Nonparametric Estimation of Time-Varying Characteristics of Intertemporal Asset Pricing Models

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Abstract

Nonlinear techniques to estimate dynamic general equilibrium (DSGE) models relying on implications of the first–order condition like the generalized method of moments, or building on simulations of numerical solutions have been found to converge rather slowly. We want to contribute to those studies by presenting a new estimation technique based on parameterized expectations of Den Haan and Marcet (1990), which considers full structural information of the DSGE model, but does not require numerical solutions. A Monte Carlo study for the stochastic growth model reveals precise estimation results. Since we account for full structural information of the DSGE model, the method is well suited to assess the time–varying Sharpe ratio. Notice that the research using linearized dynamic models as a standard, led to different conclusions about the time–varying behaviour of the risk/return trade–off.

1 Introduction

Most nonlinear estimation schemes for DSGE models rely on either moment conditions arising from first–order conditions, or they consider information efficiently by relying on numerical solu-

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We want to contribute to those studies by proposing a nonlinear estimation technique using full structural information while not requiring numerical solutions. Inspired by parameterizing expectations in Den Haan and Marcet (1990), we approximate conditional expectations in the Euler equation by a polynomial function in closed–form. We propose to choose structural parameters so as to minimize Euler equation errors. Standard statistical tests apply directly to check accuracy of the estimated Euler equation. In a Monte Carlo study of the stochastic growth model we get precise estimates of the structural parameters for polynomials of order higher than one.

Since we use full structural information of the DSGE model, the new estimation technique is well suited to assess time–varying asset market facts, and we also apply the estimation method to assess the time–varying Sharpe ratio which is an open issue in the asset pricing literature. In particular, based on linearized DSGE models there are different conclusions.

Macroeconomic asset pricing literature is concerned with many puzzling aspects in the financial market. Most prominent are the equity premium puzzle, the risk–free rate puzzle, and the volatility puzzle. Another puzzle arises with respect to the risk–to–reward measure, i.e. the Sharpe ratio, over the business cycle. It has been shown that one can improve on some of those puzzles by the introduction of habit formation, see Campbell and Cochrane (1999), or Boldrin, Christiano and Fisher (1997, 2001). This paper is concerned less with the level, but rather with the time variation of the financial measures. Following the above literature we in particular focus on the time varying Sharpe ratio. Campbell and Cochrane (1999) obtain a Sharpe–ratio moving counter–
cyclically with respect to the business cycle for an exchange economy. Yet, Lettau and Ludvigson (2003) have pointed out that the risk–return trade–off changes over time with a standard deviation considerably exceeding that suggested by Campbell and Cochrane (1999). In this contribution we demonstrate that the model with habit formation is able to match the observed standard deviation of the Sharpe ratio, since (i) the variables regarding production exhibit a larger standard deviation than consumption, and (ii) the restrictive assumptions associated with the log–linear solution of Campbell and Cochrane are relaxed. We estimate the model based on the technique discussed above. Based on quarterly U.S. data we estimate the structural parameters of the model and investigate its Sharpe ratio for preferences with habit persistence. We find indication that the model is able to capture the time variation of the Sharpe ratio, i.e. its standard deviation. We also obtain a negative relation between the Sharpe ratio and the business cycle.

The remainder is organized as follows. In Section 2 we present the estimation technique, and provide the results of the Monte Carlo study for the stochastic growth model. In Section 3 we estimate time–varying asset market characteristics of the inter–temporal business cycle model with habit formation. Section 4 concludes.

2 DSGE Model Estimation With Parameterized Expectations

2.1 A Full Structural Information Estimation Technique for DSGE Models without Simulation

We build an estimation procedure for dynamic optimization problems based on first-order conditions of the form

\[ z_t = E_t [f(z_{t+1}, z_{t+2}, \ldots; \theta)] , \tag{1} \]

where \( z_t \) is a vector of state and control variables, and \( \theta \) refers to the structural (deep) parameters of the economic model. Function \( f \) is supposed to satisfy usual regularity conditions. Den Haan and Marcet (1994) claim, that transversality conditions or the assumption of time-invariant solutions may ensure a unique solution. For instance, this is shown for a dynamic asset pricing model satisfying a multi–period fund separation condition in Gerber, Hens and Woehrmann (2008).

To solve dynamic optimization problems numerically, Den Haan and Marcet (1990) suggest
to parameterize expectations by a linear or preferably nonlinear function $\psi$, which depends on a parameter vector $\omega$ and the partition of the states of nature $F_t$, i.e.

$$E_t[f(z_{t+1}, z_{t+2}, \ldots ; \theta)] = E[f(z_{t+1}, z_{t+2}, \ldots ; \theta)|F_t] = \psi(F_t; \omega). \tag{2}$$

Hence, determining expectations given the trajectories of $z$ is a stochastic approximation problem,

$$\min_\omega \|f(z; \theta) - \psi(F; \omega)\|, \tag{3}$$

where $\|\cdot\|$ denotes the Euclidean norm, which is calculated in data samples as mean squared error.

We assume that observed values of the state variables $z$ are generated by the system (1), (2) and (3). Hence, we estimate the structural parameters by choosing $\theta$, such that the parameterized expectations (2) imply a time series for $z$ which is closest to the empirical observations of $z$. In other words, we intend to minimize the Euler equation error. Formally, we estimate the structural parameters by

$$\hat{\theta} = \arg\min_\theta \|z - \psi(F; \hat{\omega})\| \quad s.t. \quad \hat{\omega} = \arg\min_\omega \|f(z; \theta) - \psi(F; \omega)\|, \tag{4}$$

where $\hat{\theta}$ can be found by grid–search, while the adequate algorithm for determining $\hat{\omega}$ is given by the concrete functional form of $\psi(\cdot)$. We propose the polynomial function which can be estimated by ordinary least squares by applying the logarithm operator.

Subsequently, it is shown by simulating time series from the neoclassical stochastic growth model that this inferential approach reveals strong results in samples as small as we are facing in real world applications.

### 2.2 A Monte Carlo Study

#### 2.2.1 The Stochastic Growth Model

In financial economics, the pure exchange economy of Lucas (1978), or the neoclassical growth model allowing for saving, see e.g. Brock and Mirman (1972), Breeden (1979), or Kydland and Prescott (1982), are widely considered.\footnote{The dimension of the vector $\omega$ equals the number of parameters in the function chosen to specify $\psi$.}

The representative agent is assumed to choose consumption, $C_t$, either habit formation or loss aversion may be applied, see e.g. Campbell and Cochrane (1999) and Barberis, Huang, and Santos (2001), respectively.
\( t = 1, 2, \ldots, T \), so as to maximize current and discounted future utilities (using discount factor \( \beta \in [0, 1] \)) arising from consumption. In the model of Lucas (1978) dividends have to be consumed instantaneously. The stochastic growth model allows for saving by introducing a capital stock \( K_t \), \( t = 1, 2, \ldots, T \). The representative investor solves

\[
\max_C E_t \sum_{\tau=0}^{\infty} \beta^\tau U(C_{t+\tau}).
\]

The choice of optimal policies, \( C_t \) is constrained by a budget equation (5), where capital stock is decreased by consumption and depreciation, denoted by \( \rho \in [0, 1] \), and is increased by output, \( Y_t \), \( t = 1, 2, \ldots, T \), obtained from the Cobb–Douglas production function (6) with capital share \( \alpha \),

\[
K_{t+1} = (1 - \rho)K_t + Y_t - C_t, \tag{5}
\]

\[
Y_t = A_t K_t^\alpha. \tag{6}
\]

In the context of this model business cycles are assumed to be driven by an exogenous stochastic technology shock, \( A_t \), \( t = 1, 2, \ldots, T \), following the autoregressive process

\[
\ln A_t = \phi \ln A_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2) \tag{7}
\]

with persistence \( \phi \in [0, 1] \). The power utility function

\[
U(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}
\]

with constant relative risk aversion \( \gamma > 0 \) is a common choice for the utility function.

The Euler equation derived from the first order condition of this inter–temporal optimization problem has the form

\[
1 = E_t [M_{t+1} R_{t+1}] \tag{8}
\]

with stochastic discount factor

\[
M_{t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\gamma}
\]

and gross return on capital

\[
R_{t+1} = \alpha A_{t+1} K_t^{\alpha-1} + 1 - \rho.
\]
2.2.2 Monte Carlo Study of the Estimation Procedure

To test the reliability of the estimation scheme (4) for DSGE models, we are generating sample paths for the neoclassical stochastic growth model in Section 2.2.1 using the numerical simulation of Den Haan and Marcet (1990). We set the number of replications in the Monte Carlo study to 500, and simulate trajectories of consumption and capital stock for different sample sizes with risk aversion $\gamma = 0.5$, discount factor $\beta = 0.95$, $\phi = 0.95$, $\alpha = 0.33$, and $\sigma_c = 0.1$. We do not consider depreciation in this model and set $\rho = 0$.

We let $K_{t-1}$ and $A_t$ represent the state of nature $F$, and apply a polynomial function $\psi_n(K_{t-1}, A_t; \omega)$ of degree $n$ to approximate conditional expectations in the Euler equation. To be concrete, we estimate the logarithms of

$$\psi_1(K_{t-1}, A_t; \omega) = \omega_1 K_{t-1}^{\omega_2} A_t^{\omega_3},$$

$$\psi_2(K_{t-1}, A_t; \omega) = \psi_1(K_{t-1}, A_t; \omega) K_{t-1}^{2\omega_4} A_t^{\omega_5} A_t^{2\omega_6},$$

and

$$\psi_3(K_{t-1}, A_t; \omega) = \psi_2(K_{t-1}, A_t; \omega) K_{t-1}^{3\omega_7} (K_{t-1} A_t)^{\omega_8} (K_{t-1} A_t)^2 A_t^{3\omega_{10}},$$

using ordinary least squares. Den Haan and Marcet (1994) show numerically using accuracy checks that the stochastic growth model described in Section 2.2.1 is solved by $\psi_3(K_{t-1}, A_t; \omega)$ with the values for $\omega$ reported in Table 1.

Assuming that consumption, capital stock, output, and technology shocks are observable in reality, only two parameters must be estimated, constant relative risk aversion $\gamma$ and discount factor $\beta$. Note that depreciation is blinded out in the benchmark model.

Table 2 shows summary statistics of the Monte Carlo study of the estimation of the estimator (4) applied to the stochastic growth model,$$
(\hat{\gamma}, \hat{\beta}) = \arg\min_{\theta, \beta} \|C_t^{-\gamma} - \beta \psi_n(K_{t-1}, A_t; \hat{\omega})\| \quad (9)
\text{s.t. } \hat{\omega} = \arg\min_{\omega} \|C_{t+1}^{-\gamma}(\alpha Y_t + 1) - \psi_n(K_{t-1}, A_t; \omega)\|, \quad n = 1, 2, 3, \quad (10)
$$
where the Euclidean norm is calculated by the mean squared distance. We estimate the structural parameters on a grid with step size 0.05. It is evident, that both parameters are estimated with
Table 1: $\omega$ solving the growth model in Section 2.2.1.

<table>
<thead>
<tr>
<th></th>
<th>$n = 1$</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_1$</td>
<td>2.0359</td>
<td>1.8106</td>
<td>1.8151</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>-0.4063</td>
<td>-0.3212</td>
<td>-0.3252</td>
</tr>
<tr>
<td>$\omega_3$</td>
<td>-0.1157</td>
<td>-0.2243</td>
<td>-0.2747</td>
</tr>
<tr>
<td>$\omega_4$</td>
<td>-0.0152</td>
<td>-0.0130</td>
<td></td>
</tr>
<tr>
<td>$\omega_5$</td>
<td>0.0388</td>
<td>0.0725</td>
<td></td>
</tr>
<tr>
<td>$\omega_6$</td>
<td>-0.0294</td>
<td>-0.0846</td>
<td></td>
</tr>
<tr>
<td>$\omega_7$</td>
<td></td>
<td>-0.0004</td>
<td></td>
</tr>
<tr>
<td>$\omega_8$</td>
<td></td>
<td>0.0055</td>
<td></td>
</tr>
<tr>
<td>$\omega_9$</td>
<td></td>
<td>0.0193</td>
<td></td>
</tr>
<tr>
<td>$\omega_{10}$</td>
<td></td>
<td>-0.0117</td>
<td></td>
</tr>
</tbody>
</table>

high precision if higher order approximations of expectations in the Euler equation ($n > 1$) are conducted. First–order approximations ($n = 1$) provide weak estimation results. This suggests that linearizing the solution to DSGE models would not be useful in empirical applications.

It has been shown by Pozzi (2003), that the bias of the the GMM–estimate of relative risk aversion is 3 times to 10 times its true value. The standard deviation of the GMM estimate is as high as the true parameter value.

In the following we document the results of accuracy checks of the estimated Euler equation. In Table 3 we study the Euler equation errors based on a $t$–test of the hypothesis that the Euler equation error is zero. In the case of higher-order approximation of expectations ($n > 1$), the Euler equation errors are fairly small, while first–order approximations lead to less accurate estimation results.
Table 2: Results for estimation scheme (9), (10).

<table>
<thead>
<tr>
<th></th>
<th>n = 3</th>
<th></th>
<th>n = 2</th>
<th></th>
<th>n = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>γ</td>
<td>β</td>
<td>γ</td>
<td>β</td>
<td>γ</td>
</tr>
<tr>
<td>True</td>
<td>0.5</td>
<td>0.95</td>
<td>0.5</td>
<td>0.95</td>
<td>0.5</td>
</tr>
<tr>
<td>T = 25 Mean</td>
<td>0.47732</td>
<td>0.95153</td>
<td>0.44924</td>
<td>0.95164</td>
<td>0.39998</td>
</tr>
<tr>
<td>Stdev</td>
<td>0.08353</td>
<td>0.00292</td>
<td>0.10709</td>
<td>0.00251</td>
<td>0.09994</td>
</tr>
<tr>
<td>MSE</td>
<td>0.00749</td>
<td>0.00001</td>
<td>0.01404</td>
<td>0.00009</td>
<td>0.01999</td>
</tr>
<tr>
<td>T = 50 Mean</td>
<td>0.43366</td>
<td>0.95171</td>
<td>0.42048</td>
<td>0.95176</td>
<td>0.35882</td>
</tr>
<tr>
<td>Stdev</td>
<td>0.05183</td>
<td>0.00250</td>
<td>0.06349</td>
<td>0.00255</td>
<td>0.04937</td>
</tr>
<tr>
<td>MSE</td>
<td>0.00709</td>
<td>0.00009</td>
<td>0.01035</td>
<td>0.00010</td>
<td>0.02237</td>
</tr>
<tr>
<td>T = 100 Mean</td>
<td>0.42934</td>
<td>0.95204</td>
<td>0.42626</td>
<td>0.95213</td>
<td>0.30542</td>
</tr>
<tr>
<td>Stdev</td>
<td>0.05375</td>
<td>0.00258</td>
<td>0.05773</td>
<td>0.00259</td>
<td>0.04427</td>
</tr>
<tr>
<td>MSE</td>
<td>0.00788</td>
<td>0.00001</td>
<td>0.00877</td>
<td>0.00001</td>
<td>0.03982</td>
</tr>
</tbody>
</table>

Table 3: Accuracy check based on Euler equation errors.

<table>
<thead>
<tr>
<th></th>
<th>n = 3</th>
<th></th>
<th>n = 2</th>
<th></th>
<th>n = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>γ</td>
<td>β</td>
<td>γ</td>
<td>β</td>
<td>γ</td>
</tr>
<tr>
<td>T = 25 Mean</td>
<td>0.19488</td>
<td>0.94453</td>
<td>0.77982</td>
<td></td>
<td></td>
</tr>
<tr>
<td>p–value</td>
<td>0.84706</td>
<td>0.35394</td>
<td>0.44282</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T = 50 Mean</td>
<td>-0.47141</td>
<td>-0.28106</td>
<td>-1.30641</td>
<td></td>
<td></td>
</tr>
<tr>
<td>p–value</td>
<td>0.63746</td>
<td>0.77983</td>
<td>0.19739</td>
<td></td>
<td></td>
</tr>
<tr>
<td>T = 100 Mean</td>
<td>-1.97852</td>
<td>-2.23552</td>
<td>-6.10526</td>
<td></td>
<td></td>
</tr>
<tr>
<td>p–value</td>
<td>0.05062</td>
<td>0.02761</td>
<td>0.00000</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
3 Estimating the Time–varying Sharpe Ratio

Next, we want to apply our estimation technique to the habit formation model that predicts a time varying Sharpe ratio.

3.1 The Stochastic Growth Model with Habit Formation

Economic research in the past has attempted to link macroeconomic fundamentals to asset prices in the context of inter–temporal models. The inter–temporal asset pricing literature has relied either on models of a pure exchange economy such as Lucas (1978) and Breeden (1979) or on the stochastic growth model with production as in Brock and Mirman (1972) and Kydland and Prescott (1982). These models are referred to as the consumption based CAPM and the stochastic growth model, respectively. In the pure exchange model asset prices are computed in an economy where there is an exogenous dividend stream for the representative agent. Given the observed low variability in consumption it has been shown that the risk–free interest rate is too high and the mean equity premium as well as the volatility of stock returns too low. These phenomena are referred to as the risk–free rate puzzle, the equity premium puzzle, and the volatility puzzle, respectively. For a survey on these problems, see e.g., Mehra and Prescott (1985), Kocherlakota (1996) and Cochrane (2001).

Among others, Rouwenhorst (1995) and Lettau and Uhlig (1997a) have argued that it is crucial how consumption is modelled. In models with production, e.g., the production and investment based Capital Asset Pricing Model by Cochrane (1991, 1996) or the stochastic growth model of Kydland and Prescott (1982) the fundamental shock is to the production function of firms and consumption is not an exogenous process as consumers can optimize their consumption path in response to production shocks. They thus can smooth consumption via savings and labor input if the latter is in the model. If consumption is modeled as a choice variable and endogenous the inter–temporal marginal rate of substitution\(^3\) may become even less variable and asset market facts are even harder to match, see e.g. Rouwenhorst (1995), Lettau (1998), Lettau and Uhlig (1997), and Lettau, Gong and Semmler (2001).

In order to allow to match asset price characteristics with data economic research has extended standard inter–temporal models. Those extensions include the use of different utility functions,

\(^3\) This is also referred to as stochastic discount factor or pricing kernel.
in particular habit formation, see e.g. Heaton (1993, 1995), Campbell and Cochrane (1999) and Boldrin, Christiano and Fisher (1997, 2001), consider incomplete markets, see e.g. Telmer (1993), Heaton and Lucas (1996), Luttmer (1996) and Lucas (1994), introduce heterogenous agents as in Constantinides and Duffie (1996), or replace the stochastic discount factor with a nonparametric function as in Chapman (1997). Other approaches, for example, have focused on the variation of the dividend stream rather than on the discount factor to explain the asset price characteristics, see e.g. Bansal and Yaron (2004). Although some progress has been made to match asset price characteristics with the data none of the models is able to resolve all the puzzles at once.

We investigate whether the dynamic stochastic growth model with habit formation is able to replicate time variation in asset price characteristics, in particular the counter-cyclical movement of the Sharpe ratio over the business cycle as well as its variability. To be able to spell out time series behavior of asset market facts of inter-temporal asset pricing models empirically we apply computational efficient estimation strategies based on numerical solutions of the nonlinear first-order conditions using the full structure of the model. We use the expectations approach of Den Haan and Marcet (1990) to incorporate nonparametric expectations in our numerical solution method and show how estimation schemes are obtained. Based on Monte Carlo simulations we show its dominance over the standard GMM approach in terms of small sample performance which is crucial for empirical economics.

Preferences with habit formation provide a better solution to most of those puzzles than the models with power utility, see for instance Campbell and Cochrane (1999). In particular they imply a counter-cyclical Sharpe ratio. Extensions to replicate more stylized facts are provided by, e.g., Brandt and Wang (2003), Wachter (2004), or Buraschi and Jiltsov (2007). Most of the recent literature supports the view that there is a negative relationship between the risk-return trade-off and the business cycle. Lettau and Ludvigson (2003) have pointed out that the risk-return trade-off

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4 Note, that path dependence of consumption choices in habit formation models imply the possibility of negative marginal utility of consumption and equivalently (implausible) negative Arrow–Debreu prices – these may be prohibited by imposing rather strong assumptions regarding to distributions of asset returns, see Chapman (1998) for details.

5 Bansal, Hsieh and Viswanathan (1993) and Bansal and Viswanathan (1993) use a similar approach to estimate the consumption based capital asset pricing model.

6 This is strongly supported by Kuan and White (1994), Brown and Newey (1998) and Chen and White (1998) since it is likely to end up with incorrect belief equilibria if incorrect parameterizations are applied.
changes over time with a variability considerably exceeding that of the Sharpe ratio produced by the estimated model of Campbell and Cochrane (1999). In this contribution we demonstrate that the model with habit formation is able to match the observed standard deviation of the Sharpe ratio, if the restrictive assumptions associated with the log–linear solution of Campbell and Cochrane are relaxed. Therefore, we use the estimation procedure introduced above. In our estimation approach we are able to use full structural information and, consequently, Monte Carlo simulations show that our estimations are less biased and more efficient than the widely applied GMM procedure. Based on quarterly U.S. data we obtain significant estimates of the structural parameters of the model and investigate its Sharpe ratio for preferences with habit persistence. We provide support for the hypothesis that the Sharpe ratio moves counter–cyclically. Moreover, we find indication that the model is able to capture the size of the time variation of the Sharpe ratio. We still obtain a negative relation between the Sharpe ratio and the business cycle.

The dynamic model we use to investigate the time-varying Sharpe ratio is the stochastic growth model described in Section 2.1 extended by habit formation. The household has a utility function that not only depends on current consumption but also on the habit \( X_t = C_{t-1} \):

\[
U(C_t, X_t) = \frac{(C_t - X_t)^{1-\gamma} - 1}{1 - \gamma}. \tag{11}
\]

Campbell and Cochrane (1999) define the surplus consumption ratio as \( S_t = (C_t - X_t)/C_t \) and specify an autoregressive process for the log of the surplus ratio:

\[
s_{t+1} = (1 - \phi^H)S_t + \phi_H s_t + \lambda(s_t)(c_{t+1} - c_t - g). \tag{12}
\]

The impact of consumption on habit may be state dependent, described by

\[
\lambda(s_t) = \frac{1}{S} \sqrt{1 - 2(s_t - \overline{s})} - 1, \quad S = \sigma_c \sqrt{\frac{\gamma}{1 - \phi^H}} \tag{13}
\]

The above utility function (5) provides us with a stochastic discount factor incorporating habit formation such as

\[
M_{t+1} = \beta \left( \frac{S_{t+1} C_{t+1}}{S_t C_t} \right)^{-\gamma}. \tag{14}
\]
In order to solve the Euler equation numerically for the model variant with habit formation we employ the discount factor (14) in the Euler equation (8). Yet since the gross return on capital are the same, it remains the same in the Euler equation.

3.2 Asset Pricing Implications of the Habit Formation Model

From the above outlined model one can spell out the following asset market implications. The (maximal) Sharpe ratio can be obtained from the derivation of volatility bounds in Hansen and Jagannathan (1991) as

$$\delta_t^{\max} = \frac{\sigma_t [\mathcal{M}_{t+1}]}{E_t [\mathcal{M}_{t+1}]}.$$  \hfill (15)

In recent research asset market characteristics of inter–temporal models are mostly derived under the crucial assumption of jointly log–normally distributed asset prices and consumption.\(^7\)

In order to evaluate (15) without imposing distributional assumptions on consumption and the constancy of the equity premium and Sharpe–ratio we aim to determine \(E_t [\mathcal{M}_t]\) and \(\sigma_t [\mathcal{M}_t]\) via polynomial regression nonparametrically, where, now in the present case, today’s expectations are determined nonparametrically based on the relevant observable state of the economy: present capital stock and technology shock.\(^8\) Expectations of the stochastic discount factor are obtained by

$$E_t [\mathcal{M}_{t+1}] = E_t [\mathcal{M}_{t+1}|S_t] = g(S_t; \theta_M),$$  \hfill (16)

where \(g\) is implemented by third-order polynomial regression,

$$g(S_t; \omega_M) = \omega_{M,1} S_t^{\omega_{M,2}} S_t^{\omega_{M,3}} S_t^{\omega_{M,4}}.$$  

Notice that \(\omega_M\) is estimated by ordinary least squares.

To proceed in this way is in line with Den Haan and Marcet (1990) and Duffy and McNelis (1997) who determine expectations in the Euler equation and model the stochastic discount factor of the first order conditions of the stochastic growth model based on capital stock and the technology shock as conditional variables. Application of nonparametric expectations is recommended by

\(^{7}\)See Campbell, Lo and MacKinlay (1997). In the framework of the stochastic growth model with power utility this implies a time invariant equity premium and Sharpe ratio.

\(^{8}\) Note, that the utility function in our model is time separable.
Kuan and White (1994), Brown and Newey (1998) and Chen and White (1998).\footnote{Yet, we want to note, however, as a referee has pointed out the Den Haan and Marcet (1990) procedure may become nonparametric as the order of polynomial increases.}

The standard deviation of the stochastic discount factor (SDF) is estimated by
\[ \sigma_t(M_{t+1}) = \sqrt{E_t [M_{t+1}^2] - E_t [M_{t+1}]^2}. \] (17)
Therefore, expectations of the squared SDF are also determined nonparametrically via
\[ E_t [M_{t+1}^2] = E [M_{t+1}^2 | S_t] = g(S_t; \theta_{M^2}). \] (18)
Function \( g \) is implemented by third–order polynomial regression,
\[ g(S_t; \omega_{M}) = \omega_{M,1}S_t^2 + \omega_{M,2}S_t^3 + \omega_{M,3}S_t^4. \]
Notice that \( \omega_{M} \) is estimated by ordinary least squares.

### 3.3 Estimation of the Habit Formation Model

We aim at estimating the structural parameters of the U.S. Real Business Cycle Model with habit formation according to (4) in the same way as we have estimated the stochastic growth model in Section 2, as well as the time–varying asset market characteristics as discussed above. In particular, we want to test whether the Sharpe ratio as obtained by our new methodology turns out to be more volatile than in the log–linear version of the pure exchange economy. We estimate the risk aversion coefficient \( \gamma \) and the discount factor \( \beta \) using quarterly data from 1951:4 to 2005:4 since most studies use this time period for matching the model to the data.\footnote{Data are taken from Datastream.}

The estimator (4) applied to the stochastic growth model with habit formation yields
\[
(\hat{\gamma}, \hat{\beta}, \hat{\rho}) = \arg\min_{\theta, \beta, \rho} \| C_t^{-\gamma} - \beta \psi_3(S_t; \hat{\omega}) \| \\
\text{s.t. } \hat{\omega} = \arg\min_{\omega} \| \frac{S_{t+1}}{S_t} C_{t+1}^{-\gamma} (\alpha Y_t + 1) - \psi_3(S_t; \omega) \|,
\] (19) (20)
where the Euclidean norm is calculated by the mean squared distance. We estimate the structural parameters on a grid with step size 0.05. Estimated parameters are reported in Table 4.\footnote{Note that the technology parameters, the depreciation rate and the capital share, are fixed, because their instances are well agreed in the literature, \( \phi = 0.9750 \) and \( \alpha = 0.33 \). This is equivalent to knowing the output time series.}
standard errors for $\hat{\gamma}$ and $\hat{\beta}$ were obtained by Monte Carlo simulations with 1,000 replications where the starting values for $\theta$ were drawn from normally distributed random variables.

Table 4: Parameter estimates of the stochastic growth model (standard errors in brackets).

<table>
<thead>
<tr>
<th>$\hat{\gamma}$</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\rho}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.2352 (0.0260)</td>
<td>0.9713 (0.0020)</td>
<td>0.9316 (0.0074)</td>
</tr>
</tbody>
</table>

We estimate the average growth rate of consumption as 0.0174, and its standard deviation as 0.0064. The sample counterparts found in the data are 0.0178 and 0.0077, respectively.

To explore the time-varying asset market characteristics for the implications of the habit formation model we employ the above discussed stochastic volatility model of Härdle and Tsybakov (1997) to achieve conditional expectations and variances of the stochastic discount factor, $M_t$, based on the lagged surplus consumption ratio as described in above. We estimate the average Sharpe ratio as 0.3854, and its standard deviation as 0.1739. The time–varying stylized facts such as the maximal Sharpe–ratio can be computed by (15), and Figure 1 illustrates how it moves over time. It can be seen that its variability considerably exceeds that in the log–linear version of the pure exchange economy.
The graph illustrates the estimated Sharpe–ratio from our model, grey line, and the Sharpe–ratio in the log–linear version of the pure exchange economy of Campbell and Cochrane (1999), black line.

Figure 1: Maximal Sharpe ratio in the estimated stochastic growth model with habit formation.

To obtain evidence how the Sharpe–ratio relates to the phase of the business cycle we plot it against the surplus consumption ratio. The latter takes on small numbers in recessions and large numbers in booms. With respect to this measure of the phase of the business cycle Figure 2 shows that the Sharpe–ratio moves counter–cyclically over the business cycle.
The graph illustrates the relationship between the estimated Sharpe–ratio (vertical axis) and surplus consumption ratio (horizontal axis).

Figure 2: Counter–cyclical variation of the maximal Sharpe ratio.

4 Conclusions

We are proposing a computationally tractable recipe to estimate DSGE models. Because we use full structural information of the dynamic model, the new estimation technique is well suited to assess time–varying asset market facts, and we also apply the estimation method to assess the time–varying Sharpe ratio. Notice that the literature has come to different conclusions based on linearized models.

It has been shown that one can improve on the asset pricing puzzles, related to the level of financial variables, by the introduction of habit persistence, see Campbell and Cochrane (1999), or Boldrin, Christiano and Fisher (1997, 2001). To estimate a time–varying Sharpe–ratio we follow the latter since they explicitly model the business cycle. As Campbell and Cochrane (1999) in the case of a pure exchange economy for quarterly U.S. data we obtain a Sharpe–ratio moving counter–cyclically with respect to the business cycle.

However, Lettau and Ludvigson (2003) have pointed out that the risk–return trade–off changes
over time with a standard deviation considerably exceeding that suggested by Campbell and Cochrane (1999). In this contribution we demonstrate that the model with habit formation is able to match the variability of the Sharpe ratio, since (i) the variables regarding production exhibit a larger standard deviation than consumption, and (ii) the restrictive assumptions associated with the log-linear solution of Campbell and Cochrane are relaxed. We have proposed a new estimation procedure based on the solution technique of Den Haan and Marcet (1990). In our estimation approach we are able to use full structural information and, consequently, Monte Carlo simulations show that our estimations are less biased and more efficient than the widely applied GMM procedure. Based on quarterly U.S. data we estimate the structural parameters of the model and investigate its Sharpe ratio for preferences with habit persistence. We find indication that the model is able to capture the time variation of the Sharpe ratio, i.e. its standard deviation. We still obtain a negative relation between the Sharpe ratio and the business cycle.

**Literature**


Boldrin, M., L. J. Christiano, and J. D. M. Fisher (2001), Habit Persistence, Asset Returns and


