoco (1998); Basak and Croitoru (2000); Gallmeyer and Hollifield (2008) and Pavlova and Rigobon (2007).
if and only if there are no bubbles in the equilibrium prices. While this may sound like a technical subtlety, the results of this paper show that the difference between local and true martingales should not be overlooked as it can have severe consequences for the computation of equilibrium prices.\footnote{Other papers where this difference matters include Sin (1998) who studies stochastic volatility models, Fernholz, Karatzas, and Kardaras (2005) who study the modeling of relative arbitrages in equity markets and Heston, Loewenstein, and Willard (2007) who study the impact of bubbles on the pricing of derivatives in partial equilibrium.} In particular, I illustrate the general result by presenting some examples of economies with seemingly innocuous portfolio constraints in which bubbles, and hence the failure of the weighting process to be a true martingale, are a necessary condition for the existence of equilibrium.

The first example I consider is a slight generalization of the restricted participation model of Basak and Cuoco (1998) where there is a single stock, agents have logarithmic utility and the constrained agent can neither short the risky asset nor invest more than a fixed fraction of his wealth into it. For an equilibrium to exist in this model, the unconstrained agent must find it optimal to hold a leveraged position in the risky asset. As a result, the interest rate must decrease and the risk premium must increase compared to an otherwise similar unconstrained economy. These local effects of the portfolio constraint go in the right direction, but they are not sufficient to reach an equilibrium. Indeed, I show that for this to be the case two conditions must be satisfied. First, the stock and the riskless asset must each include a non trivial bubble component. Second, the bubble on the riskless must be larger in relative terms than that on stock. The intuition behind this result is a follows. Since exploiting the bubble on one asset means going long in the other, the agent cannot benefit from both bubbles at the same time. Taking into account the nonnegativity constraint on wealth, the agent should exploit the arbitrage opportunity on the riskless asset because it requires less collateral per unit of initial profit. The fact that the stock price also includes a bubble increases its collateral value and, hence, allows
the agent to increase the size of his short position in the riskless asset to the level required by market clearing.

A convenient feature of this limited participation example is that the transition density of the associated weighting process can be computed in closed form. This allows me to conduct an extensive analysis of its behavior. If this process was a true martingale, then its expectation would be constant through time. In contrast, I show that, depending on its initial value, the weighting process can be expected to decrease by as much as 75 percent over a ten year horizon. Since agents have homogenous logarithmic preferences this property translates into a decrease of the constrained agent’s expected consumption share. This property is intuitive because constrained agents should suffer more than unconstrained ones from the lack of perfect risk sharing. However, what is unique to the presence of bubbles is the speed of this decrease. Indeed, I show that for reasonable parameter configurations the constrained agent’s consumption share should be expected to decrease by 50 percent over a ten year horizon. This figure should be compared to those found in the recent literature studying the impact of irrational beliefs on equilibrium asset prices, see Kogan, Ross, Wang, and Westerfield (2006); Berrada (2008) and Dumas, Kurshev, and Uppal (2008). These models show that the consumption share of irrational agents tends to decrease over time, but this decline is much slower than that implied by the presence of bubbles. For example, Kogan et al. (2006) and Berrada (2008) find that in their respective models it takes approximately eighty years for the consumption share of the constrained agent to be divided by two.

When there is a single risky asset in the economy, as is the case in the limited participation example, the equilibrium price of that security is given by the sum the agents’ wealth. In that case, the bubble, if it is present, can be computed as the difference between the price of the stock and the cost of replicating its cash flows for the unconstrained agent. When there are multiple risky assets in the economy, the situation is more complicated. The
total value of the economy, i.e. the equilibrium value of the market portfolio, is still given by the sum of the agents’ wealth but it is not clear a priori how this aggregate value should be split among the individual stocks. If there are no bubbles, then the existence of an equilibrium is sufficient to guarantee that the unconstrained agent’s marginal utility can be used as a state price density to compute the individual prices. On the contrary, when bubbles are present such a pricing kernel fails to exist and I show that, in this case, there are infinitely many ways to split the aggregate value of the economy among the risky assets. In other words, the presence of bubbles may give rise to multiplicity and real indeterminacy of equilibrium. This implication of asset pricing bubbles is, to the best of my knowledge, novel to this paper.

While the role of portfolio constraints in expanding the set of equilibria has been recently pointed out by Basak, Cass, Licari, and Pavlova (2007), it is important to note that the nature of the multiplicity in their model is different from that which occurs in mine. In the model of Basak et al. (2007) there are multiple goods and the multiplicity of equilibria arises from the fact that the agents can partially alleviate their portfolio constraints by trading in the spot market for goods. Furthermore, none of the equilibria identified by Basak et al. (2007) include asset pricing bubbles.

To illustrate the indeterminacies that can arise due to the presence of portfolio constraints, I consider an economy with two risky assets, two agents with logarithmic preferences and assume that one of the agents faces a risk constraint which limits the volatility of his wealth. As in the limited participation example, this constraint prevents the agent from investing as much as desired in the risky assets and, thus, forces the unconstrained agent to hold a leveraged position in the risky assets. In such a setting, I show that the existence of a bubble on both the market portfolio and the riskless asset is a necessary condition for the existence of equilibrium. Relying on this result, I demonstrate that there exist infinitely many equilibria which differ in the repartition of the aggregate bubble among the two stocks. In order
to gain some insight on the differences between these multiple equilibria I conduct a comparative static analysis of key equilibrium quantities such as stock volatilities, correlations and equity premia. My results suggest that variations across the set of equilibria can be significant. For example, I show that for reasonable parameter configurations the equity premium on a given risky asset ranges from 6 to 12 percent per year while the correlation between the two risky assets ranges from $-0.5$ to $0.5$.

The rest of this paper is organized as follows. In Section 2, I present my assumptions about the economy, the traded assets and the agents and define the notion of equilibrium that I use in this paper. In Section 3, I solve the agents’ individual optimization program and provide a general characterization of equilibrium. In Section 4, I state necessary and sufficient conditions for the existence of bubbles in equilibrium and show how such bubbles can give rise to multiplicity of equilibrium. Sections 5 and 6 contain the two examples. Finally, Appendix A presents a detailed study of the properties of the weighting process that arises in the examples and Appendix B contains all the proofs.

2 The economy

A. Information structure

I consider a continuous time economy on the finite time span $[0, T]$. The uncertainty in the economy is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ supporting a $n$–dimensional standard Brownian motion which I denote by $B$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ is the usual augmentation of the filtration generated by the Brownian motion and I let $\mathcal{F} = \mathcal{F}_T$ so that the true state of nature is completely determined by the path of the Brownian motion up to the terminal time of the model.

All processes are assumed to be adapted with respect to $\mathbb{F}$, and all statement involving random quantities are understood to hold either almost
surely or almost everywhere depending on the context.

B. Securities market

There is a single consumption good (the numéraire) in units of which all quantities are expressed.

The financial market consists of \( n + 1 \) assets: a locally riskless bond in zero net supply, and \( n \geq 1 \) risky assets in positive supply of one unit each. The price of the riskless asset evolves according to

\[
S^0_t = 1 + \int_0^t S^0_s r_s ds
\]

for some instantaneous interest rate process \( r \in \mathbb{R} \) which is to be determined in equilibrium. The \( i \)-th risky asset is a claim to a strictly positive dividend process of the form

\[
e^i_t = e^i_0 + \int_0^t e^i_s a^i_s ds + \int_0^t e^i_s (v^i_s)^\top dB_s
\]

for some exogenously given drift and volatility processes \( (a^i, v^i) \in \mathbb{R} \times \mathbb{R}^n \) such that the above integrals are well defined. The price process of the risky assets is denoted by \( S \) and evolves according to

\[
S^i_t = S^i_0 + \int_0^t S^i_s \left( \mu^i_s ds + (\sigma^i_s)^\top dB_s \right) - \int_0^t e^i_s ds
\]

for some initial value \( S^i_0 \in \mathbb{R}_+ \) and some drift and volatility processes \( (\mu^i, \sigma^i) \in \mathbb{R} \times \mathbb{R}^n \) which are to be determined endogenously in equilibrium. In what follows I denote by

\[
e_t \equiv \sum_{i=1}^n e^i_t = e_0 + \int_0^t e_s a_s ds + \int_0^t e_s v_s^\top dB_s
\]

the aggregate dividend process, by \( \mu \) the drift of the vector \( S \) and by \( \sigma \) the matrix obtained by stacking up the individual volatilities.
C. Agents

The economy is populated by two agents who have homogenous beliefs about the state of the economy.\(^8\) The preferences of agent \(a \in \{1, 2\}\) over strictly positive consumption plans are represented by

\[
U_a(c) \equiv E\left[ \int_0^T u_a(c_s)ds \right].
\]

for some utility function \(u_a : (0, \infty) \to \mathbb{R}\). I let \(u_2(c) = \log c\) and assume that \(u_1\) satisfies textbook regularity, monotonicity and concavity assumptions as well as the Inada conditions.\(^9\) As a result, \(u_1c\) admits a strictly decreasing inverse \(I_1\) which maps the positive real line onto itself. To guarantee that the first agent’s optimization problem is well defined I further assume that \(u_1\) satisfies a technical condition which is stated in Appendix B.

Agent \(a\) is initially endowed with \(\beta_a \in \mathbb{R}\) units of the riskless asset and a positive fraction \(\alpha_a \leq 1\) of the market portfolio.\(^10\) Initial short positions in the riskless asset are allowed as long as the initial wealth

\[
w_a = \beta_a + \sum_{i=1}^{n} \alpha_a S_{0}^i = \beta_a + \alpha_a S_{0}^{*}
\]

is strictly positive when computed at the equilibrium prices. I denote by \(\alpha \equiv \alpha_2\) the fraction of the market portfolio initially held by the second agent and let \(\alpha_1 = 1 - \alpha\) and \(-\beta_1 = \beta_2 \equiv \beta\) so that markets clear.

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\(^8\)The assumption of homogenous beliefs is made for simplicity of exposition and does restrict the generality of the model. Under appropriate modifications, all the results in the paper can be shown to hold even if the agents have heterogenous beliefs.

\(^9\)Since the economy has a finite horizon, the assumption that agents do not discount utility is without loss of generality. The equilibrium for the case where agents have a common strictly positive discount rate can be obtained from that of the undiscounted case by simply adding the discount rate to the interest rate.

\(^10\)The adoption of a collinear endowment structure is meant to simplify the presentation of the results. It is by no means a restriction in the setting of this paper and can be relaxed at the cost of more complicated notations. See Remark 3 in Section 6 for a discussion of this assumption in the setting of an economy with multiple stocks.
D. Trading strategies and feasible plans

Trading takes place continuously on the interval $[0, T]$ and there are no market frictions such as transaction costs or taxes.

An admissible trading strategy is a process $(\phi, \pi) \in \mathbb{R}^{1+n}$ satisfying

$$\int_0^T \|\sigma_t^\top \pi_t\|^2 dt + \int_0^T |\phi_t r_t + \pi_t^\top \mu_t| dt < \infty$$

and $\phi + \pi^\top 1 \geq 0$ where $1 \in \mathbb{R}^n$ is a vector of ones. The scalar process $\phi$ represents the amounts invested in the riskless asset while the vector $\pi$ represents the amounts invested in the risky assets.

The first agent is unconstrained in his portfolio choice whereas the second one is constrained. More precisely, I assume that the trading strategy of the second agent must belong to the set

$$C \equiv \{ (\phi, \pi) : \pi_t \in W_t C_t \text{ for all } t \in [0, T] \}$$

where $W_t = \phi_t + \pi_t^\top 1$ denotes the agent’s wealth and $(C_t)$ is a family of closed convex sets such that $0 \in C_t$ for all $t \in [0, T]$. As is easily seen this amounts to a constraint on the proportion of wealth invested in each of the risky assets. A wide variety of portfolio constraints can be modeled in this way, see Cvitanić and Karatzas (1992) for various examples.

A consumption plan $c$ is feasible for Agent 1 if there exists an admissible trading strategy $(\phi, \pi)$ such that the associated wealth process

$$W_t = W_t(\phi, \pi) \equiv \phi_t + \pi_t^\top 1 \geq 0$$

satisfies the dynamic budget constraint

$$W_t = w_1 + \int_0^t \left( \phi_s r_s + \pi_s^\top \mu_s - c_s \right) ds + \int_0^t \pi_s^\top \sigma_s dB_s.$$ 

Feasible plans for agent 2 are defined similarly with the additional constraint that the admissible trading strategy which finances the consumption plan
belongs to the constraint set $C$. In what follows, I denote by $\mathcal{X}_a$ the set of consumption plans which are feasible for agent $a$.

E. Equilibrium

Let $\mathcal{E} \equiv (\Omega, \mathcal{F}, \mathbb{F}, \{e^t\}, \{u_a, \alpha_a, \beta_a, P_a, X_a\})$ denote the primitives for the above continuous time economy. The concept of equilibrium that I use throughout this paper is similar to that of equilibrium of plans, prices and expectations which was introduced by Radner (1972):

**Definition 1.** An equilibrium for the continuous time economy $\mathcal{E}$ is a pair of security price processes $(D, S)$ and a set of consumption plans and trading strategies $\{c_a, (\phi_a, \pi_a)\}$ such that:

1. The consumption plan $c_a$ maximizes $U_a$ over the feasible set $\mathcal{X}_a$ and is financed by the admissible trading strategy $(\phi_a, \pi_a)$.

2. The securities and goods markets clear at all times in the sense that $\phi_1 + \phi_2 = 0$, $\pi_1 + \pi_2 = S$ and $c_1 + c_2 = e$.

In the model there are as many risky assets as there are sources of risks. As a result, one naturally expects that none of the stocks are redundant in equilibrium and, hence, that markets are dynamically complete for the unconstrained agent. Unfortunately, and as shown by Cass and Pavlova (2004) and Berrada, Hugonnier, and Rindisbacher (2007), this is not the case in general even if the dividends of the risky assets are assumed to be linearly independent processes. In order to facilitate the presentation of my results, and to simplify the definition of an asset pricing bubble in the next section, I will restrict the analysis to the class of equilibria in which the volatility matrix of the stocks has full rank at all times. Since none of the stocks are redundant in any such equilibrium, I will refer this class as that of non redundant equilibria.
3 Optimality and equilibrium

In this section I gather some results about individual optimality and provide a characterization of non redundant equilibria. These results will serve as a basis for the discussion of asset pricing bubbles in the next sections.

A. Individual optimality

Let \((S^0, S)\) denote the asset prices in a given non redundant equilibrium and assume that there are no arbitrage opportunities for otherwise the market could not be in equilibrium. As is well known, this assumption implies that there exists a \(n\)-dimensional process \(\theta\) such that

\[
\mu_t - r_t = (\sigma_t^{\top} \theta_t) \quad \text{and} \quad \int_0^T \|\theta_t\|^2 dt < \infty.
\]

This process is referred to as the risk premium and is uniquely defined since the volatility matrix \(\sigma\) has full rank. Now consider the state price density process defined by

\[
\xi_{1t} = \frac{1}{S_t^0} \exp \left[ -\int_0^t \theta_s^\top dB_s - \frac{1}{2} \int_0^t \|\theta_s\|^2 ds \right].
\] (1)

The following well-known proposition shows that the process \(\xi_1\) serves as the driving economic state variable in the unconstrained agent’s consumption and portfolio choice problem.

**Proposition 1.** The optimal consumption plan and trading strategy of the unconstrained agent are given by

\[
\begin{align*}
c_{1t} &= I_1(y_{1t} \xi_{1t}) \\
\pi_{1t} &= (\sigma_t^{-1})^{\top} (W_{1t} \theta_t + \xi_{1t}^{-1} \theta_{1t})
\end{align*}
\]

where the nonnegative process

\[
W_{1t} = \frac{1}{\xi_{1t}} E \left[ \int_t^T \xi_{1s} c_{1s} ds \left| F_t \right. \right] = \frac{1}{\xi_{1t}} \left[ M_{1t} - \int_0^t \xi_{1s} c_{1s} ds \right].
\] (2)
represents the agent’s wealth along the optimal path, \( \vartheta_1 \) is the integrand in the representation of the martingale \( M_1 \) as a stochastic integral and the strictly positive constant \( y_1 \) is chosen in such a way that \( W_{10} = w_1 \).

When the agent is subject to portfolio constraints, the situation is more complicated since \( \xi_1 \) no longer identifies the unique arbitrage free state price density. To circumvent this difficulty, I use the duality approach of Cvitanić and Karatzas (1992) which allows to characterize the solution of the constrained problem as if the agent faced the unique state price density of a fictitious unconstrained economy with complete financial markets.

In order to present the result, let \( \delta_t : \mathbb{R}^n \to \mathbb{R} \cup \{ \infty \} \) be the support function of the set \( C_t \), that is the convex function defined by

\[
\delta_t(b) \equiv \sup_{c \in C_t} (-c^\top b),
\]

and denote by \( B_t \) the set of points where this function is finite. The following proposition provides a complete solution to the constrained agent’s portfolio and consumption choice problem.

**Proposition 2.** The optimal consumption plan and trading strategy of the constrained agent are given by

\[
c_{2t} = 1/(y_2 \xi_{2t}) = W_{2t}/(T-t) \tag{3}
\]

\[
\pi_{2t} = (\sigma_t^{-1})^\top W_{2t} \theta_{2t}.
\]

In the above expressions, the implicit risk premium \( \theta_2 \) faced by the agent is defined by the relation

\[
\sigma_t(\theta_{2t} - \theta_t) = b_t \equiv \arg \min_{b \in B_t} \left\{ \delta_t(b) + \frac{1}{2} \| \theta_t + \sigma_t^{-1} b \|^2 \right\}, \tag{4}
\]

the nonnegative process

\[
\xi_{2t} \equiv \frac{1}{S_t} \exp \left[ -\int_0^t \delta_s(b_s) ds - \int_0^t \theta_{2s}^\top dB_s - \frac{1}{2} \int_0^t \| \theta_{2s} \|^2 ds \right]
\]
is the implicit state price density faced by the agent, and the nonnegative process $W_2$ is the agent’s wealth along the optimal path.

The last result in this section results establishes a useful property of the ratio of the agents’ marginal utilities of consumption. This property follows from the assumption of logarithmic utility for the constrained agent and will be crucial for our characterization of bubbles in the next section.

**Proposition 3.** Let $c_a$ denote the optimal consumption of agent $a$. The strictly positive process $\lambda = u_{1c}(c_1)/u_{2c}(c_2)$ is a local martingale.\(^\text{\footnote{A process $X$ is a martingale on the time interval $[0,T]$ if it is integrable and such that $E[X_T|\mathcal{F}_t] = X_t$ for all $t \in [0,T]$. A process $Y$ is a local martingale if there exists an increasing sequence of stopping times $\tau_n \uparrow T$ such that the stopped process $Y_{\cdot \wedge \tau_n}$ is a martingale for each $n$. A martingale is always a local martingale but the converse is not true in general, see Karatzas and Shreve (1998); Elworthy, Li, and Yor (1999) and the appendix for necessary and sufficient conditions.}}\)

**B. Characterization of equilibrium**

Because one of the agents faces portfolio constraints, the usual construction of a representative agent as a linear combination of the individual utility functions with constant weights is impossible. Nevertheless, the aggregation of individual preferences into a representative agent is still possible if one allows for stochastic weights in the definition of the representative utility function. This construction, which is originally due to Cuoco and He (1994), allows to easily account for the market clearing conditions and reduces the search for an equilibrium to the specification of the weights.\(^\text{\footnote{One should be cautious with the interpretation of this construction. The utility function of the representative agent is an allocational device which allows to easily account for the goods market clearing condition but it is not always the case that an equilibrium of the two agents economy is a no-trade equilibrium for the representative agent endowed with the sum of the securities. The reason why these may differ is precisely that the equilibrium price system of the two agents economy may include bubbles.}}\)

Before stating the main characterization result, I provide a heuristic discussion of the different steps which lead to the equilibrium. Consider the representative agent with utility function given by

$$u(c, \lambda_t) \equiv \max_{c_1 + c_2 = c} u_1(c_1) + \lambda_t u_2(c_2)$$
where $\lambda$ is a strictly positive process which represents the time varying weight of the constrained agent. Comparing the first order condition of the representative agent’s problem to the result of Proposition 1 shows that, in equilibrium, the process of normalized marginal rate of substitution

$$\xi_t = \frac{u_c(e_t, \lambda_t)}{u_c(e_0, \lambda_0)}$$

must identify the unconstrained agent’s state price density. On the other hand, the agent’s optimal consumption plans must solve the representative agent’s utility maximization problem and it follows that

$$c_{1t} = I_1(y_1\xi_{1t}) = I_1(u_c(e_t, \lambda_t)), \quad c_{2t} = \frac{\lambda_t}{u_c(e_t, \lambda_t)}.$$

Plugging this back into the results of the previous section shows that the weighting process is given by the ratio of marginal utilities:

$$\lambda_t = \frac{u_{1c}(c_{1t})}{u_{2c}(c_{2t})} = c_{2t}u_{1c}(c_{1t})$$

Applying Itô’s lemma to the unconstrained state price density process and comparing the result with equation (1) allows to pin down the interest rate and the risk premium. Finally, accounting for the budget constraint of one of the agents and putting everything back together yields:

**Proposition 4.** In equilibrium, the state price density processes and optimal consumption plans of the two agents are given by

$$\xi_{1t} = \frac{u_c(e_t, \lambda_t)}{u_c(e_0, \lambda_0)}, \quad c_{1t} = I_1(u_c(e_t, \lambda_t)), \quad (5)$$

$$\xi_{2t} = \frac{\lambda_0}{\lambda_t}\xi_{1t}, \quad c_{2t} = \frac{\lambda_t}{u_c(e_t, \lambda_t)}, \quad (6)$$

where the strictly positive constant $\lambda_0$ is chosen to satisfy

$$\frac{\lambda_0 T}{u_c(e_0, \lambda_0)} = \beta + \alpha E \left[ \int_0^T \frac{u_c(e_t, \lambda_t)e_t + \lambda_0 - \lambda_t}{u_c(e_0, \lambda_0)} dt \right]. \quad (7)$$
Furthermore, the equilibrium risk premium and the equilibrium interest rate are given by

\[ \theta_t = R_t (v_t - s_2 t \Gamma_t), \]

\[ r_t = a_t R_t + (P_t - R_t) (s_2 t \Gamma_t)^\top \theta_t + \frac{P_t R_t}{2} (\|s_2 t \Gamma_t\|^2 - \|v_t\|^2), \]

where \( \Gamma_t \) denote the volatility of the logarithm of the weighting process, \( s_2 t \) is the consumption share of the constrained agent and \( R_t \) denote, respectively, the relative risk aversion and the relative prudence of the representative agent’s utility function.

The structure revealed by the above proposition is typical of equilibrium models with portfolio constraints, see Detemple and Murthy (1997), Cuoco (1997), Basak and Cuoco (1998) and Shapiro (2002) for various examples. In particular, the equilibrium excess returns are given by

\[ \mu^i_t - r_t = R_t \left[ \text{cov} \left( \frac{dS^i_t}{S^i_t} , \frac{d e_t}{e_t} \right) - s_2 t \times \text{cov} \left( \frac{dS^i_t}{S^i_t} , \frac{d \lambda_t}{\lambda_t} \right) \right] \]

and satisfy a two factor capital asset pricing model where the weighting process plays the role of the second factor. This process encapsulates the differences in wealth across agents and also accounts for the presence of portfolio constraints in the economy.

If there are no portfolio constraints, then the weighting process \( \lambda \) is constant and we recover the usual consumption based capital asset pricing model. In this case, the stock prices are given by the familiar formula

\[ S^i_t = E \left[ \int_t^T \frac{u_c(e_s, \lambda_s)}{u_c(e_t, \lambda_t)} e^s d\lambda | \mathcal{F}_t \right] \]

and the existence of a non redundant equilibrium amounts to the existence of a strictly positive constant \( \lambda_0 \) such that equation (7) holds and the volatility
of the stock price process is invertible. In the general case where portfolio constraints are present, the situation is more complicated because, as shown by the following corollary, the volatility of the weighting process can no longer be determined independently of the equilibrium prices.

**Corollary 1.** In equilibrium, the volatility of the logarithm of the weighting process satisfies

$$\Gamma_t = -R_t (v_t - s_2 \Gamma_t) + \Pi \left[ R_t (v_t - s_2 \Gamma_t) | \sigma_t^\top C_t \right] \quad (12)$$

where $\Pi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ stands for the projection operator and $(s_2, R)$ denote the constrained agent’s share of consumption and the relative risk aversion of the representative agent as defined in Proposition 4.

The above corollary identifies one channel of dependence between the weighting process and the equilibrium stock prices, but there is another one. Indeed, standard intuition suggests that the equilibrium stock prices depend on the conditional expectation of future dividends discounted by the unconstrained state price density process

$$\xi_{1t} = \frac{u_{1c}(c_{1t})}{u_{1c}(c_{10})} = \frac{u_c(c_t, \lambda_t)}{u_c(c_0, \lambda_0)}.$$ 

In conjunction with equation (12) this implies that the stock prices and the weighting process form a forward–backward system and, hence, that they must be solved for simultaneously. In order to unveil the precise nature of this system, it is necessary to determine whether the equilibrium price system admits a linear representation as in equation (11) holds, or if it includes a bubble. This is the issue I address in the next section.

## 4 Equilibrium pricing bubbles

In this section I determine conditions under which the equilibrium prices include bubbles, and study the basic properties of these deviations from the
law of one price. Furthermore, I show that the presence of bubbles in the aggregate price system can give rise to multiplicity of equilibria.

A. Characterization

If the equilibrium is non redundant, then the state price density $\xi_1$ of the unconstrained agent is the unique nonnegative process such that

$$M_i^t \equiv \xi_1 S_i^t + \int_0^t \xi_1 e^s d\lambda_s = \frac{u_c (e_t, \lambda_t)}{u_c (e_0, \lambda_0)} S_i^t + \int_0^t \frac{u_c (e_s, \lambda_s)}{u_c (e_0, \lambda_0)} e^s d\lambda_s$$

is a local martingale for each risky asset. Since both the dividends and the stock prices are nonnegative, the processes $M^t$ are nonnegative local martingales and because of this they are also supermartingales (see Karatzas and Shreve (1998, p.36)). This implies that

$$S_i^t \geq E \left[ \int_t^T \frac{u_c (e_s, \lambda_s)}{u_c (e_t, \lambda_t)} e^s d\lambda_s \bigg| \mathcal{F}_t \right].$$

(14)

Since the market is dynamically complete for the unconstrained agent, the expected present value of future dividends which appears on the right of the above inequality coincides with the minimal amount necessary to replicate the pay–off of the asset by trading in the market (see Karatzas and Shreve (1999)). If one thinks of this replication cost as defining the fundamental value of the asset, then the above inequality requires that the current value of the asset be no smaller than its fundamental value.

What is important about inequality (14) is that it may be strict. This implies that, in general, one should think of the price of an asset as being composed of two parts. The first part is given by

$$f_i^t \equiv E \left[ \int_t^T \frac{u_c (e_s, \lambda_s)}{u_c (e_t, \lambda_t)} e^s d\lambda_s \bigg| \mathcal{F}_t \right].$$

(15)

and corresponds to the fundamental value of the asset; while the second part

17
is given by
\[
b_i^t \equiv S_i^t - f_i^t = S_i^t - E \left[ \int_t^T \frac{u_t(e_s, \lambda_s)}{u_t(e_t, \lambda_t)} e^s_d s \right] \mathcal{F}_t \tag{16}
\]
and corresponds to a bubble. According to this decomposition, the price of an asset includes a bubble as soon as it exceeds the fundamental value of the asset. This is consistent with the traditional definition of bubbles used by Blanchard (1979), Blanchard and Watson (1982), Santos and Woodford (1997) and Loewenstein and Willard (2000a; 2007) among others.\(^{13}\)

At first glance, it would seem that bubbles should be ruled out by the absence of arbitrages since they imply that two assets with the same pay-offs have different prices. To see that this is not the case, assume that asset \(i\) includes a bubble and consider the following strategy: sell the asset, buy the portfolio that replicates it and invest the proceeds in the bank account. This strategy has initial value zero and terminal value \(b_i^0 S_T^0 > 0\) but it is not admissible because the corresponding wealth process
\[
W_t \equiv f_t^i - S_t^i + b_i^0 S_t^0 = b_i^0 S_t^0 - b_i^t
\]
can take negative values with strictly positive probability at any time before maturity. In order to make this strategy admissible, the agent must hold enough collateral to absorb the potential losses. This implies that starting from some initial wealth the agent will be able to implement the arbitrage strategy up to a certain scale but will not be able to indefinitely increase the scale of the arbitrage trade due to the requirement of nonnegative wealth. In other words, the existence of a bubble does imply the existence of an arbitrage opportunity but the agents in our model cannot exploit it fully because they are required to maintain nonnegative wealth at all times. This implies that the presence of bubbles is not incompatible with the existence

\(^{13}\)An alternative definition involving the presence of pure charges in the price system was proposed by Gilles and LeRoy (1992). See Jarrow, Protter, and Shimbo (2007) for details on the connection between these two closely related definitions.
of an equilibrium. A similar argument, albeit with more general wealth constraints, appears in Loewenstein and Willard (2000a; 2007).

In order to determine the conditions under which the equilibrium prices include bubbles I will use a comparison argument. According to Propositions 1, 2 and 4, the wealth processes are given by

$$W_{1t} = E\left[\int_t^T \frac{u_c(e_s, \lambda_s)}{u_c(e_t, \lambda_t)} I_1(u_c(e_s, \lambda_s)) ds \middle| F_t\right]$$ (17)

for the unconstrained agent, and

$$W_{2t} = \frac{\lambda_t(T - t)}{u_c(e_t, \lambda_t)} = E\left[\int_t^T \frac{\lambda_t u_c(e_s, \lambda_s)}{\lambda_s u_c(e_t, \lambda_t) u_c(e_s, \lambda_s)} ds \middle| F_t\right]$$ (18)

for the constrained agent. Since financial markets clear in equilibrium, the sum of the agents’ wealth processes equals the sum of the equilibrium stock price processes, i.e. the market portfolio. This implies

$$\mathbb{S}_t = \sum_{i=1}^n S_t^i = \sum_{a=1}^2 W_{at}$$

$$= E\left[\int_t^T \frac{u_c(e_s, \lambda_s)}{u_c(e_t, \lambda_t)} \left( I_1(u_c(e_s, \lambda_s)) + \frac{\lambda_t}{u_c(e_s, \lambda_s)} \right) ds \middle| F_t\right]$$

$$= E\left[\int_t^T \frac{u_c(e_s, \lambda_s)}{u_c(e_t, \lambda_t)} \left( e_s + \frac{\lambda_t - \lambda_s}{u_c(e_s, \lambda_s)} \right) ds \middle| F_t\right]$$

$$= \sum_{i=1}^n f_t^i + E\left[\int_t^T \frac{\lambda_t - \lambda_s}{u_c(e_t, \lambda_t)} ds \middle| F_t\right]$$ (19)

where the fourth equality follows from Proposition 4 and the clearing of the goods market, and the last equality follows from the definition of the fundamental value processes. The result of Proposition 3 shows that the equilibrium weighting process is a nonnegative local martingale. If this process is a true martingale then the last term in the above expression
vanishes and it follows that
\[ S_t = \sum_{i=1}^{n} f^i_t = \sum_{i=1}^{n} (S^i_t - b^i_t) = S_t - \sum_{i=1}^{n} b^i_t. \]

Since bubbles are nonnegative by definition (recall equations (14) and (16)) this further implies that \( b^i_t = 0 \) for all \( i \). In other words, the equilibrium price system does not include a bubble as long as the weighting process is a true martingale. Using the fact that a nonnegative local martingale with constant expectation is a true martingale shows that the converse implication is also true; and putting everything together yields the following:

**Theorem 1.** The equilibrium stock prices are free of bubbles if and only if the weighting process \( \lambda = c_2u_1c_1 \) is a true martingale.

The above result shows that bubbles can occur in equilibrium only if the weight of the constrained agent fails to be a true martingale.\(^{14}\) Since this process is a nonnegative local martingale, it is a supermartingale and will be a true martingale only if it is constant in expectation. As a result, bubbles occur in equilibrium if and only if the weight of the constrained agent is strictly decreasing in expectation. This suggest that bubbles arise in equilibrium if the portfolio constraint is so costly to the constrained agent that it unambiguously benefits the unconstrained agent. This intuition will be confirmed in Sections 5 and 6 where I present two explicit examples of models where bubbles are necessary for the existence of an equilibrium.

As discussed in the previous section, the weighting process is a true martingale in the absence of portfolio constraints since it is then constant. In conjunction with Theorem 1 this immediately gives the following:

**Corollary 2.** In the absence of portfolio constraints there can be no asset pricing bubbles in a non redundant equilibrium.

\(^{14}\) A local martingale which is not a martingale is called a strict local martingale, see Elworthy et al. (1999). Apart from the study of asset pricing bubbles, strict local martingales play an important role in stochastic volatility models (Sin (1998)) and in the modeling of relative arbitrages (Fernholz et al. (2005)).
Now assume that the weighting process is not a true martingale so that there are bubbles in the equilibrium price system. Equation (19) implies that the aggregate bubble value is given by

$$b_t^i \equiv \sum_{i=1}^{n} b_t^i = E \left[ \int_t^T \frac{\lambda_t - \lambda_s}{u_c(e_t, \lambda_t)} ds \bigg| \mathcal{F}_t \right].$$  

(20)

In conjunction with Itô’s lemma this shows that, just like the weighting process, the discounted bubble value process

$$\frac{u_c(e_t, \lambda_t)}{u_c(e_0, \lambda_0)} b_t^i = E \left[ \int_t^T \frac{\lambda_t - \lambda_s}{u_c(e_0, \lambda_0)} ds \bigg| \mathcal{F}_t \right]$$

$$= \int_0^t T - s \frac{\lambda_0 - \lambda_s}{u_c(e_0, \lambda_0)} d\lambda_s + E \left[ \int_0^T \frac{\lambda_0 - \lambda_s}{u_c(e_0, \lambda_0)} ds \bigg| \mathcal{F}_t \right]$$

is a nonnegative local martingale and a supermartingale but not a martingale since otherwise it would be identically equal to zero. In particular, this implies that the bubble can burst before the terminal time of the model and that in that case it can never be reborn.

Another consequence of equation (20) is that, contrary to what happens in discrete time models, bubbles can be uniformly bounded across time and states of nature. To see this it suffices to observe that

$$b_t^i \leq \max_{1 \leq i \leq n} b_t^i \leq b_t \leq e_t(T - t)$$

where the second inequality follows from Proposition 4 and the clearing of the goods market. In particular, all the bubbles are bounded as soon as the aggregate dividend is bounded. This shows that the intuition according to which assets with bounded payoffs, such as coupon bonds or put options, cannot have bubbles is not true in continuous time, see Heston et al. (2007) for various examples. However, one should be careful with this result: the bubble itself can be bounded but its discounted value

$$\xi_{1t} b_t^i = \frac{u_c(e_t, \lambda_t)}{u_c(e_0, \lambda_0)} b_t^i$$

21
cannot if the model contains a non zero bubble. Indeed, the discounted value of the bubble is a local martingale. If it was uniformly bounded then it would be a true martingale and hence would be identically equal to zero since its terminal value is

$$\frac{u_e(e_T, \lambda_T)}{u_e(e_0, \lambda_0)} b_T = \frac{u_e(e_T, \lambda_T)}{u_e(e_0, \lambda_0)} b_T = 0.$$ 

As argued by Loewenstein and Willard (2000a), the crucial element is not so much that the bubble be large by itself, but that it be large relative to the unconstrained state price density.

Remark 1 (Bubbles on the riskless asset). The above results focus exclusively on stock price bubbles but bubbles may also arise on the price of the riskless asset. As shown Heston et al. (2007), the absence of bubble on the riskless asset is equivalent to the requirement that the local martingale

$$\xi_t S_t^0 = \exp \left[ - \int_0^t \theta_s^T d B_s - \frac{1}{2} \int_0^t \|\theta_s\|^2 ds \right]$$

is a true martingale and is thus closely related to the absence of (generalized) arbitrage opportunities, see Delbaen and Schachermayer (1994) and Loewenstein and Willard (2000a;b) for details. In Section 5 below I present an example of an economy with a single risky asset where the presence of a limited participation constraint induces a bubble on both the stock and the riskless asset in equilibrium.

B. Bubbles and multiplicity of equilibria

Having identified the conditions under which the equilibrium price system includes bubbles, I now turn to the determination of the equilibrium stock prices. The following result gives necessary and sufficient condition for a nonnegative process to be an equilibrium price process.

**Proposition 5.** Let $S$ be a nonnegative process with values in $\mathbb{R}^n$, assume
that its volatility matrix $\sigma$ is non singular and let

$$
\lambda_t = \exp \left[ -\frac{1}{2} \int_0^t \| \Gamma_s \|^2 ds + \int_0^t \Gamma_s^\top dB_s \right]
$$

where $\Gamma$ is defined by equation (12). Then $S$ is the stock price process in a non redundant equilibrium if and only if

$$
1^\top S_t = E \left[ \int_t^T \frac{u_c(e_s, \lambda_s)}{u_c(e_t, \lambda_t)} e_s ds \right] \mathcal{F}_t + E \left[ \int_t^T \frac{\lambda_t - \lambda_s}{u_c(e_t, \lambda_t)} ds \right] \mathcal{F}_t
$$

(21)

and the nonnegative process $M^i$ defined by equation (13) is a local martingale for each of the risky assets.

The conditions of the above proposition can be explained as follows. The second condition in the statement says that the stock price process under consideration offers the risk premium of Proposition 4. This guarantees the optimality of the equilibrium consumption allocation and implies that the wealth processes of the agents are given by equations (2)–(3). The first condition then implies that the sum of these wealth processes coincides with the sum of the stock prices and guarantees that the market for the riskless asset clears. This further implies that the agents’ portfolios satisfy

$$
(\pi_{1t} + \pi_{2t} - S_t)^\top \sigma_t = (0, \ldots, 0)
$$

(22)

and, since the volatility matrix of the stock price process is non singular by assumption, it follows that the market for the risky assets clears.

If the volatility matrix of the stocks is such that the weighting process is a true martingale, then the equilibrium prices are bubble free and Proposition 5 implies that the stock price is uniquely given by the expected value of future discounted dividends. Indeed, in that case the second term on the
right hand side of equation (21) vanishes and, as a result, the process

$$M_t \equiv \sum_{i=1}^n M^i_t = \mathbb{E} \left[ \int_0^T \frac{u_c(e_s, \lambda_s)}{u_c(e_0, \lambda_0)} e_s ds \bigg| \mathcal{F}_t \right]$$

is a nonnegative martingale. Since $M^i \leq M$ this implies that the local martingale $M^i$ is a true martingale and it follows that the stock prices are given by equation (11).\(^{15}\) On the contrary, if the weighting process fails to be a true martingale, then the conditions of Proposition 5 pin down the sum of the stock prices but are not sufficient to determine the individual stock prices since at least one of the $M^i$ is a strict local martingale.

This suggests that the presence of asset pricing bubbles may give rise to multiplicity of equilibria as soon as the model includes more than one risky asset. To see this, let $(A_t)_{t \in [0, T]}$ denote a family of convex subsets and assume that the portfolio constraint set is given by

$$C \equiv \left\{ \pi \in \mathbb{R}^n : \sigma_t^\top \pi/W_t \in A_t \right\}$$

so that the projection operator in equation (12) does not depend on the stock volatility matrix. In this case, the volatility of the weighting process is independent of the stock prices and, as a result, the presence of bubbles can be assessed from the primitives of the economy by determining whether $\lambda$ is a true martingale or not. Assume the latter so that any equilibrium includes bubbles in the price of at least one asset. Since equation (7) does not depend on the stock prices, the initial value of the weighting process, the risk free rate, the risk premium and the consumption allocations are also determined uniquely from the primitives of the economy.\(^{16}\) The only equilibrium quantities that are undetermined are the individual stock prices.

\(^{15}\)Combining this result with Corollary 1 leads to a constrained forward-backward stochastic differential equation for the pair $(S, \lambda)$. This system characterizes the set of a non redundant, bubble free equilibria and can be used to study the existence problem.

\(^{16}\)This occurs because the initial endowments of the agents only depend on the total value of the economy. If this was not the case, then all of the equilibrium quantities would vary across the different equilibria. See Remark 3 below for details.
But, since the aggregate bubble value

\[ b_t = S_t - \sum_{i=1}^{n} f_t^i = \sum_{i=1}^{n} b_t^i = E \left[ \int_t^T \frac{\lambda_t - \lambda_s}{u_c(e_t, \lambda_t)} ds \bigg| \mathcal{F}_t \right] \]

can be spread across the different stocks arbitrarily as long as the conditions of Proposition 5 hold true, there are infinitely many stock price processes that result in equilibrium. Section 6 provides an explicit example of an economy with two stocks and portfolio constraints which admits a continuum of non redundant equilibria due to the presence of a bubble.

5 Limited market participation

In this section I study a single stock economy which generalizes the restricted market participation model of Basak and Cuoco (1998). Using the results of the previous sections, I show that the equilibrium of this economy is unique and includes a bubble on both the stock and the riskless asset.

A. The economy

Consider an economy with a single risky asset and assume that the dynamics of the dividend process are given by

\[ e_t = e_0 + \int_0^t a e_s ds + \int_0^t v e_s dB_s. \]

for some constants \( e_0 > 0, a \in \mathbb{R} \) and \( v > 0 \). Agents have logarithmic utility\(^{17}\) and I assume that the portfolio constraint set is given by

\[ C_t \equiv C_0 = \left\{ p \in \mathbb{R} : 0 \leq p \leq 1 - \varepsilon \right\} \]

for some \( 0 < \varepsilon \leq 1 \). This is a limited participation constraint which implies that the agent cannot short the risky asset and must keep at least \( \varepsilon \)% of

\(^{17}\)This assumption is made for simplicity of exposition. The model of this section can also be solved, with similar conclusions, under the assumption that the unconstrained agent has a power utility function. However, in that case equilibrium may fail to exist under some parameter configurations.
his wealth in the riskless asset at all times. In particular, the limiting case
where $\varepsilon = 1$ corresponds to the restricted stock market participation model
of Basak and Cuoco (1998).\(^{18}\)

To complete the description of the economy, I assume that the initial
wealth of the agents are given by $w_2 = \alpha S_0 + \beta$ and $w_1 = S_0 - w_2$ where
$(\alpha, \beta)$ are nonnegative constants such that

$$
\frac{\varepsilon \alpha S_0 T}{1 - \varepsilon} \leq \beta < (1 - \alpha) e_0 T.
$$

These parametric restrictions can be justified as follows. The first inequality
guarantees that, in equilibrium, the initial portfolio of agent 2 satisfies the
portfolio constraint. The second inequality restricts the first agent to not
be so deeply in debt at the initial time that he can never pay back from
the dividend supply. As in Basak and Cuoco (1998, Equation (6)) this
restriction is necessary to guarantee the existence of an equilibrium.

B. Equilibrium

Under the assumption of homogenous logarithmic utility, the representative
agent’s utility function can be computed explicitly and is given by

$$
u(c, \lambda) = (1 + \lambda)(\log c - \log(1 + \lambda)) + \lambda \log \lambda.
$$

Combining this expression with equations (17)–(18) gives the agents’ wealth
processes and summing over the agents shows that the equilibrium stock
price is uniquely given by

$$
S_t = \sum_{a=1}^{2} W_{at} = \frac{e_t(T - t)}{1 + \lambda_t} + \frac{e_t \lambda_t (T - t)}{1 + \lambda_t} = e_t(T - t).
$$

\(^{18}\)The other limiting case $\varepsilon = 0$ models a situation where agent 2 can neither borrow nor
short the risky asset. Since agents have homogenous preferences, they would not invest in
the riskless asset in the absence of portfolio constraint and it follows that the case $\varepsilon = 0$
corresponds to an unconstrained equilibrium.
This implies that the volatility of the stock equals that of the aggregate dividend and it follows that the volatility of the weighting process can be computed independently of the stock price by solving equation (12). This, in turn, gives an autonomous stochastic differential equation for the weighting process and plugging the result in the formulas of Proposition 4 yields:

**Proposition 6.** Assume that equation (23) holds and define the weighting process as the unique solution to

\[ \lambda_t = (w_2/w_1) - \int_0^t \lambda_s(1 + \lambda_s)\hat{v}dB_s. \tag{25} \]

with \( \hat{v} \equiv \varepsilon v \). There exists a unique non redundant equilibrium in which the consumption plans and trading strategies are given by

\[
\begin{align*}
    c_{1t} &= \frac{e_t}{1 + \lambda_t}, & \pi_{1t} &= (1 + \varepsilon \lambda_t)W_{1t}, & \phi_{1t} &= -\varepsilon \lambda_tW_{1t}, \\
    c_{2t} &= \frac{\lambda_t e_t}{1 + \lambda_t}, & \pi_{2t} &= (1 - \varepsilon)W_{2t}, & \phi_{2t} &= \varepsilon W_{2t}.
\end{align*}
\]

Furthermore, the equilibrium relative risk premium, interest rate and stock price processes are given by

\[
\begin{align*}
    \theta_t &= v(1 + \varepsilon \lambda_t), \\
    r_t &= a - v\theta_t = a - v^2(1 + \varepsilon \lambda_t), \\
    S_t &= e_t(T - t).
\end{align*}
\tag{26}
\]

Finally, the stock price and the riskless asset price both include a non trivial bubble components. These bubbles are given by

\[
\begin{align*}
    b_{0t} &= h^0(t, \lambda_t)S_t^0, \\
    b_t &= h(t, \lambda_t)S_t,
\end{align*}
\]

for some \([0,1)\)-valued functions \(h^0\) and \(h \leq h^0\) defined in the appendix.

Because agents have homogenous logarithmic preferences, the weighting
process has a physical interpretation as the ratio of the agents’ consumptions or wealth processes. This ratio fluctuates randomly to reflect the impact of the portfolio constraint. The resulting changes in the distribution of wealth affect both the equilibrium risk free rate and the risk premium but, as usual with logarithmic preferences, leave the stock price and its local characteristics unchanged.

The effect of a redistribution of wealth on the equilibrium depends on the severity of the constraint and the magnitude of \( v \). However, it is worth noting that, as long as the volatility of the dividend process is positive, limited participation always implies a higher risk premium and a lower interest rate compared to the case of an unconstrained economy. To understand this feature, note that in an unconstrained economy the agents do not invest in the riskless asset because they have homogenous preferences. In the constrained economy, however, the second agent is forced to invest a positive fraction of his wealth in the riskless asset. The unconstrained agent must therefore be induced to become a net borrower and it follows that the interest rate must decrease and the risk premium must increase compared to the unconstrained economy.

These local effects of the portfolio constraint go in the right direction but they are not sufficient to reach an equilibrium. Indeed, the second part of the proposition shows that both the stock and the riskless asset include a bubble in equilibrium. Since exploiting the bubble on one asset means going long in the other, the agent cannot benefit from both bubbles at the same time. Taking into account the nonnegativity constraint on wealth, the agent should exploit the limited arbitrage opportunity on the riskless asset because it requires less collateral per unit of initial profit. This intuition can be confirmed by decomposing the agent’s trading strategy as

\[
(\pi_{1t}, \phi_{1t}) = W_{1t} (1, 0) + \varepsilon \lambda_t W_{1t} (1, -1).
\]

The first part is the trading strategy that the agent would have used in the
absence of portfolio constraints. The second part is a continuously resettled arbitrage strategy which exploits the bubble on the riskless asset by going short in the riskless asset and long in the stock. As explained in Section 4, this strategy is not admissible by itself because it entails the possibility of negative wealth. However, when coupled with a large enough collateral investment in the stock this strategy becomes admissible and allows the unconstrained agent to benefit from the limited arbitrage opportunity which is induced by the bubble on the riskless asset. The fact that the stock price also includes a bubble increases its collateral value and, hence, allows the agent to increase the size of his short position in the riskless asset to the level required by market clearing.

Remark 2. In the case ε = 1, the result of Proposition 6 provides a negative answer to a question left open in Basak and Cuoco (1998, Remark 4 p.322) since it shows that the equilibrium stock price is not given by the expectation of future discounted dividends.

C. Comparative static analysis

In order to illustrate the properties of the above equilibrium, I now analyze the impact of the horizon, the severity of the constraint and the weight of constrained agents on key equilibrium quantities. In order to facilitate this analysis I start by discussing the behavior of the weighting process.

C.1. The weighting process

Using the results Appendix A (see Lemma 3) I obtain that the conditional expected value of the weighting process is given by

\[ E[\lambda_s|\mathcal{F}_t] = \lambda_t \Phi \left( \frac{1}{\hat{v}\sqrt{s-t}} \log \frac{1 + \lambda_t}{\lambda_t} - \frac{\hat{v}\sqrt{s-t}}{2} \right) \]

\[ - (1 + \lambda_t) \Phi \left( \frac{1}{\hat{v}\sqrt{s-t}} \log \frac{\lambda_t}{1 + \lambda_t} - \frac{\hat{v}\sqrt{s-t}}{2} \right) \]
where $\Phi$ is the Gaussian cumulative distribution function. Since there is a bubble on the stock, we know from Theorem 1 that the weighting process is a nonnegative local martingale and a supermartingale, but not a true martingale. This implies that its conditional expectation is strictly smaller than the initial value and decreases with the horizon.

To assess the magnitude of these effects, Figure 1 plots the expected value of the weighting process as a function of the horizon and the initial value. The right panel of the figure shows that the expected value of the weighting process is much lower than the initial value and decreases very fast as a function of time. For example, given an initial value of four, the value of the process is expected to be divided by more than four after eight years. The left panel shows that the default of martingality

$$\delta(t) \equiv \lambda_0 - E[\lambda_t]$$

is an increasing function of both the time horizon and the initial value of the weighting process. This should be intuitive. As time passes, the weighting process is expected to decrease more because it is a supermartingale and, since it is an exponential process, this effect increases as the initial value increases. What is striking in the right panel of the figure is the speed at which the default of martingality increases.

In order to gain more insights on the behavior of the weighting process, I also compute its transition density (see Lemma 5 in Appendix A) and compare it to that of the exponential martingale

$$\Psi_t \equiv \lambda_0 \exp \left[ -\hat{v} B_t - \frac{1}{2} \hat{v}^2 t \right].$$

The upper panels of Figure 2 plot these two transitions densities at a one year horizon for different initial values. As expected the density of the
exponential martingale is nicely centered around the initial value of the process. On the contrary, the density of the weighting process peaks to the left of the initial value to reflect the fact that the process is a local martingale and a supermartingale but not a true martingale. The lower panels of Figure 2 plot the transition densities of the two processes at different horizons. As the horizon increases, the density of $\Psi$ flattens to reflect the increase in volatility but it stays centered around the initial value because the process is a martingale on any finite time interval. By contrast, the density of the weighting process moves to the left and piles up its mass against the vertical axis as the horizon increases. This is due to the fact that, being a supermartingale, the weighting process is expected to decrease more and more as time passes.

The reason why the weighting process fails to be a true martingale is that it is not uniformly integrable. In particular, the expected value of its supremum is infinite on any finite time interval. Since the transition density piles up towards zero as the time horizon increases, it must be the case that the right tail of the distribution is rather wide. In fact, it can be shown (see Lemma 4 in Appendix A) that the default of martingality is related to the right tail of the distribution by the formula

$$\delta(t) = \lim_{x \to \infty} p(t, x) = \lim_{x \to \infty} xP\left[\max_{s \leq t} \lambda_s \geq x\right].$$

(28)

If the weighting process were a true martingale then the right hand side of the above expression would be equal to zero as is the case for the process $\Psi$. Since this is not the case, one naturally expects that the right tail which is associated with the weighting process is much larger than that which corresponds to the exponential martingale.

To confirm this intuition, I compare in Figure 3 the difference between
the tail probabilities \( p(t, x) \) associated with the two processes as functions of the threshold for various horizons. As expected, the tail probabilities of the weighting process are always above those of the corresponding exponential martingale. Furthermore, the figure shows that the difference between the two increases as the time horizon increases. This is due to the fact that the default of martingality is zero for \( \Psi \) but increases as a function of the horizon for the weighting process.

C.2. The asset pricing bubbles

Having studied the behavior of the weighting process, I now turn to the asset pricing bubbles. To facilitate the economic interpretation of the results, I will express the value of the bubbles as functions of time, the aggregate dividend and the auxiliary state variable

\[
\Sigma_t \equiv \frac{W_{2t}}{W_{1t} + W_{2t}} = \frac{c_{2t}}{c_{1t} + c_{2t}} = \frac{\lambda_t}{1 + \lambda_t}
\]

which represents both the share of the total wealth held by the constrained agent and his consumption share.

Since \( \Sigma \) is an increasing concave function of \( \lambda \), it follows from Jensen’s inequality and Proposition 3 that it is a supermartingale. This implies that the consumption share of the constrained agent is expected to decline over time. Because the weighting process is always a local martingale this would be the case even if there was no bubble, but one expects that the presence of a bubble increases the speed at which the constrained agent’s consumption share decreases. This intuition is confirmed by Figure 4 which compares the expected consumption share to that computed under the assumption that
the weighting process is given by the exponential martingale of equation (27). In particular, the middle curve in the figure shows that starting from an initial share of 50% the constrained agent is expected to consume about a quarter of the total endowment after twenty years.\textsuperscript{19} This rapid decrease of the constrained agent’s consumption share shows that explanations of asset pricing puzzles based on limited participation (see e.g. Basak and Cuoco (1998)) should be taken with care as they can only be transitory.

As explained above, the bubble arises because the unconstrained agent must find it optimal to hold a leveraged position in the stock. On the other hand, Proposition 6 shows that the risky part of the unconstrained agent’s equilibrium portfolio can be written as

\[ \pi_{1t} = W_{1t} + \varepsilon e_t \Sigma_t (T - t) \]

and it follows that the optimal leverage increases with the consumption share of the constrained agent, the horizon of the model and the tightness of the constraint. Since the optimal leverage is determined by the relative size of the bubbles, this suggest that the size of the bubbles in percentage of the underlying assets should be increasing functions of \((\varepsilon, T, \Sigma)\). This intuition is confirmed by Figure 5 which plots the contribution of the bubbles to the stock and the riskless asset as a functions of the model horizon and the constrained agent’s consumption share. The figure also shows that the bubble component can be quite large. For example, with an horizon of ten years and \(\Sigma_0 = 60\%\) the bubbles can account for as much as 18\% of the

\textsuperscript{19}This figure should be compared to those found in the recent literature studying the impact of bounded rationality on asset prices, see Kogan et al. (2006), Berrada (2008) and Dumas et al. (2008) among others. These models show that the consumption share of irrational traders decreases over time but the speed of this decline is usually rather slow. For example, Kogan et al. (2006) and Berrada (2008) find that in their respective models it takes approximately 80 years for the consumption share of the irrational agent to decrease from 50\% to 25\% when both agents have logarithmic utility functions.
equilibrium stock price and close to 60% of the riskless asset value depending on the tightness of the portfolio constraint.

The presence of a bubble contributes not only to the value of the stock but also to its volatility. Using Itô’s lemma I obtain that the fraction of the stock volatility which is due to the bubble is given by

\[
\zeta(t, \Sigma_t) = \frac{\text{cov}(\text{db}_t, \text{dS}_t)}{\text{var}(\text{dS}_t)} = h\left(t, \frac{\Sigma_t}{1 - \Sigma_t}\right) - \varepsilon \Sigma_t \frac{\partial h}{\partial \lambda}\left(t, \frac{\Sigma_t}{1 - \Sigma_t}\right)
\]

where the function \( h \) is defined as in Proposition 6. Figure 6 plots this contribution as a function of the constrained agent’s consumption share for different constraint levels. As shown by the figure, the function \( \zeta \) is negative and decreases with both \( \varepsilon \) and \( \Sigma \). The intuition for this result comes from the decomposition of the stock prices as the sum of the bubble and the fundamental value. The stock provides the same risk premium and the same dividends as the fundamental value but has a higher price. This implies that the stock should have a lower expected return and hence also a lower volatility if the risk premium is nonnegative as is the case here.

The figure also shows that the magnitude of this volatility dampening effect increases as the portfolio constraint tightens. This is due to the fact that, as shown by Figure 5, the relative contribution of the bubble to the stock price increases with the level of the portfolio constraint.

6 Multiplicity in a two stocks model

In this section I study a model of a two stock economy where one of the agent is subject to a constraint on the volatility of his wealth. Using the results of the previous sections I establish that any non redundant equilibrium includes a bubble and show that this gives rise to a continuum of equilibria.
A. The economy

Consider an economy with two risky assets, assume that the uncertainty is generated by a two dimensional Brownian motion and let the aggregate dividend process be given by

\[ e_t = e_0 + \int_0^t e_s ds + \int_0^t e_s v^\top dB_s \]

for some constants \((e_0, a) \in \mathbb{R}_+ \times \mathbb{R} \) and \(v \in \mathbb{R}^2 \). Rather than modeling the dividends of the two stocks, I will assume that the share of consumption produced by the first stock follows a process of the form

\[ x_t = x_0 + \int_0^t x_s (1 - x_s) \sigma^\top dB_s \]

for some \(x_0 \in (0, 1) \) and \(\sigma_x \in \mathbb{R}^2 \). In order to simplify the computation I will further assume that \(\sigma_x \) and \(v\) are orthogonal.\(^{20}\) This cash flow model is a special case of that used by Menzly, Santos, and Veronesi (2004) in their study of return predictability and I refer to these authors for details on the properties of the induced dividend processes.

Agents have homogenous logarithmic preferences and I assume that the portfolio constraint set of the second agent is given by

\[ C_t \equiv \left\{ p \in \mathbb{R}^2 : \|\sigma^\top_t p\| \leq (1 - \varepsilon)\|v\| \right\} \]

for some \(\varepsilon \in (0, 1) \). This is a risk constraint which implies that the volatility of the agent’s wealth cannot exceed a fixed threshold. Since both agents are myopic, the volatility of the market as a whole is equal to that of the aggregate dividend and it follows that the constraint forces the agent to choose a portfolio which is less volatile than the market.\(^{21}\)

\(^{20}\)This specification is adopted for analytical convenience only. The crucial properties needed for the validity of the results of this section are that the volatility of the dividend share be linearly independent from that of the aggregate dividend and that the latter be bounded away from zero. Details are available from the author upon request.

\(^{21}\)Because agents have homogenous preferences, the case where \(\varepsilon = 0\) leads to an unconstrained equilibrium where both agents invest the totality of their wealth in the
To complete the description of the economy, assume that the initial wealth of the agents are given by $w_2 = \beta + \alpha S_0$ and $w_1 = S_0 - w_2$ for some nonnegative constants $\alpha, \beta$ satisfying equation (23). As in the previous section, this parametric restriction has a dual purpose. It implies that the initial portfolio of the second agent satisfies the constraint and guarantees that the set of non redundant equilibria is non empty.

B. Existence and multiplicity of equilibria

As in the previous section, the representative agent’s utility function is given by equation (24). Combining this expression with equations (17)–(18) shows that the value of the market portfolio is given by

$$S_t \equiv \sum_{i=1}^2 S_i^t = \sum_{a=1}^2 W_{at} = e_t(T - t)$$

in any equilibrium and it follows from equation (7) that the initial value of the weighting process is uniquely given by

$$\lambda_0 \equiv \frac{w_2}{w_1} = \frac{\alpha e_0 T + \beta}{(1 - \alpha)e_0 T - \beta}.$$

On the other hand, since the set $\sigma^\top C_t$ is independent from the stock volatility it follows from Corollary 1 that the volatility of the weighting process can be computed independently of the stock prices by solving equation (12). Combining this property with the results of Proposition 4 yields the following characterization of the set of equilibria.

**Proposition 7.** Assume that equation (23) holds and define the weighting process as the unique solution to

$$\lambda_t = \lambda_0 - \int_0^t \lambda_s (1 + \lambda_s)\varepsilon v^\top dB_s.$$

risky asset. The other limiting case where $\varepsilon = 1$ corresponds to a restricted participation model similar to that of Basak and Cuoco (1998) albeit with two risky assets.
In any non redundant equilibrium, the consumption plans, the risk premium and the interest rate are given by

\[
\begin{align*}
    c_{1t} &= \frac{e_t}{1 + \lambda_t}, \\
    c_{2t} &= \frac{e_t \lambda_t}{1 + \lambda_t}, \\
    \theta_t &= (1 + \varepsilon \lambda_t) v, \\
    r_t &= a - (1 + \varepsilon \lambda_t) \|v\|^2.
\end{align*}
\]

Furthermore, the market portfolio and the riskless asset both include a bubble component as soon as \(\varepsilon \|v\| > 0\). These bubbles are given by

\[
\begin{align*}
    b_0^t &= h^0(t, \lambda_t) S_t^0, \\
    b_t &= h(t, \lambda_t) \overline{S}_t,
\end{align*}
\]

where \(h^0\) and \(h \leq h^0\) are defined as in Proposition 6 albeit with \(\hat{v} = \varepsilon \|v\|\).

The first part of the proposition shows that, if an equilibrium exists, the consumption allocations, the risk premium and the interest rate are constant across the set equilibria since they do not depend on the volatility of the stocks. More importantly, the second part shows that the equilibrium prices include bubbles as soon as there are portfolio constraints \((\varepsilon \neq 0)\) and randomness at the aggregate level \((\|v\| \neq 0)\). This implies that, unless the economy is unconstrained or deterministic, the presence of bubbles is a necessary condition for equilibrium.

To establish the existence of an equilibrium, I need to construct a stock price process which supports the consumption plans, interest rate and risk premium of Proposition 7. Since the weighting process is independent of the stock prices, it follows from equations (15) and (20) that the aggregate bubble and the fundamental value of the stocks are constant across the set of equilibria. Furthermore, the assumed independence between \(x\) and \(e\) imply that the fundamental value of stock \(i\) is given by

\[
f_t^i = e_t^i (1 - h(t, \lambda_t))(T - t),
\]

where \(h^0\) and \(h \leq h^0\) are defined as in Proposition 6 albeit with \(\hat{v} = \varepsilon \|v\|\).
where the function $h$ represents the share of the bubble in the value of the market portfolio. Since both the discounted bubble value
\[ N_t \equiv \frac{u_c(e_t, \lambda_t)}{u_c(e_0, \lambda_0)} b_t = e_0 E \left[ \int_t^T \frac{\lambda_t - \lambda_s}{1 + \lambda_0} ds \bigg| \mathcal{F}_t \right], \]
and the processes
\[ \frac{u_c(e_t, \lambda_t)}{u_c(e_0, \lambda_0)} f_t^i + \int_0^t \frac{u_c(e_s, \lambda_s)}{u_c(e_0, \lambda_0)} e_s^i ds = E \left[ \int_0^T \frac{u_c(e_s, \lambda_s)}{u_c(e_0, \lambda_0)} e_s^i ds \bigg| \mathcal{F}_t \right] \]
are local martingales by construction, the result of Proposition 5 suggests that any constant repartition of the bubble among the two stocks leads to an equilibrium. The following proposition confirms this intuition and, hence, shows that there exists infinitely many equilibria.

**Proposition 8.** Assume that equation (23) holds. Then
\[ S_t(\phi) \equiv f_t + \left( \frac{\phi}{1 - \phi} \right) b_t \]
\[ = S_t \left[ (1 - h(t, \lambda_t)) \begin{pmatrix} x_t \\ 1 - x_t \end{pmatrix} + h(t, \lambda_t) \begin{pmatrix} \phi \\ 1 - \phi \end{pmatrix} \right] \]
is an equilibrium price process for each constant $\phi \in [0, 1]$. In particular, the set of non redundant equilibria is non empty.

Since the consumption allocation, the interest rate, and the relative risk premia do not depend on the stock prices, they remain constant across the set of equilibria.\textsuperscript{22} As a result, the only equilibrium quantities which vary as $\phi$ ranges through the unit interval are the stock prices, their volatility and the corresponding equity premia and correlations.\textsuperscript{23} In order to investigate

\textsuperscript{22}As explained in Footnote 16 this result is due to the assumption of collinear initial endowments. If this assumption fails to hold then the weighting process, and hence also the consumption allocation, the interest rate, and the relative risk premia, will vary across the set of equilibria. See Remark 3 below for details.

\textsuperscript{23}The equilibria characterized by Proposition 8 are based on a constant repartition of the bubble among the stocks. Since the aggregate bubble is always strictly positive this implies that a given stock either always has a bubble or never does. In other words, these

38
the variations of these quantities across the set of equilibria, I start by computing the volatility matrix of the stocks.

Let \( \phi \) be a constant representing the fraction of the aggregate bubble included in the price of the first stock. Using equations (29) and (30) in conjunction with Itô’s lemma I obtain that the volatility of stock \( i \) is

\[
\sigma_i^t(\phi) = v + \frac{a_i S_t}{S_t^i(\phi)} \left[ \varepsilon v (x_t - \phi) \Sigma_i \frac{\partial h}{\partial \lambda} + x_t (1 - x_t) (1 - h) \sigma_x \right]
\]

where \( a_i = 1 \) for \( i = 1 \) and minus one otherwise, On the other hand, in the unconstrained economy the equilibrium volatility of stock \( i \) is

\[
\sigma_i^{t,u} = \sigma_i^t(\phi|\varepsilon = 0) = v + a_i \left( 1 - \frac{e^i}{\varepsilon_i} \right) \sigma_x
\]

and is independent from \( \phi \) since the bubble component is equal to zero in that case. Combining these two expressions shows that the volatility vector of stock \( i \) can be decomposed as \( \sigma_i^{t,u} + \sigma_i^{t,c}(\phi) \) where

\[
\sigma_i^{t,c}(\phi) \equiv \frac{S_t}{S_t^i(\phi)} \left[ a_i \varepsilon v (x_t - \phi) \Sigma_i \frac{\partial h}{\partial \lambda} + \phi \left( v - \sigma_i^{t,u} \right) h \right]
\]

is the part of the volatility which is due to the presence of constrained agents in the economy. To get some insight on the behavior of the stock volatility across the set of equilibria, Figure 7 plots the standard deviation \( \| \sigma_1^t(\phi) \| \) of the return of the first stock as a function of \( \phi \). As shown by the left

Insert Figure 7 here

panel of the figure, the standard deviation decreases as the bubble share increases and the magnitude if this effect decreases as the dividend share of the stock increases. To understand this effect observe that the bubble value is negatively correlated with the aggregate dividend and hence also with the fundamental value of the stock since it is independent from the equilibria do not allow for bubbles to burst. This can be remedied by constructing non redundant equilibria where the repartition of the bubble among the stocks is allowed to be time and state dependent.
dividend share. As a result, the presence of the bubble lowers the standard deviation of the stock return compared to that of its fundamental value and this effect becomes more important as the fraction of the bubble that is attributed to the stock increases. A decrease in the dividend share increases the relative weight of the bubble in the price of the stock and hence magnifies the decrease in the standard deviation of the stock return.

The right panel of Figure 7 shows that the effect of the constrained agent’s consumption share on the standard deviation of the stock return depends on the bubble share: it increases with $\Sigma$ for low values of $\phi$ and decreases otherwise. This is due to the conflicting effect of the constrained agent’s consumption share on the volatilities of $b$ and $f$. As discussed in the previous section, the contribution of the bubble to the stock volatility is negative and decreases with $\Sigma_0$. On the other hand, the volatility of the fundamental value process is increasing in $\Sigma_0$. When the fraction of the bubble that is attributed the stock is small then the second effect dominates and the standard deviation of the stock return increases with $\Sigma_0$. On the contrary, when $\phi$ is high enough it is the first effect that dominates and the standard deviation of the stock return decreases as the consumption share of the constrained agent increases.

Figure 8 plots the correlation between the stock returns as a function of the share of the bubble that is attributed to the first stock. The figure shows that the correlation between the stocks changes dramatically and can even change signs across the set of equilibria. For example, the left panel of the figure shows that with 75% of constrained agents in the economy and a dividend share of 25%, the correlation ranges from +65% to −50% depending on the repartition of the bubble. Comparing the two panels of the figure shows that the magnitude of the changes and their direction depends heavily on the relative contribution of each stock to the aggregate dividend.
With equal dividend shares the correlation is bell shaped as a function of \( \phi \) and increases with the consumption share of the constrained agent. On the other hand, when the first stock accounts for a quarter of the aggregate dividend, the correlation is monotone decreasing in \( \phi \) and bell shaped as a function of the constrained agent’s consumption share.

Since the volatility matrix of the stocks changes with the repartition of the bubble, the equity premia associated with the stocks

\[
p_t^i(\phi) \equiv \mu_t^i(\phi) - r_t = \sigma_t^i(\phi)^\top \theta_t = (1 + \varepsilon \lambda_t) \|v\|^2 \left[ 1 + \varepsilon \frac{a_i S_t}{S_t^i(\phi)} (x_t - \phi) \Sigma_t \frac{\partial h}{\partial \lambda} \right]
\]

also vary even though the market prices of risk are constant across the set of equilibria. On the other hand, the market portfolio is independent of the repartition of the bubble and this implies that the corresponding market equity premium

\[
p_t \equiv \frac{p_t^1(\phi) S_t^1(\phi) + p_t^2(\phi) S_t^2(\phi)}{S_t^1(\phi) + S_t^2(\phi)} = (1 + \varepsilon \lambda_t) \|v\|^2
\]

is constant across the set of non redundant equilibria. In the absence of portfolio constraints (\( \varepsilon = 0 \)), the stocks and the market as a whole offer the same equity premium which is given by \( \|v\|^2 \). The introduction of a portfolio constraint increases the equity premium on the market portfolio by the amount \( \varepsilon \lambda_t \|v\|^2 \) but can either increase or decrease the premium associated with the individual stocks depending on the level of the dividend share and the repartition of the bubble.

Figure 9 shows that the equity premium increases with the dividend share and decreases with the share of the bubble which is attributed to that stock. The first result is intuitive: as the dividend share increases, the instantaneous covariance between the stock and the aggregate dividend

Insert Figure 9 here
increases and this triggers an increase of the equity premium since the only remunerated source of risk in this economy is the aggregate dividend. To understand the second effect one has to go back to the definition of the equity premium as the product of the risk premium by the volatility. Since the risk premium only depends on aggregate quantities it is constant across the set of equilibria. On the other hand, and as shown by Figure 7, the volatility of the stock decreases as the bubble share increases and this implies the equity premium also decreases.

*Remark 3 (Non collinear endowments).* The fact that consumption plans, interest rate and risk premia are constant across the set of equilibria is entirely due to the assumption of collinear initial endowments. Indeed, this assumption implies that the weighting process only depends on the aggregate stock price and, since the latter is uniquely defined, it follows that all the equilibrium quantities except the prices are uniquely defined.

To see what happens when this restriction fails, assume that the initial portfolio of the first agent can be represented by the vector \((\beta, \alpha_{11}, \alpha_{12})\) for some \(\alpha_{11} \neq \alpha_{12}\). Going through the proofs of Propositions 7 and 8, it is easily deduced that for each \(\phi \in [0, 1]\) the process

\[
S_t(\phi, \lambda_t) \equiv \sum_t \left[ h(t, \lambda_t) \left( \frac{\phi}{1 - \phi} \right) + (1 - h(t, \lambda_t)) \left( \frac{x_t}{1 - x_t} \right) \right]
\]

gives rise to a different non redundant equilibrium in which the consumption allocation, the interest rate and the risk premium are given by equations (5), (6), (9) and (8). As before, the initial value of the weighting process must be determined in such a way that the budget constraint of one of the agents,

\[^{24}\text{According to equation (10), the excess return is generated from two factors: the aggregate dividend and the weighting process. However, the assumptions of the present model imply that the weighting process is perfectly correlated with the aggregate dividend and locally independent from the dividend share.}\]
say the first one, is saturated:

\[ w_1 \equiv \beta + \sum_{i=1}^{2} \alpha_{1i} S_i(\phi, \lambda_0) = e_0^{\frac{T}{1+\lambda_0}}. \]

The only difference with the case of collinear initial endowments, is that the left hand side of this equation, and hence also the initial value of the weighting process, depends on the chosen repartition of the bubble among the two stocks. This in turn implies that the path of the weighting process depends on \( \phi \) and it follows that the consumption allocation, the interest rate and the risk premium vary across the set of equilibria.

Summing up, the above discussion shows that, with non collinear initial endowments, the presence of portfolio constraints gives rise to real rather then nominal indeterminacy.

7 Conclusion

In this article I study a continuous-time, pure exchange economy populated by two groups of agents. Agent’s in the first group have logarithmic utility and face portfolio constraints while agents in the second group have arbitrary utility functions and are unconstrained apart from a standard no bankruptcy condition which prevents them from having negative wealth.

In this setting, I show that the presence of portfolio constraints may give rise to bubbles in equilibrium even though there are unconstrained agents in the economy who can exploit the induced arbitrage opportunity. Furthermore, I demonstrate that the presence of bubbles can be assessed at the aggregate level by analyzing the behavior of a single economic state variable which is the ratio of the agents’ marginal utility of consumption. I illustrate the result by studying a generalization of the restricted participation model of Basak and Cuoco (1998). In this model, the unconstrained agent must find it optimal to borrow and I show that this forces the prices of both the risky and riskless asset to include a bubble in equilibrium.
An important implication of asset pricing bubbles that is novel to this paper is that, when there are multiple traded risky assets, the presence of bubbles in the aggregate price system can lead to multiplicity and real indeterminacy of equilibrium. I illustrate this implication by studying the set of equilibria in a two stock economy where one of the agent’s faces a risk constraint that limits the volatility of his wealth. In particular, I show that, across the set of equilibria, the variations of key equilibrium quantities, such as stock volatilities, correlation and equity premia, can be substantial.

A Technical results

This appendix is devoted to the study of the nonnegative local martingale which appears in the examples in Sections 5 and 6. I start by establishing the existence of this process for arbitrary parameters.

Lemma 1. Let \((x, b) \in (0, \infty) \times \mathbb{R}^n\). Then the one dimensional stochastic differential equation defined by

\[
X_t = x - \int_0^t X_s (1 + X_s) b^\top dB_s
\]  

admits a unique strong solution. This solution is a strictly positive local martingale but fails to be martingale unless \(b\) is the zero vector.

Proof. The existence of unique strictly positive strong solution to equation (31) follows from Lemma 1 of Basak and Cuoco (1998). Since the solution is a stochastic integral it is a local martingale. Assume that it is a true martingale so that \(Q(A) \equiv E \left[ 1_{\{A\}}(X_T/x) \right] \) defines an equivalent probability measure. Girsanov’s theorem then implies that the process

\[
W_t \equiv B_t + \int_0^t (1 + X_s) b ds
\]

is a Brownian motion under the probability measure \(Q\) and it follows that
process $X$ solves the stochastic differential equation

$$dX_t = X_t(1 + X_t)^2 \|b\|^2 dt - X_t(1 + X_t)b^\top dW_t. \quad (32)$$

Let $p(x) = 1 - 1/x$ denote the scale function associated with this stochastic differential equation and consider the function

$$v(x) \equiv \int^x_1 p'(\xi) \int^\xi_1 \frac{2/p'(\theta)}{\theta^2(1 + \theta)^2 \|b\|^2} d\theta d\xi$$

According to Feller's test for explosions (see Karatzas and Shreve (1998, Theorem 5.5.29)) the solution to equation (32) explodes with strictly positive probability under $Q$ since

$$\lim_{x \to \infty} v(x) = \frac{\log(4) - 1 \|b\|^2}{\|b\|^2} < \infty.$$ 

On the other hand, since $X$ is a nonnegative supermartingale under $P$ it is almost surely finite under that probability measure. This contradicts the equivalence between $P$ and $Q$ and establishes the desired result. Q.E.D.

**Lemma 2.** Let $M$ be a strictly positive, uniformly integrable martingale and consider the nonnegative local martingale defined by

$$Y_t = e^{-\langle \log M, \log X \rangle_t} M_t X_t$$

where $\langle \cdot, \cdot \rangle$ denotes the quadratic covariation. Then $E[Y_T] = (Y_0/X_0)E[X_T]$. In particular, the process $Y$ is a strict local martingale.

**Proof.** The definition of $Y$ and the martingale property of $M$ imply

$$E[Y_T] = (Y_0/X_0)E \left[ e^{-\langle \log M, \log X \rangle_T} X_T (M_T/M_0) \right]$$

$$= (Y_0/X_0)E \left[ e^{-\langle \log M, \log X \rangle_T} X_T \right] = (Y_0/X_0)E \left[ X_T \right]$$

where $E$ denotes the expectation operator under the equivalent probability measure defined by $\mathbb{P}(A) = E \left[ 1_A (M_T/M_0) \right]$ and $X \equiv Y/M$. Using Itô’s
lemma and the definition of $X$ I obtain that

$$
dX_t = X_t \left[ \frac{dX_t}{X_t} - \frac{d\langle M, X \rangle_t}{M_t X_t} \right] = -X_t(1 + X_t)b^\top dB_t
$$

where $B$ is a Brownian motion under $P$ by Girsanov theorem. In conjunction with the uniqueness in law of solutions of stochastic differential equation (see (Karatzas and Shreve 1998, Chapter 5)) this implies that the law of $X$ under $P$ is the same as the law of $X$ under $P$. In particular,

$$
E[Y_T] = Y_0 E_\bar{X} \left[ \frac{X_T}{X_0} \right] = Y_0 E[X_T/X_0] < Y_0.
$$

where the inequality follows from Lemma 1. Q.E.D.

**Lemma 3.** The expectation function of $X$ is given by

$$
E[X_t] = \sum_{k=1}^{2} a_{k+1} \left( x + \frac{1 + a_k}{2} \right) \Phi \left[ -\frac{\|b\| \sqrt{t}}{2} - \frac{a_k}{\|b\| \sqrt{t}} \log \frac{1 + x}{x} \right]
$$

where $a_k = [-1]^k$ and $\Phi$ is the cumulative distribution function of a standard Gaussian random variable.

In order to establish the validity of Lemma 3 we will rely on the following useful result due to Elworthy et al. (1999).

**Lemma 4.** Let $M$ be a nonnegative continuous local martingale and $\tau$ be a stopping time. Then

$$
E[M_\tau] = M_0 - \lim_{m \to \infty} m P \left[ \max_{t \leq \tau} M_t \geq m \right]. \tag{33}
$$

*Proof of Lemma 3.* In order to apply the result of Lemma 4 to the local martingale $X$ we need to compute the probability

$$
p(t, x, m) \equiv P \left[ \max_{t \leq \tau} X_t \geq m \mid X_0 = x \right]
$$

Well–known results on one dimensional diffusion processes (see for example...
Borodin and Salminen (2002, II.10) show that

\[
\int_0^\infty e^{-\alpha t} p(t, x, m) dt = \frac{1}{\alpha} E_x \left[ e^{-\alpha T_m} \mid X_0 = x \right] = \frac{1}{\alpha E_x} \left( e^{-\alpha T_m} \right)_{X_0 = x} = \frac{1}{\alpha} \phi(x) \phi(m)
\]

where \( T_m \) denotes the first hitting time of the level \( m \) and \( \phi \) is the unique increasing solution to the Sturm–Liouville problem

\[
\frac{1}{2} \|b\|^2 x^2 (1 + x)^2 \phi''(x) = \alpha \phi(x)
\]

with \( \lim_{x \to 0} \phi(x) = 0 \). Solving this ordinary differential equation I obtain

\[
\phi(x) = \sqrt{x(1 + x)} \left( \frac{x}{1 + x} \right)^\frac{1}{4} \sqrt{\frac{\alpha}{\|b\|}}
\]

Laplace transform inversion formulae (see for example Erdelyi (1954)) and tedious algebra then show that

\[
p(t, x, m) = \left( \frac{x}{m} \right) \Phi \left[ \frac{\|b\|}{2} \sqrt{t} - \frac{1}{\|b\| \sqrt{t}} \log \frac{m(1 + x)}{x(1 + m)} \right] \\
+ \left( \frac{1 + x}{1 + m} \right) \Phi \left[ -\frac{\|b\|}{2} \sqrt{t} - \frac{1}{\|b\| \sqrt{t}} \log \frac{m(1 + x)}{x(1 + m)} \right]
\]

Multiplying both sides of this expression by \( m \), plugging the result into equation (33) and letting \( m \) go to infinity gives the desired expression after some straightforward simplifications. Q.E.D.

**Lemma 5.** The transition density of \( X \) is given by

\[
p_t(x, u) = \sum_{k=1}^2 e^{-\|b\|^2 k} \left[ \frac{x(1 + x)}{u^3(1 + u)^3} \right]^{\frac{1}{2}} \varphi \left[ \log \left( \frac{x}{1 + x} \left[ \frac{u}{1 + u} \right]^a_k \right) \right] ; \|b\| \sqrt{t}
\]

where \( a_k = (-1)^k \) and \( \varphi(x; \sigma) \) is the density of Gaussian random variable with mean zero and standard deviation \( \sigma \).

**Proof.** Let \( (\gamma, u) \) be a pair of strictly positive constants, and consider the
strictly positive bounded function defined by
\[ g(x) = \int_0^\infty e^{-\gamma t} P[X_t \leq u \mid X_0 = x] dt = E \left[ \int_0^\infty e^{-\gamma t} 1_{\{X_t \leq u\}} dt \right]. \]

Using Itô’s lemma it is easily deduced that \( g \) is the unique bounded and once continuously differentiable solution to
\[ \frac{1}{2} \|b\|^2 x^2 (1 + x)^2 g''(x) = \gamma g(x) - 1_{\{x \leq u\}} \]
with boundary condition \( g(0) = 1/\gamma \). Solving this differential equation I obtain that the function \( g \) is given by
\[ g(x) = 1_{\{x \leq u\}} \frac{1}{\gamma} - \frac{1 + 2u + 2\theta}{4\gamma \theta} \left[ \frac{x(1 + x)}{u(1 + u)} \right]^{\frac{1}{2}} \left[ \frac{xu}{(1 + u)(1 + x)} \right]^\theta + \frac{1 + 2u + \text{sign}(x-u)2\theta}{4\gamma \theta} \left[ \frac{x(1 + x)}{u(1 + u)} \right]^{\frac{1}{2}} \max \left[ \frac{x(1 + u)}{u(1 + x)} ; \frac{u(1 + x)}{x(1 + u)} \right]^\theta \]
where \( \theta \equiv \sqrt{1/4 + 2\gamma/\|b\|^2} \). In order to obtain the transition density from the expression of the function \( g \) it suffices to observe that
\[ \frac{\partial g(x)}{\partial u} = \int_0^\infty e^{-\gamma t} p_t(x, u) dt. \]

Computing the derivative on the left hand side of the above equation shows that the Laplace transform of the transition density is given by
\[ p_{\gamma}(x, u) = \frac{1}{\sigma^2 \theta} \left[ \frac{x(1 + x)}{u^3(1 + u)^3} \right]^{\frac{1}{2}} \max \left[ \frac{x(1 + u)}{u(1 + x)} ; \frac{u(1 + x)}{x(1 + u)} \right]^\theta - \frac{1}{\sigma^2 \theta} \left[ \frac{x(1 + x)}{u^3(1 + u)^3} \right]^{\frac{1}{2}} \left[ \frac{xu}{(1 + u)(1 + x)} \right]^\theta \]
Finally, using standard formulae (see Erdelyi (1954)) to invert this Laplace transform and simplifying the expression yields the desired result. Q.E.D.

**Lemma 6.** For every constant \( \tau \geq 0 \) we have
\[ \tau X_t - E \left[ \int_t^{t+\tau} X_s ds \mid \mathcal{F}_t \right] = \frac{2X_t}{\|b\|^2} \beta \left[ \tau, \log \frac{X_t}{1 + X_t} \right]. \]
In the above equation, the function $\beta$ is defined by

$$
\beta(\tau, x) \equiv 2 \sum_{k=1}^{2} e^{-(1+a_k)\frac{\tau}{2}} \left( \frac{\|b\|^2 \tau}{2} - a_k x \right) \Phi \left[ -\frac{a_k \|b\| \sqrt{\tau}}{2} + \frac{x}{\|b\| \sqrt{\tau}} \right]
$$

where $a_k = [-1]^k$ and $\Phi$ is the cumulative distribution function of a standard Gaussian random variable.

**Proof.** This follows from Lemma 3, the Markov property of $X$ and tedious algebra, I omit the details. Q.E.D.

**Lemma 7.** The bounded process $\Sigma \equiv X/(1 + X)$ converges to zero as time goes to infinity and its expected value is given by

$$
E[\Sigma_t] = \Sigma_0 - \frac{\Sigma_0}{1 - \Sigma_0} \left( H_t(1, \Sigma_0) - e^{t\|b\|^2} H_t(3, \Sigma_0) \right).
$$

In the above equation, the function $H$ is defined by

$$
H_t(a, x) \equiv x^{-\frac{a-1}{2}} \Phi \left( \frac{\log x}{\|b\| \sqrt{t}} + \frac{a\|b\|}{2 \sqrt{t}} \right) + x^{-\frac{a+1}{2}} \Phi \left( \frac{\log x}{\|b\| \sqrt{t}} - \frac{a\|b\|}{2 \sqrt{t}} \right) - x
$$

for all $x > 0$ where $\Phi$ is the cumulative distribution function of a standard Gaussian random variable.

**Proof.** Using the dynamics of $X$ and Itô’s lemma I obtain that

$$
d\Sigma_t = \frac{-X_t}{1 + X_t} b^\top \left[ dB_t + bX_t dt \right] = -\Sigma_t b^\top \left[ dB_t + bX_t dt \right]
$$

where the second equality follows from the definition of $\Sigma$. The unique solution to this stochastic differential equation is given by

$$
\frac{\Sigma_t}{\Sigma_0} = \exp \left[ -\int_0^t \|b\|^2 X_s ds - b^\top B_t - \frac{1}{2} \|b\|^2 t \right] \leq \exp \left[ -b^\top B_t - \frac{1}{2} \|b\|^2 t \right]
$$

where the inequality follows from the fact that $X$ is nonnegative. Since the right hand side of the above inequality converges to zero as $t$ increases to infinity I conclude that the process $\Sigma$ converges to zero.

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To complete the proof, let $\gamma$ be a strictly positive constant and consider the nonnegative function defined by

$$g(x) \equiv E\left[ \int_0^\infty e^{-\gamma t} \Sigma_t dt \right] = E\left[ \int_0^\infty e^{-\gamma t} \frac{X_t}{1 + X_t} dt \right].$$

Using Itô’s lemma it can be shown that $g$ is the unique bounded solution to the Sturm–Liouville problem

$$\frac{1}{2} x^2 (1 + x)^2 \|b\|^2 = \gamma g(x) - \frac{x}{1 + x}$$

with $\lim_{x \to 0} g(x) = 0$. Solving this differential equation and imposing the appropriate boundary conditions shows that

$$g(x) = \frac{x}{1 + x} \left( \frac{1}{\gamma} \right) \left[ 1 + \frac{x\|b\|^2}{\|b\|^2 - \gamma} + \frac{(1 + x)\|b\|^2}{\gamma - \|b\|^2} \left( \frac{x}{1 + x} \right)^{\theta - \frac{1}{2}} \right]$$

where the constant $\theta$ is defined as in the proof of Lemma 5. Using standard formulae (see Erdelyi (1954)) to invert this Laplace transform, simplifying the resulting expression and making the change of variable $\Sigma_0 = x/(1 + x)$ gives the desired conclusion. Q.E.D.

B Proofs

Proof of Proposition 1. Assume that the market is in equilibrium. Then the value function of the unconstrained agent is finite and the result follows from the first order conditions of optimality provided that

$$\limsup_{x \to \infty} \frac{x u_1'(x)}{u_1(x)} < 1.$$ 

This growth condition is referred to as reasonable asymptotic elasticity and can be shown to hold for all the standard utility functions, see Kramkov and Schachermayer (1999) for details. Q.E.D.

Proof of Proposition 2. Assume that the market is in equilibrium. Then
the value function of the agent is finite and the proposition follows from the results of Cvitanić and Karatzas (1992).

Q.E.D.

Proof of Proposition 3. Using the definition of $\lambda$ in conjunction with Itô's lemma and the results of Propositions 1 and 2 gives

$$
\frac{d\lambda_t}{\lambda_t} = \left[ \delta_t(b_t) + \theta_{2t}(\theta_{2t} - \theta_t) \right] dt + \left[ \theta_{2t} - \theta_t \right]^\top dB_t
$$

and so all there is to prove is that the term in the first square bracket is equal to zero. The definition of the support function and Fenchel's duality theorem (see Urruty and Lemaréchal (2001, p.228)) imply that

$$
\inf_{b \in B_t} \left\{ \delta_t(b) + \frac{1}{2} \|\theta_t + \sigma_t^{-1} b\|^2 \right\} = \max_{p \in C_t} \left\{ p^\top \sigma_t \theta_t - \frac{1}{2} \|p\|^2 \right\}.
$$

Since $C_t$ is a closed convex subset of $\mathbb{R}^n$ the maximization problem on the right hand side admits a unique solution which is given by

$$
p_t = (\sigma_t^{-1})^\top \Pi \left[ \theta_t \mid \sigma_t \ C_t \right].
$$

Furthermore, the variational characterization of the projection operator (see Urruty and Lemaréchal (2001, p.47)) shows that

$$
\max_{a \in C_t} \left\{ (p_t - a)^\top \sigma_t \left( \sigma_t^{-1} p_t - \theta_t \right) \right\} = 0.
$$

In conjunction with the definition of the support function, this implies that the vector $k_t \equiv \sigma_t (\sigma_t^{-1} p_t - \theta_t)$ belongs to the set $B_t$ and satisfies

$$
\delta_t(k_t) = -p_t^\top k_t = -\left[ \theta_t + \sigma_t^{-1} k_t \right]^\top \sigma_t^{-1} k_t.
$$

Plugging this relation back into Fenchel’s identity we easily deduce that the vector $k_t$ attains the infimum on the right hand side of equation (4) and it follows that $k_t = b_t$. Finally, using the definition of the implicit risk premium $\theta_2$ yields the desired result. Q.E.D.

Proof of Proposition 4. The result follows from Propositions 1 and 2, the
definition of the representative agent’s utility function and Itô’s lemma, I omit the details. Q.E.D.

Proof of Corollary 1. Assume that the market is in equilibrium. The definition of \( \Gamma \) and equation (34) show that \( \Gamma_t = \theta_{2t} - \theta_t \) where \( \theta_2 \) is the risk premium faced by the constrained agent. On the other hand, the result of Proposition 2 and the proof of Proposition 3 show that \( \theta_{2t} = \Delta_t + \Pi[\theta_t]_t^T \). Combining these two equations with the expression of the unconstrained risk premium in equation (34) shows that \( \Gamma \) is a solution to equation (12). Q.E.D.

Proof of Theorem 1. Assume that the equilibrium is free of bubbles so that \( f_i^t = S_i^t \) for all \( 1 \leq i \leq n \). This implies that \( S^t = \sum_{i=1}^n f_i^t \) and comparing this identity with equation (19) I conclude that

\[
0 = E \left[ \int_t^T \frac{\lambda_t - \lambda_s}{u_c(e_0, \lambda_0)} ds \bigg| \mathcal{F}_t \right] = \int_t^T \frac{\lambda_t - E[\lambda_s|\mathcal{F}_t]}{u_c(e_0, \lambda_0)} ds.
\]

According to Proposition 3 the weighting process is a nonnegative local martingale and hence a supermartingale. Since the marginal utility of the representative agent is strictly positive, this implies that the term inside the integral on the right side is equal to zero at all times and it follows that the weighting process is a true martingale.

In order to complete the proof, assume that the weighting process is a strict local martingale so that the discounted bubble value

\[
N_t = \frac{u_c(e_t, \lambda_t)}{u_c(e_0, \lambda_0)} b_t = E \left[ \int_t^T \frac{\lambda_t - \lambda_s}{u_c(e_0, \lambda_0)} ds \bigg| \mathcal{F}_t \right]
\]

has a strictly positive initial value. Applying Itô’s product rule on the righthand side of the above expression I obtain

\[
u_c(e_0, \lambda_0) N_t = \lambda_0 T + \int_0^t (T - s) d\lambda_s - E \left[ \int_0^T \lambda_s ds \bigg| \mathcal{F}_t \right]
\]

and it follows that \( N \) is a local martingale. Because it is nonnegative, this process is also a supermartingale and it will be a true martingale if and
only if $N_0 = E[N_T]$. Since $N_T = 0$ this can only be the case if $N_0 = 0$ or equivalently if the weighting process is a true martingale. Q.E.D.

**Proof of Proposition 5.** Assume that $S$ is the stock price in a non redundant equilibrium. The invertibility of the volatility matrix, the definition of the risk premium and Proposition 4 imply

$$dS_i^t = r_t S_i^t dt + S_i^t (\sigma_t^i)^\top (dB_t + \theta_t \, dt) - e_i^t \, dt.$$ 

where $(r, \theta)$ are defined as in equations (8)–(9). Combining these dynamics with the definition of the unconstrained state price density and applying Itô’s lemma shows that the process $M^i$ is a local martingale. On the other hand, Propositions 1, 2 and 4 show that the consumption allocation of Proposition 4 is optimal. This in turn implies that the agents’ wealth satisfy

$$\sum_a W_{at} = \sum_i S_i^t$$

and it follows that the market for the riskless asset clears. Finally, applying Itô’s lemma on both sides of this equality and equating the volatility terms shows that the agents’ portfolios satisfy equation (22) and the invertibility of $\sigma$ now implies that the stock market clears. Q.E.D.

**Proof of Proposition 6.** Equations (7), (24) and (26) imply that the initial value of the weighting process is uniquely given by

$$\lambda_0 = \frac{\alpha e_0 T + \beta}{(1 - \alpha) e_0 T - \beta} = \frac{w_2}{w_1}.$$
On the other hand, since the volatility of the stock is equal to that of the aggregate dividend it follows from Corollary 1 and the definition of $C_t$ that the volatility of the weighting process satisfies

$$v + \frac{\Gamma_t}{1 + \lambda_t} = \left[ v - \frac{\lambda_t \Gamma_t}{1 + \lambda_t} \right]^+ - \left[ \varepsilon v - \frac{\lambda_t \Gamma_t}{1 + \lambda_t} \right]^+.$$

Solving this equation gives $\Gamma_t = -(1 + \lambda_t) \varepsilon v$ and it now follows from Lemma 1 that the weighting process is the unique solution to equation (25).

The expressions for the consumption plans, trading strategies, interest rate and risk premium follow from the definition of $\Gamma$ and Propositions 1, 2 and 4. I omit the details.

Using the definition of $\Gamma$ and Lemma 1 I deduce that $\lambda$ is a strict local martingale. As a result, it follows from Theorem 1 and equations (20), (26) that the stock price includes a bubble which satisfies

$$\frac{b_t}{S_t} = h(t, \lambda_t) \equiv \frac{\lambda_t}{1 + \lambda_t} - \frac{1}{\tau(1 + \lambda_t)} E \left[ \int_t^T \lambda_s ds \bigg| \mathcal{F}_s \right]$$

with $\tau = T - t$. Using Lemma 6 to compute the expectation on the right hand side and simplifying the result gives

$$h(t, \lambda) = \frac{\lambda}{1 + \lambda} \left[ 1 + \frac{2}{\tau \|\dot{v}\|^2} \beta \left( \tau, \log \frac{\lambda}{1 + \lambda} \right) \right]$$

where $\beta$ is defined as in Lemma 6 albeit with $\|b\|^2 = \|\dot{v}\|^2$. Using Remark 1 I have that the riskless asset includes a bubble if and only if the nonnegative local martingale $Y \equiv \xi_1 S^0$ is a strict local martingale in which case

$$\frac{b_t^0}{S_t^0} = h^0(t, \lambda_t) \equiv 1 - E \left[ \frac{Y_T}{Y_t} \bigg| \mathcal{F}_t \right].$$

The strict martingale property follows by observing that $Y$ satisfies the conditions of Lemma 2 with $X = \lambda$ and

$$M_t = \exp \left[ (\varepsilon - 1) u B_t - \frac{1}{2} [(\varepsilon - 1) u]^2 t \right].$$
On the other hand, using the results of Lemmas 3 and 2 I obtain that the relative bubble value is given by

\[ h^0(t, \lambda_t) = 1 - E[\lambda_T / \lambda_t | \mathcal{F}_t] = \Phi \left( -\frac{\hat{\nu}}{2} \sqrt{\tau} - \frac{1}{\hat{\nu} \sqrt{\tau}} \log \frac{\lambda}{1 + \lambda} \right) \]

where \( \Phi \) denotes the standard Gaussian cumulative distribution function.

To complete the proof it remains to show that \( h \leq h^0 \). This easily follows from equations (35), (36), the supermartingale property of the weighting process and Fubini’s theorem, I omit the details. Q.E.D.

**Proof of Proposition 7.** The proof of this proposition is similar to that of Proposition 6, I omit the details. Q.E.D.

**Proof of Proposition 8.** Let \( \phi \in [0, 1] \) be fixed. In order to establish the result it suffices to show that the process \( S(\phi) \) satisfies the conditions of Proposition 5. The definition of \( (b, f) \) and the fact that \( \phi \) is constant imply that \( (M^i)^2 \) are local martingales. On the other hand, the definition of \( \phi \) and equations (29)–(30) imply that \( \sum_i S_i(\phi) = \overline{S}_t \).

To complete the proof, let \( \sigma(\phi) \in \mathbb{R}^{2 \times 2} \) denote the volatility of the candidate price process. Using the dynamics of \( (e, x) \) in conjunction with the definition of the function \( h \) and Itô’s lemma I obtain that

\[
\det(\sigma_t(\phi)) = x_t(1 - x_t)(1 - h(t, \lambda_t)) \frac{\overline{S}_t^2}{S_t^2(\phi)S_t^2(\phi)} \det \begin{pmatrix} v_1 & \sigma_{x1} \\ v_2 & \sigma_{x2} \end{pmatrix}.
\]

Since \( x \in (0, 1) \), \( h < 1 \) and the vectors \( v \) and \( \sigma_x \) are linearly independent by assumption, the above expression implies that the matrix \( \sigma(\phi) \) is almost everywhere non singular. Q.E.D.
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Figure 1: Expected value of the weighing process

Notes: This figure plots the expected value of the weighting process as a function of the initial value (panel (a)) and the horizon (panel (b)). The 45° line in panel (a) gives the expected value under the assumption that the process is a true martingale. In both panels the constraint weighted volatility is $\hat{\sigma} = 20\%$. 
Figure 2: Transition densities

Notes: This figure compares the transition density of the weighting process (panels (a) and (c)) and of the corresponding exponential martingale (panels (b) and (d)). In all panels the constraint weighted volatility is \( \hat{\sigma} = 20\% \).
Figure 3: Tail probabilities

Notes: This figure compares the tail probabilities (equation (28)) of the weighting process (panel (a)) and of the corresponding exponential martingale (panel (b)). The initial value of the processes is set to $\Lambda_0 = \Psi_0 = 1$ and the constraint weighted volatility is $\hat{v} = 20\%$. 
Figure 4: Expected consumption share

Notes: This figure plots the expected consumption share of the constrained agent as a function of the horizon for different values of the initial consumption share. The circled lines give the same expected values under the assumption that the weighting process equals the exponential martingale of equation (27). The constraint weighted volatility is $\hat{v} = 20\%$. 
Figure 5: Relative size of the bubbles

Notes: This figure plots the relative size of the bubble on the stock (panels (a) and (c)) and on the riskless asset (panels (b) and (d)) as functions of the horizon and the initial consumption share of the constrained agent. In the top panels, the constraint weighted volatility is set to $\hat{\nu} = 20\%$. In the bottom panels, the model has an horizon of ten years and the volatility of the dividend is equal to $\nu = 20\%$. 
Figure 6: Bubble and stock volatility

Notes: This figure plots the contribution of the bubble to the volatility of the stock as a function of the constrained agent’s initial consumption share. The horizon of the model is ten years and the volatility of the dividend is $v = 20\%$. 
Figure 7: Standard deviation of stock returns

Notes: This figure plots the standard deviation of the first stock return as a function of the bubble weight \( \phi \). In both panels, the volatility of the dividend is 20\%, the volatility of the dividend share is 15\%, the model has a ten year horizon and the portfolio constraint is \( \varepsilon = 50\% \).
Figure 8: Correlation between stocks

Notes: This figure plots the correlation between the two stocks as a function of the bubble weight $\phi$. The volatility of the aggregate dividend is 20%, the volatility of the dividend share is 15%, the model has an horizon of ten years and the level of the portfolio constraint is set to $\varepsilon = 50\%$. 

(a): Div. share $x_0 = 25\%$

(b): Div. share $x_0 = 50\%$
Figure 9: Equilibrium equity premia

Notes: This figure plots the equity premia on both stocks as functions of $\phi$ for various levels of the dividend share. In both panels, the horizontal line gives the equity premium on the market portfolio. The volatility of the aggregate dividend is 20%, the volatility of the dividend share is 15%, the consumption share of the constrained agent is 50%, the model has an horizon of ten years and the level of the portfolio constraint is set to $\varepsilon = 50\%$. 