Closedness in the Semimartingale Topology For Spaces of Stochastic Integrals With Constrained Integrands

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Abstract
Let $S$ be an $\mathbb{R}^d$-valued semimartingale and $(\psi^n)$ a sequence of $C$-valued integrands, i.e., predictable, $S$-integrable processes taking values in some given closed set $C(\omega, t) \subseteq \mathbb{R}^d$ which may depend on the state $\omega$ and time $t$ in a predictable way. Suppose that the stochastic integrals $(\psi^n \cdot S)$ converge to $X$ in the semimartingale topology. We provide a necessary and sufficient condition (on $S$ and $C$) that $X$ can be represented as stochastic integral with respect to $S$ of some $C$-valued integrand, and we explain the relation to the sufficient conditions introduced earlier in [6], [20] and [21]. The existence of such representations is equivalent to the closedness (in the semimartingale topology) of the space of stochastic integrals of $C$-valued integrands, which is crucial for the existence of solutions to most optimisation problems under trading constraints in mathematical finance. Moreover, we show that a predictably convex space of stochastic integrals is closed in the semimartingale topology if and only if it is a space of stochastic integrals of $C$-valued integrands, where each $C(\omega, t)$ is convex.

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1 Introduction

In mathematical finance, proving the existence of a solution to optimisation problems like superreplication, utility maximisation and quadratic hedging in most cases boils down to the same abstract problem: One must show that a subsequence of (predictably) convex combinations of an optimising sequence of wealth processes, i.e., stochastic integrals with respect to the underlying price process $S$, converges and that the limit is again a wealth process, i.e., can be represented as a stochastic integral with respect to $S$. As the space of all stochastic integrals is closed in the semimartingale topology, this is the suitable topology to work with.

For applications, it is also natural to include trading constraints by requiring the strategy (integrand) to lie pointwise in some set $C$; this is usually convex to keep the above procedure applicable, and one would like to have it depend on the state and time as well. Examples of interest include no shortselling, no borrowing or nonnegative wealth constraints; see e.g. [4] and [15]. As pointed out by Delbaen [8] and Karatzas and Kardaras [15], a natural and convenient description of constraints is best formulated in terms of correspondences, i.e., set-valued functions. This is the approach we also advocate and use here.

To get some motivation, consider as above a sequence of (predictably convex combinations of) strategies and suppose (as usually happens by the convexification trick) that this converges pointwise. Each strategy is predictable, so any constraints should also be “predictable” in some sense. To have the limit still satisfy the same restrictions as the sequence elements, the constraints should moreover be of the form “closure of a sequence $(\psi^n(\omega, t))$ of random points”, since this is where the limit will lie. But if each $\psi^n(\omega, t)$ is a predictable process, the Castaing representation (see Proposition 2.3) says that the above closure is then a predictable correspondence. This explains why correspondences come up naturally.

Going back now to our constrained optimisation problem (and assuming that we have predictable, convex, closed constraints), the same procedure as in the unconstrained case yields a sequence of wealth processes (integrals) converging to some limit which is a candidate for the solution of our problem. (We have cheated a little in the motivation — it is the integrals which usually converge, not the integrands.) This limit process is again a stochastic integral, but it still remains to check that the corresponding trading strategy also satisfies the constraints. In more abstract terms, one asks whether the limit of a sequence of stochastic integrals of some constrained integrands can again be represented as a stochastic integral of some constrained integrand or, equivalently, if the space of stochastic integrals of constrained integrands is closed in the semimartingale topology. We illustrate by a counterexample that this is not true in general, since it might happen that some assets become redundant, i.e., can be replicated on some predictable set by trading in the remaining ones. This phenomenon occurs when there is linear dependence between the different components of $S$.

One way to deal with this issue is (as in [4], [3], [18] and [20]) to simply assume that there are no redundant assets in the market, which implies that the closedness result is
true for all constraints which are formulated via closed (and convex) sets. Especially in Itô process models with a Brownian filtration, such a non-redundancy condition is useful (e.g. when working with artificial market completions), but it can also be restrictive. An alternative approach is to study only constraints given by polyhedral or continuous convex sets as in [14] and [6]. While most constraints of practical interest are indeed polyhedral, this is conceptually unsatisfactory as one does not recover all results from the first case when there are no redundant assets. A good formulation should therefore account for the interplay between the constraints $C$ and potential redundancies in the assets $S$.

To realise this idea and overcome the above gap, we use the projection on the predictable range of $S$. This is a predictable process taking values in the orthogonal projections in $\mathbb{R}^d$, which has been introduced in [23], [9] and [8]. It allows us to uniquely decompose each integrand into one part which contains all the relevant information for its stochastic integral and another part which has stochastic integral zero. As a consequence, our problem reduces to the question whether or not the projection of the constraints on the predictable range is closed. Convexity is not relevant for that aspect. Since this approach turns out to give a necessary and sufficient condition, we recover all the previous results in [4], [18], [20], [14] and [6] as special cases; and in addition we obtain for constant constraints $C(\omega, t) \equiv C$ closedness of the space of $C$-constrained integrands holds for all semimartingales if and only if all projections of $C$ in $\mathbb{R}^d$ are closed. The well-known characterisation of polyhedral cones thus implies in particular that the closedness result for constant convex cone constraints is true for an arbitrary semimartingale if and only if the constraints are polyhedral.

For a general constraint set $C(\omega, t)$ which is closed and convex, the set of stochastic integrals of $C$-constrained integrands is the prime example of a predictably convex space of stochastic integrals. By adapting arguments from [8], we show that this is in fact the only class of predictably convex spaces of stochastic integrals which are closed in the semimartingale topology. So this paper makes both mathematical contributions to stochastic calculus and financial contributions in the modelling and handling of trading constraints for optimisation problems from mathematical finance.

The remainder of the article is organised as follows. We formulate the problem in the terminology of stochastic processes in Section 2 and provide there some results on measurable correspondences and measurable selectors, which we need to introduce and handle the constraints. Section 3 contains a counterexample which illustrates where the difficulties arise and motivates in a simple setting the definition of the projection on the predictable range. The main results are established in Section 4. Section 5 gives the construction of the projection on the predictable range as well as two proofs omitted in the previous section. Finally, Section 6 briefly discusses some related work.

## 2 Problem formulation and preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t < \infty}$ satisfying the usual conditions of completeness and right-continuity. For all unexplained notations concerning
stochastic integration we refer to the book of Jacod and Shiryaev [13]. Set \( \mathfrak{P} := \Omega \times [0, \infty) \).

The space of all \( \mathbb{R}^d \)-valued semimartingales is denoted by \( S^0(P) := S^0(P; \mathbb{R}^d) \) or simply \( S(P) \), if the dimension is clear from the context. The Ėmery distance (see [10]) of two semimartingales \( X \) and \( Y \) is

\[
d(X, Y) = \sup_{|\vartheta| \leq 1} \left( \sum_{n \in \mathbb{N}} 2^{-n} E\left[1 \wedge \left|\vartheta \cdot (X - Y)\right|_n\right]\right),
\]

where \((\vartheta \cdot X)_t := \int_0^t \vartheta_s dX_s\) stands for the vector stochastic integral, which is by construction a real-valued semimartingale, and the supremum is taken over all \( \mathbb{R}^d \)-valued predictable processes \( \vartheta \) bounded by 1. With this metric, \( S(P) \) is a complete topological vector space and the corresponding topology is called the semimartingale topology. For brevity, we say “in \( S(P) \)” for “in the semimartingale topology”. For a given \( \mathbb{R}^d \)-valued semimartingale \( S \), we write \( \mathcal{L}(S) \) for the space of \( \mathbb{R}^d \)-valued, \( S \)-integrable, predictable processes \( \vartheta \) and \( \mathcal{L}(S) \) for the space of equivalence classes \([\vartheta] = [\vartheta]^S = \{\varphi \in \mathcal{L}(S) \mid \varphi \cdot S = \vartheta \cdot S\}\) of processes in \( \mathcal{L}(S) \) which yield the same stochastic integral with respect to \( S \), identifying processes equal up to \( P \)-indistinguishability. By Theorem V.4 in [19] the space of stochastic integrals \( \{\vartheta \cdot S \mid \vartheta \in \mathcal{L}(S)\}\) is closed in \( S(P) \). Equivalently, \( L(S) \) is a complete topological vector space with respect to \( d_S([\vartheta], [\varphi]) = d(\vartheta \cdot S, \varphi \cdot S) \), where \( \vartheta \) and \( \varphi \) are representatives of the equivalence classes \([\vartheta]\) and \([\varphi]\). So if \((\vartheta^n)\) is a sequence in \( \mathcal{L}(S) \) such that \((\vartheta^n \cdot S)\) converges to some semimartingale \( X \) in \( S(P) \), there exists \( \vartheta \in \mathcal{L}(S) \) such that \( X = \vartheta \cdot S \).

In this paper we establish a generalisation of the above result to integrands which are restricted to lie in a given closed set, in the following sense. Let \( C(\omega, t) \) be a non-empty, closed subset of \( \mathbb{R}^d \), which is allowed to depend on the state \( \omega \) and time \( t \) in a predictably measurable way. Definition 2.2 below makes this precise: \( C \) should be a predictable correspondence with closed values. Denote by

\[
C = C^S = \{\psi \in \mathcal{L}(S) \mid \psi(\omega, t) \in C(\omega, t) \text{ for all } (\omega, t)\}
\]

the set of \( C \)-valued, \( S \)-integrable, predictable processes, simply called \( C \)-valued or \( C \)-constrained integrands for \( S \). If \((\psi^n)\) is a sequence of \( C \)-valued integrands such that 

\((\psi^n \cdot S)\) converges to some \( X \) in the semimartingale topology, does there exist a \( C \)-valued integrand \( \psi \) such that \( X = \psi \cdot S \)? Maybe surprisingly, the answer is negative in general; this is due to the linear dependence between the different components of \( S \), as a simple counterexample in the next section illustrates. So we reformulate our basic question and rather ask: Under which conditions does there exist a \( C \)-valued integrand \( \psi \) such that \( X = \psi \cdot S \)? By the closedness in \( S(P) \) of the space of all stochastic integrals, the limit \( X \) can always be represented as some stochastic integral \( \vartheta \cdot S \). Therefore it is sufficient to decide whether or not there exists a representative \( \psi \) of the limit class \([\vartheta]\) which is \( C \)-valued. Equivalently, one can ask whether \( C \cdot S \) is closed in \( S(P) \) or if the corresponding set

\[
[C] = [C]^S = \{[\vartheta] \in L(S) \mid \exists \psi \in [\vartheta] \text{ such that } \psi(\omega, t) \in C(\omega, t) \text{ for all } (\omega, t)\}
\]

of equivalence classes of elements of \( C \) is closed in \((L(S), d_S)\).
As already explained, this question arises naturally in mathematical finance for various optimisation problems under trading constraints; see [11], [20], [21], [18], [14] and [5]. But not all papers make it equally clear whether the procedure outlined in the introduction can be or is being used. For [18] and [14] this is clarified in [5]. Under additional assumptions, the closedness of \( C \cdot S \) in the semimartingale topology is sufficient to apply the results of Föllmer and Kramkov [11] on the optional decomposition under constraints, which give a dual characterisation of payoffs that can be superreplicated by constrained trading strategies. This is used to prove the existence of solutions to constrained utility maximisation problems in [20], [21] and [16]. The results in [11] are formulated more generally for sets of (special) semimartingales which are predictably convex.

**Definition 2.1.** A set \( \mathcal{S} \) of semimartingales is predictably convex if \( h \cdot X + (1 - h) \cdot Y \in \mathcal{S} \) for all \( X \) and \( Y \) in \( \mathcal{S} \) and all \([0, 1]\)-valued predictable processes \( h \). Analogously, a set of integrands \( C \subseteq L(S) \) is predictably convex if \( h \vartheta + (1 - h) \varphi \in C \) for all \( \vartheta \) and \( \varphi \) in \( C \) and all \([0, 1]\)-valued predictable processes \( h \).

The prime example of predictably convex sets of integrands is given by \( C \)-constrained integrands when \( C \) is convex-valued. Theorem 4.10 below shows that all predictably convex spaces of integrands \( C \) must be of this form if \( C \cdot S \) is in addition closed in \( S(P) \).

To formulate precisely the assumptions on the (random and time-dependent) set \( C \), we adapt the language of measurable correspondences to our framework of predictable measurability and recall for later use some of the results in this context. Note that the general results we exploit do not depend on special properties of the predictable \( \sigma \)-field on \( \Omega \). However, we do exploit that \( \mathbb{R}^d \) is metric and \( \sigma \)-compact; this ensures by Proposition 1A in [22] or the proof of Lemma 18.2 in [1] that weak measurability and measurability for a closed-valued correspondence coincide in our setting.

**Definition 2.2.** A correspondence \( C \) is a mapping from \( \Omega \) into \( 2^{\mathbb{R}^d} \). Its domain is \( \text{dom}(C) := \{ (\omega, t) \mid C(\omega, t) = \emptyset \} \). We say that a correspondence \( C \) is predictable if \( C^{-1}(F) := \{ (\omega, t) \mid C(\omega, t) \cap F = \emptyset \} \) is a predictable set for each closed \( F \subseteq \mathbb{R}^d \). A correspondence has predictable graph if its graph \( \text{gr}(C) := \{ (\omega, t, x) \in \Omega \times \mathbb{R}^d \mid x \in C(\omega, t) \} \) is in \( \mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d) \). We call a process \( \psi \) a measurable selector of a predictable correspondence \( C \) if \( \psi \) is predictable and satisfies \( \psi(\omega, t) \in C(\omega, t) \) for all \( (\omega, t) \in \text{dom}(C) \).

The following results ensure the existence of measurable selectors in all situations relevant for us.

**Proposition 2.3** (Castaing). For a correspondence \( C : \Omega \rightarrow 2^{\mathbb{R}^d} \) with closed values, the following are equivalent:

1) \( C \) is predictable.

2) \( \text{dom}(C) \) is predictable and there exists a Castaing representation of \( C \), i.e., a sequence \( (\psi^n) \) of measurable selectors of \( C \) such that

\[
C(\omega, t) = \{ \psi^1(\omega, t), \psi^2(\omega, t), \ldots \} \quad \text{for each } (\omega, t) \in \text{dom}(C).
\]
Proposition 2.4 (Aumann). Let \( C : \Omega \to 2^{\mathbb{R}^d} \) be a correspondence with non-empty values and predictable graph and \( \mu \) a finite measure on \((\Omega, \mathcal{P})\). Then there exists a predictable process \( \psi \) such that \( \psi(\omega, t) \in C(\omega, t) \) \( \mu \)-a.e.

Proof. See Corollary 18.27 in [1]. \qed

The proof of Proposition 2.4 is based on the following result on projections, to which we refer later.

Proposition 2.5. Let \((R, \mathcal{R}, \mu)\) be a \(\sigma\)-finite measure space and \(\mathcal{R}_\mu\) the \(\sigma\)-field of \(\mu\)-measurable sets. If a set \(A\) belongs to \(\mathcal{R}_\mu \otimes \mathcal{B}(\mathbb{R}^d)\), then the projection \(\pi_R(A)\) of \(A\) on \(R\) belongs to \(\mathcal{R}_\mu\).

Proof. See Theorem 18.25 in [1]. \qed

Measurability and graph measurability of a correspondence are linked as follows.

Proposition 2.6. Let \( C : \Omega \to 2^{\mathbb{R}^d} \) be a correspondence with non-empty values. If \( C \) is predictable, its closure correspondence \( \overline{C} \) given by \( \overline{C}(\omega, t) := \overline{C(\omega, t)} \) has a predictable graph.

Proof. See Theorem 18.6 in [1]. \qed

Since we require in (2.1) for our integrands \( \psi \) that \( \psi(\omega, t) \in C(\omega, t) \) for all \((\omega, t)\), we shall assume, as motivated in the introduction, that \( C \) is predictable and takes closed values. Then Proposition 2.3 guarantees the existence of measurable selectors. Moreover, we shall use that predictable measurability of a correspondence is preserved under transformations by Carathéodory functions and stable under countable unions and intersections.

A function \( f : \Omega \times \mathbb{R}^n \to \mathbb{R}^m \) is called Carathéodory if \( f(\omega, t, x) \) is predictable with respect to \((\omega, t)\) and continuous in \(x\).

Proposition 2.7. Let \( C : \Omega \to 2^{\mathbb{R}^d} \) be a predictable correspondence with closed values and \( f : \Omega \times \mathbb{R}^n \to \mathbb{R}^d \) and \( g : \Omega \times \mathbb{R}^d \to \mathbb{R}^m \) Carathéodory functions. Then \( C' \) and \( C'' \) given by \( C'(\omega, t) = \{ y \in \mathbb{R}^m \mid f(\omega, t, y) \in C(\omega, t) \} \) and \( C''(\omega, t) = \{ g(\omega, t, x) \mid x \in C(\omega, t) \} \) are predictable correspondences with closed values.

Proof. See Corollaries 1P and 1Q in [22]. \qed

Proposition 2.8. Let \( C^n : \Omega \to 2^{\mathbb{R}^d} \) be a predictable correspondence with closed values for each \( n \in \mathbb{N} \) and define the correspondences \( C' \) and \( C'' \) by \( C'(\omega, t) = \bigcap_{n \in \mathbb{N}} C^n(\omega, t) \) and \( C''(\omega, t) = \bigcup_{n \in \mathbb{N}} C^n(\omega, t) \). Then \( C' \) and \( C'' \) are predictable and \( C' \) is closed-valued.

Proof. See Theorem 1M in [22] and Lemma 18.4 in [1]. \qed
To investigate the relation between a predictably convex space of integrands and \( C \)-valued integrands, we later use the following proposition, which is a reformulation of the contents of Theorem 5 in [8]. We view an \( \mathbb{R}^d \)-valued predictable process on \( \Omega \) as a \( \mathcal{P} \)-measurable \( \mathbb{R}^d \)-valued mapping on \( \Omega \), take some probability measure \( \mu \) on \( (\Omega, \mathcal{P}) \) and denote by \( B(0, r)^{L^\infty} \) and \( B(0, r) \) the closure of a ball of radius \( r \) in \( L^\infty(\Omega, \mathcal{P}, \mu; \mathbb{R}^d) \) and \( \mathbb{R}^d \), respectively. Predictable convexity is understood as in the second part of Definition 2.1.

**Proposition 2.9.** Let \( \mathfrak{K} \) be a predictably convex and \( \mu \)-weak\(^*\)-compact subset of \( B(0, r)^{L^\infty} \) containing zero. Then there exists a predictable correspondence \( K : \Omega \rightarrow 2^{B(0, r)} \{\emptyset\} \) with convex and compact values containing zero such that

\[
\mathfrak{K} = \left\{ \vartheta \in L^\infty(\Omega, \mathcal{P}, \mu; \mathbb{R}^d) \mid \vartheta(\omega, t) \in K(\omega, t) \ \mu\text{-a.e.} \right\}.
\]

**Proof.** In the proof of Theorem 5 in [8], the set \( C^\lambda \) defined there for \( \lambda > 0 \) is a predictably convex and weak\(^*\)-compact subset of \( B(0, \lambda)^{L^\infty} \) containing zero by Lemmas 10 and 11 in [8]. No other properties of \( C^\lambda \) are used. So we can modify the proof of Theorem 5 in [8] by replacing the use of the Radon-Nikodým theorem of Debreu and Schmeidler (Theorem 2 in [7]) with that of Artstein (Theorem 9.1 in [2]). This yields that \( K \) is predictably measurable and has not only (as argued in [8]) a predictable graph. Replacing the correspondence \( K \) coming from this construction by \( K \cap B(0, r) \) then gives that \( K \) is valued in \( 2^{B(0, r)} \). \( \square \)

### 3 A motivating example

In this section, we give a simple example of a semimartingale \( Y \) and a predictable correspondence \( C \) with non-empty, closed, convex cones as values such that \( C^Y \cdot Y \) is not closed in \( S(P) \). This illustrates where the problems in answering our basic question arise and suggests a way to overcome them. The example is the same as Example 2.2 in [6], but we use it here for a different purpose.

Let \( W = (W^1, W^2, W^3)^\top \) be a 3-dimensional Brownian motion and \( Y = \sigma \cdot W \), where

\[
\sigma = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & -1 & 1
\end{pmatrix}.
\]

The matrix \( \sigma \) and hence \( \hat{c} = \sigma \sigma^\top \) have a non-trivial kernel spanned by \( w = \frac{1}{\sqrt{2}}(0, 1, 1)^\top \), i.e., \( \text{Ker}(\hat{c}) = \text{Ker}(\sigma) = \mathbb{R}w = \text{span}\{w\} \). By construction, the stochastic integral of each \( \mathbb{R}^3 \)-valued predictable process valued in \( \text{Ker}(\hat{c}) \) \( dP \otimes dt\text{-a.e.} \) is zero, and vice versa. Thus the equivalence class \([0]^Y\) of the process \( 0 \in \mathcal{L}(Y) \) consists up to \( dP \otimes dt\text{-a.e.} \) equality of the processes \( hw \), where \( h \) is some real-valued predictable process. Since adding a
representative of 0 to some element of \( \mathcal{L}(Y) \) does not change its equivalence class, the equivalence class \([\vartheta]^Y\) of any given \( \vartheta \in \mathcal{L}(Y) \) is given by
\[
[\vartheta]^Y = \{ \vartheta + hw \mid h \text{ a real-valued predictable process} \}
\]
up to \( dP \otimes dt \)-a.e. equality. Let \( K \) be the closed and convex cone
\[
K = \{(x, y, z)^\top \in \mathbb{R}^3 \mid x^2 + y^2 \leq z^2, \ z \geq 0\}
\]
and \( C \) the (constant) predictable correspondence with non-empty and closed values given by \( C(\omega, t) = K \) for all \( (\omega, t) \in \Omega \). Define the sequence of (constant) processes \((\psi^n)\) by \( \psi^n = (1, \sqrt{n^2 - 1}, n)^\top \) for each \( n \in \mathbb{N} \). In geometric terms, \( K \) is a circular cone around the \( z \)-axis, and \((\psi^n)\) is a sequence of points on its surface going to infinity. (Instead of \( n \), any sequence \( z_n \to \infty \) in \([1, \infty)\) would do as well.) Each \( \psi^n \) is \( C \)-valued, and we obtain \( \psi^n \cdot Y = (\sigma \psi^n) \cdot W = W^1 + (\sqrt{n^2 - 1} - n)(W^2 - W^3) \) and hence for each \( t \geq 0 \)
\[
E\left[ (\psi^n \cdot Y - W^1)_t \right] = \int_0^t 2 \left[ 2n^2 \left( 1 - \sqrt{1 - \frac{1}{n^2}} \right) - 1 \right] ds
\]
\[
= \int_0^t 2 \left[ 2n^2 \left( 1 - \left\{ 1 - \frac{1}{2n^2} + O\left( \frac{1}{n^4} \right) \right\} \right) - 1 \right] ds
\]
\[
= tO\left( \frac{1}{n^2} \right)
\]
as \( n \to \infty \). This implies that \( \psi^n \cdot Y \to W^1 \) locally in \( \mathcal{M}^2(P) \) and therefore in \( \mathcal{S}(P) \) by Theorem 2 in [10]. The (constant) process \( e_1 := (1, 0, 0)^\top \) leading to the limiting stochastic integral \( e_1 \cdot Y = W^1 \) is not \( C \)-valued, but since we identify processes in \( \mathcal{L}(Y) \) yielding the same stochastic integral, it would be sufficient to find one predictable process which is equivalent to \( e_1 \) and valued in \( C \). But up to \( dP \otimes dt \)-a.e. equality, the equivalence class of \( e_1 \) is \{ \( e_1 + hw \mid h \) a real-valued predictable process \}, and so the definition of \( K \) implies that no predictable process equivalent to \( e_1 \) can be \( C \)-valued. Thus \( C^Y \cdot Y \) is not closed in \( \mathcal{S}(P) \).

To see why this causes problems, define the stopping time \( \tau := \inf \{ t > 0 \mid |W_t| = 1 \} \) and set \( S := Y^\tau \). From the explicit expression for \( \psi^n \cdot Y \) and \( \sqrt{n^2 - 1} - n = -\frac{1}{2n} + O\left( \frac{1}{n^2} \right) \), we then see that the sequence \((\psi^n \cdot Y)\) is bounded from below (uniformly in \( n, t, \omega \)) and converges in \( \mathcal{S}(P) \) to \((W^1)^\tau \), which cannot be represented as \( \psi \cdot S \) for any \( C \)-valued integrand \( \psi \). As a consequence, the set \( C^S \cdot S \) does not satisfy Assumption 3.1 of the optional decomposition theorem under constraints in [11]. But for instance the proof of Proposition 2.13 in [16] explicitly uses that result of [11] (see [16], p. 1835) in a setting where constrained integrands could be given by \( C \)-valued integrands as above. So technically, the argument in [16] is not valid without further assumptions (and Theorem 4.4 and Corollary 4.8 below show ways to fix this).

What can we learn from this counterexample? The key point is that the convergence of stochastic integrals \( \psi^n \cdot Y \) need not imply the pointwise convergence of their integrands.
Without constraints, this causes no problems; by Méméin’s theorem, the limit is still some stochastic integral of \(Y\), here \(e_1 \cdot Y\). But if we insist on having \(C\)-valued integrands only, the above example shows that we ask for too much. Since \(K\) is closed, we can deduce above that \(|(\psi^n)|\) must diverge (otherwise we should get along a subsequence a limit, which would be \(C\)-valued by closedness), and in fact \(|\psi^n| = \sqrt{2n} \to \infty\). But at the same time, \((\sigma \psi^n)\) converges to \(e_1 = (1, 0, 0)^\top\) — and this observation brings up the key idea of not looking at \(\psi^n\), but rather at suitable projections of \(\psi^n\) that are linked (via \(\sigma\)) to the integrator \(Y\).

To make this more precise, denote the orthogonal projection on \(\text{Im}(\sigma \sigma^\top)\) by

\[
\Pi^Y = (1_{d \times d} - w w^\top) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.
\]

Then \(\Pi^Y \psi^n = (1, \frac{1}{2}(\sqrt{n^2 - 1} - n), -\frac{1}{2}(\sqrt{n^2 - 1} - n))^\top\) converges to the limit integrand \((1, 0, 0)^\top = e_1\). We now might worry about the obvious fact that \(\Pi^Y \psi^n\) does not take values in \(K\); but for the stochastic integrals, this does not matter because we claim that \((\Pi^Y \psi^n) \cdot Y = \psi^n \cdot Y\). Indeed, any \(\vartheta \in \mathcal{L}(Y)\) can be written as a sum

\[
\vartheta = \Pi^Y \vartheta + (w w^\top) \vartheta
\]

of one part with values in \(\text{Im}(\sigma \sigma^\top)\) and another part orthogonal to the first one; and since

\[
((w w^\top) \vartheta) \cdot Y = (\vartheta^\top w w^\top \sigma)^\top \cdot W = 0\]

because \(\sigma^\top w = 0\), the claim follows. Going a little further, we even have for any \(\vartheta \in \mathcal{L}(Y)\) and any \(\mathbb{R}^d\)-valued predictable process \(\varphi\) that

\[
\varphi \in \mathcal{L}(Y) \quad \text{with} \quad \varphi \cdot Y = \vartheta \cdot Y \iff \Pi^Y \varphi = \Pi^Y \vartheta \quad dP \otimes dt\text{-a.e.} \quad (3.1)
\]

In fact, for “\(\Rightarrow\)” we argue as above that \((\Pi^Y (\varphi - \vartheta)) \cdot Y = (\varphi - \vartheta) \cdot Y\) which is 0 by assumption; so \(\Pi^Y (\varphi - \vartheta)\) is \(P \otimes dt\text{-a.e.} \text{ in } \ker(\sigma \sigma^\top)\), but also in \(\text{Im}(\sigma \sigma^\top) = (\text{Ker}(\sigma \sigma^\top))^\perp\) by definition of \(\Pi^Y\), and hence it is 0. For “\(\Leftarrow\”\), we first prove \(Y\)-integrability of \(\varphi\) by noting that

\[
\int \varphi^\top_s \sigma \sigma^\top \varphi^\top_s ds = \int (\Pi^Y \varphi^\top_s)^\top \sigma \sigma^\top (\Pi^Y \varphi^\top_s) ds = \int (\Pi^Y \vartheta^\top_s)^\top \sigma \sigma^\top (\Pi^Y \vartheta^\top_s) ds = \int \vartheta^\top_s \sigma \sigma^\top \vartheta^\top_s ds
\]

is finite-valued since \(\vartheta \in \mathcal{L}(Y)\); and then as above, \(\varphi \cdot Y = (\Pi^Y \varphi) \cdot Y = (\Pi^Y \vartheta) \cdot Y = \vartheta \cdot Y\).

The significance of (3.1) is that the stochastic integral \(\vartheta \cdot Y\) is uniquely determined by \(\Pi^Y \vartheta\), and so \(\Pi^Y \vartheta\) gives a “minimal” choice of a representative of the equivalence class \([\vartheta]^Y\). Moreover, \(\Pi^Y\) gives via (3.1) a simple way to decide whether or not some given \(\mathbb{R}^d\)-valued predictable process \(\varphi\) belongs to some equivalence class \([\vartheta]^Y\).

Coming back to the set \(K\), we observe that

\[
\Pi^Y K = \left\{ \left( x, \frac{1}{2}(y - z), -\frac{1}{2}(y - z) \right)^\top \left| x^2 + y^2 \leq z^2, \ z \geq 0 \right. \right\}
\]
is the projection of the cone $K$ on the plane through the origin with the normal vector $(0, 1, 1)\top$. In geometric terms, the projection of each horizontal slice of the cone transforms the circle above the $x$-$y$-plane into an ellipse in the projection plane having the origin as a point of its boundary. As we move up along the $z$-axis, the circles become larger, and so do the ellipses which in addition flatten out more and more towards the line through the origin and the point $e_1 = (1, 0, 0)\top$. But since they never reach that line although they come arbitrarily close, $\Pi^YK$ is not closed in $\mathbb{R}^d$ — and this is the source of all problems in our counterexample. It explains why the limit $e_1 = \lim_{n \to \infty} \Pi^Y\psi^n$ is not in $\Pi^YK$, which implies by (3.1) that there cannot exist any $C$-valued integrand $\psi$ such that $\Pi^Y\psi = e_1$. But the insight about $\Pi^YK$ also suggests that it should be sufficient to assume that

$$\Pi^YC(\omega, t) \text{ is closed } dP \otimes dt\text{-a.e.}$$  \hspace{1cm} (3.2)

for some predictable correspondence $C$ in order to obtain that $C^Y \cdot Y$ is closed in $S(P)$. This indeed works: After ensuring that $\lim_{n \to \infty} \Pi^Y\psi^n$ is valued in $\Pi^YC(\omega, t) dP \otimes dt$-a.e., one can use standard measurable selection theorems to produce a $C$-valued process $\psi$ such that $\Pi^Y\psi = \lim_{n \to \infty} \Pi^Y\psi^n$ (see the proof of Theorem 4.4 below). Moreover, it will turn out that condition (3.2) is not only sufficient, but also necessary.

The above explicit computations rely on the specific structure of $Y$, but they nevertheless motivate the approach for a general semimartingale $S$. We are going to define a predictable process $\Pi^S$ taking values in the orthogonal projections in $\mathbb{R}^d$ and satisfying (3.1) with $dP \otimes dt$ replaced by a suitable measure on $(\Omega, \mathcal{P})$ to control the stochastic integrals with respect to $S$. The process $\Pi^S$ will be called the projection on the predictable range and will allow us to formulate and prove our main results in the next section.

4 Main results

This section contains the main results (Theorems 4.4 and 4.10) as well as some consequences and some auxiliary results. Before we can formulate and prove them, we need some facts and results about the projection on the predictable range of $S$. For the reader’s convenience, the actual construction of $\Pi^S$ is postponed to Section 5.

Although we work with a general semimartingale $S$, we are only interested in the representation of a semimartingale $X$ which is the limit of a sequence of stochastic integrals $(\psi^n \cdot S)$ in the semimartingale topology. Since the stochastic integral and convergence in $S(P)$ are invariant under a change to an equivalent measure, we can (as in the proof of Theorem V.4 in [19]) switch to an equivalent probability measure under which $S$ is locally square-integrable. Therefore we impose without loss of generality from now on

**Assumption 1.** The semimartingale $S$ is locally square-integrable.

Let $S = S_0 + M + A$ be the canonical decomposition of $S$, where $M \in \mathcal{M}^{2,d}_{0,loc}(P)$ is an $\mathbb{R}^d$-valued locally square-integrable local martingale null at 0 and $A \in \mathcal{A}^{1,d}_{loc}(P)$ is an $\mathbb{R}^d$-valued predictable process of locally integrable variation $|A|$ also null at 0. By Propositions II.2.9 and II.2.29 in [13], there exist an increasing, locally integrable, predictable process
Definition 4.1. The projection on the predictable range of \( S \) is a predictable process \( \Pi^S : \Omega \rightarrow \mathbb{R}^{d \times d} \) which takes values in the orthogonal projections in \( \mathbb{R}^d \) and has the following property: If \( \vartheta \in \mathcal{L}(S) \) and \( \varphi \) is predictable, then \( \varphi \) is in \( \mathcal{L}(S) \) with \( \varphi \cdot S = \vartheta \cdot S \) if and only if \( \Pi^S \vartheta = \Pi^S \varphi \) \( P_B \)-a.e. We choose and fix one version of \( \Pi^S \).

As illustrated by the example in Section 3, the convergence in \( \mathcal{S}(P) \) of stochastic integrals does not imply in general that the integrands converge \( P_B \)-a.e. But like in the example, a subsequence of the projections of the integrands on the predictable range does.

Lemma 4.2. Let \( (\vartheta^n) \) be a sequence in \( \mathcal{L}(S) \) such that \( \vartheta^n \cdot S \rightarrow \vartheta \cdot S \) in \( \mathcal{S}(P) \). Then there exists a subsequence \( (n_k) \) such that \( \Pi^S \vartheta^{n_k} \rightarrow \Pi^S \vartheta \) \( P_B \)-a.e.

Lemma 4.3. Let \( C : \Omega \rightarrow 2^{\mathbb{R}^d \setminus \{\emptyset\}} \) be a predictable correspondence with closed values and such that the projection on the predictable range of \( S \) is not closed, i.e.,

\[
\tilde{F} = \{ (\omega, t) \in \Omega \mid \Pi^S (\omega, t) C(\omega, t) \text{ is not closed} \}
\]

has outer \( P_B \)-measure > 0. Then there exist \( \vartheta \in \mathcal{L}(S) \) and a sequence \( (\psi^n) \) of \( C \)-valued integrands such that \( \psi^n \cdot S \rightarrow \vartheta \cdot S \) in \( \mathcal{S}(P) \), but there exists no \( C \)-valued integrand \( \psi \) such that \( \psi \cdot S = \vartheta \cdot S \). Equivalently, there exists a sequence \( ([\psi^n]) \) in \( [C] \) such that \( [\psi^n] \xrightarrow{L(S)} [\vartheta] \) but \( [\vartheta] \notin [C] \), i.e., \( [C] \) is not closed in \( L(S) \).

Lemmas 4.2 and 4.3 as well as existence of \( \Pi^S \) will be shown in Section 5. Admitting that, we can now prove our first main result; related work in [15] is discussed in Section 6. Recall the definition of \( C = C^S \) from (2.1).

Theorem 4.4. Let \( C : \Omega \rightarrow 2^{\mathbb{R}^d \setminus \{\emptyset\}} \) be a predictable correspondence with closed values. Then \( C \cdot S \) is in \( \mathcal{S}(P) \) if and only if the projection of \( C \) on the predictable range of \( S \) is closed, i.e., \( \Pi^S (\omega, t) C(\omega, t) \) is closed \( P_B \)-a.e. Equivalently, for any sequence \( (\psi^n) \) of \( C \)-valued integrands such that \( \psi^n \cdot S \rightarrow X \) in \( \mathcal{S}(P) \), there exists a \( C \)-valued integrand \( \psi \) such that \( X = \psi \cdot S \) if and only if the projection of \( C \) on the predictable range of \( S \) is closed.
Proof. “⇒”: This implication follows immediately from Lemma 4.3.

“⇐”: Let \( \langle \psi^n \rangle \) be a sequence of \( C \)-valued integrands such that \( \psi^n \cdot S \to X \) in \( S(P) \). Then there exist \( \vartheta \in L(S) \) with \( X = \vartheta \cdot S \) and a subsequence, again indexed by \( n \), such that \( \Pi^S \psi^n \to \Pi^S \vartheta \) \( P_B \)-a.e. by Lemma 4.2 and the closedness in \( S(P) \) of the space of all stochastic integrals. So it remains to show that there exists a \( C \)-valued representative \( \psi \) of the limit class \( \vartheta = [\Pi^S \vartheta] \). To that end we observe that the \( P_B \)-a.e. closedness of \( \Pi^S(\omega,t)C(\omega,t) \) implies that \( \Pi^S \vartheta = \lim_{n \to \infty} \Pi^S \psi^n \in \Pi^S C \) \( P_B \)-a.e. By Proposition 2.7 the correspondences given by \( \{ \Pi^S(\omega,t)\vartheta(\omega,t) \} \), \( C'(\omega,t) = \{ \Pi^S(\omega,t)\vartheta(\omega,t) \} \cap \Pi^S(\omega,t)C(\omega,t) \) and \( C''(\omega,t) = \{ z \in \mathbb{R}^d \mid \Pi^S(\omega,t)z \in C'(\omega,t) \} \cap C(\omega,t) \) are predictable and closed-valued. Indeed, \( \Pi^S(\omega,t)\vartheta(\omega,t) \) is a predictable process; \( \{ z \in \mathbb{R}^d \mid \Pi^S(\omega,t)z \in C'(\omega,t) \} \) and \( \Pi^S C = \overline{\Pi^S C} \) are the preimage and (the closure of) the image of a closed-valued correspondence under a Carathéodory function, respectively; and thus \( C' \) and \( C'' \) are the intersections of two predictable and closed-valued correspondences and therefore predictable by Proposition 2.8. So there exists a measurable selector \( \psi \) of \( C'' \) on

\[
\text{dom}(C'') = \{ (\omega,t) \mid \Pi^S(\omega,t)\vartheta(\omega,t) \in \Pi^S(\omega,t)C(\omega,t) \}
\]

by Proposition 2.3. This can be extended to a \( C \)-valued integrand by using any measurable selector on \( (\text{dom}(C''))^c \), which is a \( \mathcal{P}_f \)-nullset. By construction \( \psi \) is \( C \)-valued and satisfies \( \Pi^S \psi = \Pi^S \vartheta \) \( P_B \)-a.e., which implies that \( \psi \in \vartheta \) by the definition of \( \Pi^S \). This completes the proof. \( \square \)

Theorem 4.4 gives as necessary and sufficient condition for the closedness of the space of \( C \)-constrained integrals of \( S \) that the projection of the constraint set \( C \) on the predictable range of \( S \) is closed. This takes into account information coming from both the semimartingale \( S \) and the constraints \( C \) as well as their interplay. We shall see below how this allows to recapture several earlier results as special cases.

Corollary 4.5. Let \( S = S_0 + M + A \) be in \( S^S_{row}(P) \) and suppose that

\[
[0]^M = \{ h a \mid h \text{ real-valued and predictable} \} \quad \text{ up to } P \otimes B \text{-a.e. equality} \quad (4.2)
\]

with the process \( a \) from (4.1). Then \( C \cdot S \) is closed in \( S(P) \) for all predictable correspondences \( C : \overline{\Pi} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\} \) taking closed values.

Proof. From Lemma 5.1 below we obtain that \( [0]^S = [0]^M \cap [0]^A = \{0\} \) due to (4.2) and therefore that \( \Pi^S = 1_{d \times d} \) by (5.2) below. So the projection of any closed-valued correspondence \( C \) on the predictable range of \( S \) is closed, which implies that \( C \cdot S \) is closed in \( S(P) \) by Theorem 4.4. \( \square \)

In applications from mathematical finance, \( S \) usually satisfies the so-called structure condition \( (SC) \), i.e., there exists an \( \mathbb{R}^d \)-valued predictable process \( \lambda \in \mathcal{L}^2_{loc}(M) \) such that \( A = \lambda \cdot (M,M) \) or, equivalently, \( a = c \lambda \) \( P_B \)-a.e.; this is a weak no-arbitrage type condition. In this situation, we obtain \( [0]^M \subseteq [0]^A \) by Lemma 5.1 below and therefore that condition (4.2) is valid if and only if \( [0]^M = \{0\} \) (up to \( P_B \)-a.e. equality), which means
that \( \hat{c} \) is \( P_B \)-a.e. invertible. This is the case covered in Lemma 3.1 in [20], where one has conditions only on \( S \) but not on \( C \). Basically this ensures that there are no redundant assets in the market, i.e., every stochastic integral is realised by exactly one integrand (up to \( P_B \)-a.e. equality).

The opposite extreme is to place conditions only on \( C \) that ensure closedness of \( C \cdot S \) for arbitrary semimartingales, as done in Theorem 3.5 of [6]. We recover this as a special case in the following corollary: note that in contrast to [6], the constraints need not be convex. Recall that a closed convex set \( K \subseteq \mathbb{R}^d \) is called continuous if its support function \( \delta(v|K) = \sup_{w \in K} w^\top v \) is continuous for all vectors \( v \in \mathbb{R}^d \) with \( |v| = 1 \); see [12].

**Corollary 4.6.** Let \( C : \Omega \to 2^{\mathbb{R}^d} \setminus \{\emptyset\} \) be a predictable correspondence with closed values. Then \( C^Y \cdot Y \) is closed in \( S(P) \) for all semimartingales \( Y \) if with probability 1 for all \( t \geq 0 \) all projections \( \Pi C(\omega, t) \) of \( C(\omega, t) \) are closed in \( \mathbb{R}^d \).

In particular, if with probability 1, all the \( C(\omega, t), t \geq 0 \), are compact, a polyhedral or a continuous convex set, then \( C^Y \cdot Y \) is closed in \( S(P) \) for all semimartingales \( Y \).

**Proof.** If a set is compact or polyhedral, all its projections again have the same property (see Corollary 2.15 in [17]), so they are in particular closed. Moreover, every projection of a continuous convex set is closed by Theorem 1.3 in [12]. Now if with probability 1 for all \( t \geq 0 \) all projections \( \Pi C(\omega, t) \) of \( C(\omega, t) \) are closed, then the projection \( \Pi^Y C \) of \( C \) on the predictable range of every semimartingale \( Y \) is closed \( P \otimes B^Y \)-a.e. So \( C^Y \cdot Y \) is closed in \( S(P) \) by Theorem 4.4.

Combining Theorem 4.4 with the example in Section 3 we obtain the following corollary. It is formulated for fixed sets \( K \), but can probably be generalised to predictable correspondences \( C \) by using measurable selections.

**Corollary 4.7.** Suppose \((\Omega, \mathcal{F}, P)\) is sufficiently rich. Let \( K \) be a fixed closed subset of \( \mathbb{R}^d \) and like in (2.1) define \( K^Y = \{ \psi \in L(Y) \mid \psi(\omega, t) \in K \text{ for all } (\omega, t) \} \). Then \( K^Y \cdot Y \) is closed in \( S(P) \) for all \( \mathbb{R}^d \)-valued semimartingales \( Y \) if and only if all projections \( \Pi K \) of \( K \) in \( \mathbb{R}^d \) are closed.

**Proof.** The “if” part follows immediately from Theorem 4.4. For the “only if” part, we argue by contradiction and assume that there exists a projection \( \Pi \) in \( \mathbb{R}^d \) such that \( \Pi K \) is not closed. Let \( W \) be a \( d \)-dimensional Brownian motion and set \( Y = \Pi^\top \cdot W \). Then the projection on the predictable range of \( Y \) is given by \( \Pi \) and therefore \( K^Y \cdot Y \) is not closed by Theorem 4.4, which completes the proof.

If the constraints are not only convex, but also conic, a characterisation of convex polyhedra due to Klee [17] gives an even sharper result.

**Corollary 4.8.** Let \( K \subseteq \mathbb{R}^d \) be a closed convex cone. Then \( K^Y \cdot Y \) is closed in \( S(P) \) for all \( \mathbb{R}^d \)-valued semimartingales \( Y \) if and only if \( K \) is polyhedral.

**Proof.** By Corollary 4.7, \( K^Y \cdot Y \) is closed in \( S(P) \) if and only if all projections \( \Pi K \) are closed in \( \mathbb{R}^d \). But Theorem 4.11 in [17] says that all projections of a convex cone are closed in \( \mathbb{R}^d \) if and only if it is polyhedral.
Remark 4.9. Armed with the above result, we can briefly come back to the proof of Proposition 2.13 in [16]. We have already pointed out in Section 3 that the argument in [16] uses the optional decomposition under constraints from [11], without verifying its Assumption 3.1. In view of Corollary 4.8, we can be more precise: The argument in [16] as it stands (i.e., without assumptions on $S$) only works for polyhedral cone constraints; for others one could by Corollary 4.8 construct a semimartingale $S$ giving a contradiction.

We now turn to our second main result. Recall again the definition of $C$ from (2.1) and note that for a correspondence $C$ with convex values, $C$ is the prime example of a predictably convex space of integrands. The next theorem shows that this is actually the only class of predictably convex integrands if we assume in addition that the resulting space of stochastic integrals is closed in $S(P)$. The result and its proof are inspired from Theorem 3 and 4 in [8], but nevertheless require quite a number of modifications.

Theorem 4.10. Let $\mathfrak{C} \subseteq L(S)$ be non-empty. Then $\mathfrak{C} \cdot S$ is predictably convex and closed in the semimartingale topology if and only if there exists a predictable correspondence $C : \bar{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ with closed and convex values such that the projection of $C$ on the predictable range of $S$ is closed, i.e., $\Pi^S(\omega,t)C(\omega,t)$ is closed $P_B$-a.e., and such that $\mathfrak{C} \cdot S = C^S \cdot S$, i.e.,

$$\mathfrak{C} \cdot S = \{\psi \cdot S \mid \psi \in \mathfrak{C}\} = \{\psi \cdot S \mid \psi \in L(S) \text{ and } \psi(\omega,t) \in C(\omega,t) \text{ for all } (\omega,t)\}.$$

Proof. “$\Leftarrow$”: The pointwise convexity of $C$ immediately implies that $\mathfrak{C} \cdot S$ is predictably convex, and closedness follows from Theorem 4.4.

“$\Rightarrow$”: Like at the end of Section 2, we view predictable processes on $\Omega$ as $\mathcal{P}$-measurable random variables on $\bar{\Omega} = \Omega \times [0,\infty)$. Since we are only interested in a non-empty space of stochastic integrals with respect to $S$, there is no loss of generality if we replace $\mathfrak{C}$ by $\{\vartheta - \varphi \in L(S) \mid \vartheta \in [\mathfrak{C}]\}$ for some $\varphi \in \mathfrak{C}$ and identify this with a subspace of $L^0(\bar{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d)$ which contains zero. Indeed, if the assertion is true for $\mathfrak{C} - \varphi$ with the correspondence $\hat{C}$, then it is also true for $\mathfrak{C}$ with $C = \hat{C} + \varphi$, which is a predictable correspondence by Proposition 2.7. In order to apply Proposition 2.9, we truncate $\mathfrak{C}$ to obtain

$$\mathfrak{C}^q = \{\psi \in \mathfrak{C} \mid \|\psi\|_{L^\infty} \leq q\} = \mathfrak{C} \cap B(0,q)_{L^\infty}$$

for $q \in \mathbb{Q}_+$. Then $\mathfrak{C}^q$ inherits predictable convexity from $\mathfrak{C}$ and is therefore a convex subset of $B(0,q)_{L^\infty}$. Moreover, $\mathfrak{C}^q$ is closed with respect to convergence in $P_B$-measure because its elements are uniformly bounded bounded by $q$ and $\mathfrak{C} \cdot S$ is closed in $S(P)$; this uses the fact that, easily proved via dominated convergence separately for the $M$- and $A$-integrals, that for any uniformly bounded sequence of integrands $(\psi^n)$ that converges pointwise, the stochastic integrals converge in $S(P)$. By a (non-trivial but well-known) application of the Krein-Šmulian and Alaoglu theorems, $\mathfrak{C}^q$ is therefore weak$^*$-compact, and so Proposition 2.9 gives a predictable correspondence $C^q : \bar{\Omega} \to 2^{\mathbb{R}^d} \setminus \{\emptyset\}$ with convex and compact values containing zero such that

$$\mathfrak{C}^q = \{\psi \in L^0(\bar{\Omega}, \mathcal{P}, P_B; \mathbb{R}^d) \mid \psi(\omega,t) \in C^q(\omega,t) P_B\text{-a.e.}\}.$$
By the definition of $C^q$ we obtain, after possibly modifying the sets on a $P_B$-nullset, that
\[ C^{q_2}(\omega, t) \cap B(0, q_1) = C^{q_1}(\omega, t) \quad \text{for all } (\omega, t) \in \Omega \]
for $0 < q_1 \leq q_2 < \infty$ by Lemma 12 in [8], since the graphs of $C^q$ are predictable by Proposition 2.6. As in [8] (see the argument after Lemma 12 there), this implies that the correspondence $C$ given by
\[ C(\omega, t) := \bigcup_{q \in \mathbb{Q}_+} C^q(\omega, t) \]
has convex and closed values and it remains to show that $\mathcal{E} \cdot S = C \cdot S$. To that end, suppose first that $\psi$ is in $\mathcal{E}$. By predictable convexity and since $0 \in \mathcal{E}$, $\psi^n := 1_{\{ |\psi| \leq n \}}$ is in $\mathcal{E}^n$ and therefore $C^n$- and hence $C$-valued. Since $(\psi^n)$ converges pointwise to $\psi$, the closedness of $C$ implies that $\psi$ is $C$-valued, so that $\psi \in \mathcal{E}$ and $\mathcal{E} \cdot S \subseteq C \cdot S$. Conversely, if $\psi$ is a $C$-valued integrand, then $\psi^n := 1_{\{ |\psi| \leq n \}}$ is $C^n$-valued and therefore in $\mathcal{E}^n \subseteq \mathcal{E}$. But $(\psi^n \cdot S)$ converges to $\psi \cdot S$ in $\mathcal{S}(P)$ and $\mathcal{E} \cdot S$ is closed in $\mathcal{S}(P)$. So the limit $\psi \cdot S$ is in $\mathcal{E} \cdot S$ and hence $\psi \in \mathcal{E}$ and $\mathcal{E} \cdot S \subseteq C \cdot S$. Finally, $C \cdot S = \mathcal{E} \cdot S$ is closed in $\mathcal{S}(P)$, and therefore Theorem 12 implies that $\Pi^2 C$ is closed $P_B$-a.e. This completes the proof. \(\square\)

Remark 4.11. 1) Theorem 4.10 can be used as follows. Start with any convex-valued correspondence $C$, form the space $\mathcal{C} \cdot S$ of corresponding stochastic integrals and take its closure in $\mathcal{S}(P)$. Then Theorem 4.10 tells us that we can realise this closure as a space of $\widetilde{C}$-constrained integrands, for some predictable correspondence $\widetilde{C}$ with convex and closed values. In other words, $\overline{\mathcal{C} \cdot S}^{\mathcal{S}(P)} = \overline{\mathcal{C} \cdot S}$; and one possible choice of $\widetilde{C}$ is $\widetilde{C} = (\Pi^2)^{-1}(C)$. Another possible choice would be $\widetilde{C} = \overline{C + \mathfrak{N}}$, where $\mathfrak{N}$ denotes the correspondence of null investments for $S$; see Section 6.

2) If we assume in Theorem 4.10 that $\mathcal{E} \subseteq \mathcal{L}^p_{\text{loc}}(S)$ for some $p \in [1, \infty)$, then $\mathcal{E} \cdot S \subseteq \mathcal{S}^p_{\text{loc}}(P)$, and it is closed in $\mathcal{S}^p(S)$ if and only if there exists $C$ as in the theorem. This can be useful for applications (e.g., mean-variance hedging under constraints, with $p = 2$).

5 Projection on the predictable range

In this section we construct the projection $\Pi^S$ on the predictable range of a general semimartingale $S$ in continuous time. The idea to introduce such a projection is taken from [23] and [9], where it was used to prove the fundamental theorem of asset pricing in discrete time. Moreover, it was also used for a continuous local martingale in [8] to investigate the structure of m-stable sets and in particular of the set of risk-neutral measures.

As already explained before Definition 4.1, a sufficient condition for $\varphi \cdot S = \vartheta \cdot S$ (up to $P$-indistinguishability) or, equivalently, $\varphi = \vartheta$ in $L(S)$, i.e., $[\varphi] = [\vartheta]$, is that $\varphi = \vartheta$ $P_B$-a.e. If we view again predictable processes on $\Omega$ as $\mathcal{P}$-measurable random variables on $\Omega = \Omega \times [0, \infty)$, then $\varphi = \vartheta$ $P_B$-a.e. is the same as saying that $\varphi = \vartheta$ in $L^0(\Omega, \mathcal{P}, P_B; \mathbb{R}^d)$. But in order to get a necessary and sufficient condition for $[\vartheta] = [\varphi]$,
we need to understand not only what $0 \in L(S)$ looks like, but rather the precise structure of (the equivalence class) $[0]$. This is achieved by $\Pi^S$.

The construction of $\Pi^S$ basically proceeds by generalising that of $\Pi^Y$ in the example in Section 3 and adapting the steps in [9] to continuous time. The idea is as follows. We start by characterising the equivalence class $[0]$ as a linear subspace of $L^0(\Omega, \mathcal{P}, P_B; \mathbb{R}^d)$. As this subspace satisfies a certain stability property, we can construct predictable processes $e^1, \ldots, e^d$ which form an “orthonormal basis” of $[0]$ in the sense that $[0]$ equals up to $P_B$-a.e. equality their linear combinations with predictable coefficients, i.e.,

$$[0] = \left\{ \sum_{j=1}^d h^j e^j \mid h^1, \ldots, h^d \text{ real-valued predictable} \right\} \text{ up to } P_B\text{-a.e. equality.} \quad (5.1)$$

Since these linear combinations do not contribute to the stochastic integral, we filter them out to obtain the part of the integrand which determines the stochastic integral, by defining

$$\Pi^S := \left( 1_{d\times d} - \sum_{j=1}^d e^j (e^j)^\top \right). \quad (5.2)$$

This construction of $\Pi^S$ then yields the projection on the predictable range of $S$ as in Definition 4.1.

To obtain $[0] = [0]^S$ as a linear subspace of $L^0(\Omega, \mathcal{P}, P_B; \mathbb{R}^d)$ we use the simplifying Assumption 1 that $S$ is in $S^2_{\text{loc}}(P)$ with canonical decomposition $S = S_0 + M + A$.

**Lemma 5.1.** Let $S = S_0 + M + A \in S^2_{\text{loc}}(P)$. Then:

1) $[0]^M = \{ \varphi \mid \varphi \text{ is } \mathbb{R}^d\text{-valued and predictable such that } \hat{c}\varphi = 0 \text{ } P_B\text{-a.e.} \}$

2) $[0]^A = \{ \varphi \mid \varphi \text{ is } \mathbb{R}^d\text{-valued and predictable such that } a^\top \varphi = 0 \text{ } P_B\text{-a.e.} \}$

3) $[0]^S = [0]^M \cap [0]^A$.

**Proof.** The inclusions “$\supseteq$“ follow immediately from the definition of the stochastic integral with respect to a locally square-integrable local martingale and a finite variation process, as the conditions on the right-hand side ensure that $\varphi$ is in $L^2(M)$ and $L^0(A)$. For the converse we start with $\varphi \in [0]^S$ and set $\varphi^n := 1_{\{ |\varphi| \leq n\}} \varphi$. Then $\varphi^n \cdot S = 0$ implies that $\varphi^n \cdot M = 0$ and $\varphi^n \cdot A = 0$ by the uniqueness of the canonical decomposition of $\varphi^n \cdot S$; this uses that $\varphi^n$ is bounded. Therefore we can reduce the proof of “$\subseteq$“ for 3) to that for 1) and 2). So assume now that $\varphi$ is either in $[0]^M$ or $[0]^A$ so that $\varphi^n \cdot M = 0$ or $\varphi^n \cdot A = 0$. But $\varphi^n$ is bounded, hence in $S^2_{\text{loc}}(M)$ or $L^0(A)$, for each $n$, and by the construction of the stochastic integral, we obtain that $\hat{c}\varphi^n = 0$ or $a^\top \varphi^n = 0$ $P_B$-a.e. Since $(\varphi^n)$ converges pointwise to $\varphi$, the inclusions “$\subseteq$“ for 1) and 2) follow by passing to the limit. \[\square\]

The following technical lemma which is a modification of Lemma 6.2.1 in [9] gives the announced “orthonormal basis” of $[0]^S$ in the sense of (5.1).
Lemma 5.2. Let \( U \subseteq L^0(\Omega, \mathcal{P}, P_B; \mathbb{R}^d) \) be a linear subspace which is closed with respect to convergence in probability and satisfies the following stability property:

\[
\varphi^1 \mathbb{1}_F + \varphi^2 \mathbb{1}_{F^c} \in U \quad \text{for all } \varphi^1 \text{ and } \varphi^2 \text{ in } U \text{ and } F \in \mathcal{P}.
\]

Then there exist \( e^j \in L^0(\Omega, \mathcal{P}, P_B; \mathbb{R}^d) \) for \( j = 1, \ldots, d \) such that

1) \( \{e^{j+1} \neq 0\} \subseteq \{e^j \neq 0\} \) for \( j = 1, \ldots, d - 1 \).

2) \( |e^j(\omega, t)| = 1 \) or \( |e^j(\omega, t)| = 0 \).

3) \( (e^j)^\top e^k = 0 \) for \( j \neq k \).

4) \( \varphi \in U \) if and only if there are \( h^1, \ldots, h^d \) in \( L^0(\Omega, \mathcal{P}, P_B; \mathbb{R}) \) with \( \varphi = \sum_{j=1}^d h^j e^j \), i.e.,

\[
U = \left\{ \sum_{j=1}^d h^j e^j \bigg| h^1, \ldots, h^d \text{ real-valued predictable} \right\}.
\]

Proof. The predictable processes \( e^1, \ldots, e^d \) with the properties 1) – 4) are the column vectors of the measurable projection-valued mapping constructed in Lemma 6.2.1 in [9]. Therefore their existence follows immediately from the construction given there.

Due to Lemma 5.1, the space \([0]^S\) satisfies the assumptions of Lemma 5.2. So we take a “basis” \( e^1, \ldots, e^d \) as in that result and define \( \Pi^S \) as in (5.2) by

\[
\Pi^S := \left( \mathbb{1}_{d \times d} - \sum_{j=1}^d e^j (e^j)^\top \right).
\]

Then \( \Pi^S(\omega, t) \) is the projection on the orthogonal complement of the linear space spanned in \( \mathbb{R}^d \) by \( e^1(\omega, t), \ldots, e^d(\omega, t) \) so that \( \Pi^S(\omega, t) \gamma \) is orthogonal to all \( e^i(\omega, t) \) for each \( \gamma \in \mathbb{R}^d \); and Lemma 5.2 says that each element of \([0]^S\) is a (random and time-dependent) linear combination of \( e^1, \ldots, e^d \), and vice versa. In particular, \( \vartheta - \Pi^S \vartheta \) is in \([0]^S\) for every predictable \( \mathbb{R}^d \)-valued \( \vartheta \). The next result shows that \( \Pi^S \) then satisfies the properties required in Definition 4.1. Note that \( \Pi^S \) is only defined up to \( P_B \)-nullsets since the \( e^j \) are; so we may have to choose one version for \( \Pi^S \) to be specific.

Lemma 5.3 (Projection on the predictable range of \( S \)). For a semimartingale \( S \) the projection \( \Pi^S \) on the predictable range of \( S \) exists, i.e., there exists a predictable process \( \Pi^S : \Omega \to \mathbb{R}^{d \times d} \) which takes values in the orthogonal projections in \( \mathbb{R}^d \) and has the following property: If \( \vartheta \in \mathcal{L}(S) \) and \( \psi \) is an \( \mathbb{R}^d \)-valued predictable process, then

\[
\psi \in \mathcal{L}(S) \text{ with } \psi \cdot S = \vartheta \cdot S \quad \iff \quad \Pi^S \psi = \Pi^S \vartheta \quad P_B \text{-a.e.} \tag{5.3}
\]
Proof. If we define $\Pi^S$ as above, Lemma 5.2 implies that $\Pi^S$ is predictable and valued in the orthogonal projections in $\mathbb{R}^d$, and it only remains to check (5.3). So take $\vartheta \in \mathcal{L}(S)$ and assume first that $\Pi^S \vartheta = \Pi^S \psi \ P_B$-a.e. The definition of $\Pi^S$ and Lemma 5.1 then yield that $\vartheta - \Pi^S \vartheta \psi$ and $\Pi^S \vartheta \psi$ are in $[0]^S$, which implies that $\Pi^S \vartheta = \vartheta - \left( \vartheta - \Pi^S \vartheta \right)$ and $\Pi^S \psi$ are in $\mathcal{L}(S)$ and also that $\vartheta \cdot S = (\Pi^S \vartheta) \cdot S = (\Pi^S \psi) \cdot S$. Because also $\psi - \Pi^S \psi$ is in $[0]^S \subseteq \mathcal{L}(S)$, we conclude that $\psi \in \mathcal{L}(S)$ with $\vartheta \cdot S = \psi \cdot S$. Conversely, if $\psi \cdot S = \vartheta \cdot S$, then $\psi - \vartheta \in [0]^S$, and we always have $(\psi - \vartheta) - \Pi^S (\psi - \vartheta) \in [0]^S$. Therefore $\Pi^S (\psi - \vartheta) \in [0]^S$ which says by Lemma 5.2 that for $P_B$-a.e. $(\omega, t)$, $\Pi^S (\psi - \vartheta)(\omega, t)$ is a linear combination of the $e^i(\omega, t)$. But the column vectors of $\Pi^S$ are orthogonal to $e^1, \ldots, e^d$ for each fixed $(\omega, t)$, and so we obtain $\Pi^S (\psi - \vartheta) = 0 \ P_B$-a.e., which completes the proof. 

With the existence of the projection on the predictable range established, it remains to prove Lemmas 4.2 and 4.3, which we recall for convenience.

**Lemma 4.2.** Let $(\vartheta^m)$ be a sequence in $\mathcal{L}(S)$ such that $\vartheta^m \cdot S \to \vartheta \cdot S$ in $\mathcal{S}(P)$. Then there exists a subsequence $(n_k)$ such that $\Pi^S \vartheta^m \to \Pi^S \vartheta \ P_B$-a.e.

Proof. As in the proof of Theorem V.4 in [19] we can switch to a probability measure $Q \sim P$ with $\frac{dQ}{dP}$ bounded and such that $S - S^0 = M^Q + A^Q \in \mathcal{M}^{2, d}_{loc}(Q) \oplus \mathcal{A}^{1, d}_{loc}(Q)$ and $\vartheta^m \cdot S \to \vartheta \cdot S$ in $\mathcal{M}^{2, d}_{loc}(Q) \oplus \mathcal{A}^{1, d}_{loc}(Q)$ along a subsequence, again indexed by $n$. Since $\frac{dQ}{dP}$ is bounded, $B$ from (4.1) is also locally $Q$-integrable and so there exist an $\mathbb{R}^d$-valued predictable process $a^Q$ and a predictable process $\tilde{c}^Q$ taking values in the symmetric positive semidefinite $d \times d$-matrices such that

$$(A^Q)^i = (a^Q)^i \cdot B \quad \text{and} \quad \langle (M^Q)^i, (M^Q)^j \rangle^Q = (\tilde{c}^Q)^{ij} \cdot B \quad \text{for } i, j = 1, \ldots, d.$$ 

This is seen by expressing the semimartingale characteristics of $S$ under $Q$ by those under $P$ via Girsanov’s theorem, writing $A^Q$ and $\langle M^Q, M^Q \rangle^Q$ in terms of semimartingale characteristics and then passing to differential characteristics with $B$ as predictable increasing process; see Theorem III.3.24 and Propositions II.2.29 and II.2.9 in [13]. Since $\vartheta^m \cdot S \to \vartheta \cdot S$ in $\mathcal{M}^{2, 1}_{loc}(Q) \oplus \mathcal{A}^{1, 1}_{loc}(Q)$, we obtain for a localising sequence $(\tau_k)$ that

$$E_Q \left[ \int_0^{\tau_k} (\vartheta^m - \vartheta)^\top \tilde{c}^Q (\vartheta^m - \vartheta) d B_s + \int_0^{\tau_k} |(\vartheta^m - \vartheta)^\top a^Q| d B_s \right] \to 0 \quad \text{as } n \to \infty,$$

which implies that there exists a subsequence, again indexed by $n$, such that

$$(\vartheta^m - \vartheta)^\top \tilde{c}^Q (\vartheta^m - \vartheta) \to 0 \quad \text{and} \quad |(\vartheta^m - \vartheta)^\top a^Q| \to 0 \quad Q \otimes B\text{-a.e.} \quad (5.4)$$

Like the stochastic integral, $[0]^S$ is invariant under a change to an equivalent measure. So Lemma 5.1 under $Q$ gives

$$[0]^S = \{ \varphi \mid \varphi \ \mathbb{R}^d\text{-valued predictable with } \tilde{c}^Q \varphi = 0 \text{ and } (a^Q)^\top \varphi = 0 \quad Q \otimes B\text{-a.e.} \}.$$
Let $e^1, \ldots, e^d$ be predictable processes from Lemma 5.2 which satisfy properties 1) – 4) for $[0]^S$ and set

$$U = \{ \psi \mid \psi \text{ $\mathbb{R}^d$-valued predictable with } \psi^\top \varphi = 0 \text{ $Q \otimes B$-a.e. for all } \varphi \in [0]^S \},$$

$$V = \{ \psi \mid \psi \text{ $\mathbb{R}^d$-valued predictable with } \psi^\top \varphi = 0 \text{ $Q \otimes B$-a.e. for all } \varphi \in [0]^{M^Q} \}$$

so that loosely speaking, $U^\perp = [0]^S$ and $V^\perp = [0]^{M^Q}$. Then $[0]^{M^Q} \cap U$ and $[0]^{A^Q} \cap V$ satisfy the assumptions of Lemma 5.2 and therefore there exist predictable processes $u^1, \ldots, u^d$ and $v^1, \ldots, v^d$ with the properties 1) – 4) for $[0]^{M^Q} \cap U$ and $[0]^{A^Q} \cap V$, respectively. By the definition of $U$ and $V$ we also obtain, using $[0]^S = [0]^{M^Q} \cap [0]^{A^Q}$, that

$$(e^j)^\top u^k = (e^j)^\top v^k = (u^j)^\top v^k = 0 \text{ $Q \otimes B$-a.e. for } j, k = 1, \ldots, d,$$

$$[0]^{M^Q} = \left\{ \sum_{j=1}^d h^j e^j + \sum_{k=1}^d h^{d+k} u^k \mid h^1, \ldots, h^{2d} \text{ real-valued predictable} \right\},$$

$$[0]^{A^Q} = \left\{ \sum_{j=1}^d h^j e^j + \sum_{k=1}^d h^{d+k} v^k \mid h^1, \ldots, h^{2d} \text{ real-valued predictable} \right\}$$

up to $Q \otimes B$-a.e. equality. Therefore $\Pi^{M^Q}$ and $\Pi^{A^Q}$ can be written as

$$\Pi^{M^Q} = \left( 1_{d \times d} - \sum_{j=1}^d (e^j)^\top - \sum_{k=1}^d u^k (u^k)^\top \right),$$

$$\Pi^{A^Q} = \left( 1_{d \times d} - \sum_{j=1}^d (e^j)^\top - \sum_{k=1}^d v^k (v^k)^\top \right),$$

and we have

$$\left( \sum_{k=1}^d v^k (v^k)^\top \right) \Pi^{A^Q} \vartheta^n = \left( \sum_{k=1}^d v^k (v^k)^\top \right) \vartheta^n,$$

(5.5)

all up to $Q \otimes B$-a.e. equality. Since $\Pi^{M^Q}(\vartheta^n - \vartheta)$ and $\Pi^{A^Q}(\vartheta^n - \vartheta)$ are by Lemma 5.1 $Q \otimes B$-a.e. valued in $\text{Im}(\hat{e}^Q)$ and $\text{Im}(\hat{a}^Q)^\top$, respectively, (5.4) yields $\Pi^{M^Q} \vartheta^n \rightarrow \Pi^{M^Q} \vartheta$ and $\Pi^{A^Q} \vartheta^n \rightarrow \Pi^{A^Q} \vartheta$ $Q \otimes B$-a.e. From the latter convergence and (5.5), it follows that

$$\left( \sum_{k=1}^d v^k (v^k)^\top \right) \vartheta^n \rightarrow \left( \sum_{k=1}^d v^k (v^k)^\top \right) \vartheta \text{ $Q \otimes B$-a.e., and since}$$

$$\Pi^S = \Pi^{M^Q} + \left( \sum_{k=1}^d v^k (v^k)^\top \right) \text{ $Q \otimes B$-a.e.}$$

and $Q \otimes B \sim P_B$, we obtain by combining everything that $\Pi^S \vartheta^n \rightarrow \Pi^S \vartheta$ $P_B$-a.e. 

\[\square\]
The only result whose proof is now still pending is Lemma 4.3. This provides the general (and fairly abstract) version of the counterexample in Section 3, as well as the necessity part for the equivalence in Theorem 4.4.

**Lemma 4.3.** Let \( C : \tilde{\Omega} \rightarrow 2^{\mathbb{R}^d} \setminus \{\emptyset\} \) be a predictable correspondence with closed values and such that the projection on the predictable range of \( S \) is not closed, i.e.,

\[
\tilde{F} = \{(\omega, t) \in \tilde{\Omega} \mid \Pi^S(\omega, t)C(\omega, t) \text{ is not closed}\}
\]

has outer \( P_B \)-measure > 0. Then there exist \( \vartheta \in \mathcal{L}(S) \) and a sequence \( (\psi^n) \) of \( C \)-valued integrands such that \( \psi^n \cdot S \rightarrow \vartheta \cdot S \) in \( \mathcal{S}(P) \), but there exists no \( C \)-valued integrand \( \psi \) such that \( \psi \cdot S = \vartheta \cdot S \). Equivalently, there exists a sequence \( \{[\psi^n]\} \) in \( [C] \) such that \( [\psi^n] \overset{L(S)}{\rightarrow} [\vartheta] \) but \([\vartheta] \notin [C]\), i.e., \([C]\) is not closed in \( L(S) \).

**Proof.** The basic idea is to construct a \( \vartheta \in \mathcal{L}(S) \) which is valued in \( \overline{\Pi^S C} \setminus \Pi^S C \) on some \( F \in \mathcal{P} \) with \( F \subseteq \tilde{F} \) and \( P_B(F) > 0 \), and in \( C \) on \( F^c \). Then there exists no \( C \)-valued integrand \( \psi \in [\vartheta] \) by the definition of \( \Pi^S \) since \( \Pi^S \vartheta \notin \Pi^S C \) on \( F \); but one can construct a sequence \( (\psi^n) \) of \( C \)-valued integrands with \( \Pi^S \psi^n \rightarrow \Pi^S \psi \) pointwise since \( \Pi^S \vartheta \in \Pi^S C \). However, this is technically a bit more involved for several reasons: While \( C, \Pi^S C \) and \( \overline{\Pi^S C} \) are all predictable, \( (\Pi^S C)^c \) need not be; so \( \tilde{F} \) need not be predictable, and one cannot use Proposition 2.3 to obtain a measurable selector. In addition, \( \overline{\Pi^S C} \setminus \Pi^S C \) need not be closed-valued.

We first argue that \( \tilde{F} = \mathcal{P}_{P_B} \)-measurable. Let \( \overline{B(0, n)} \) be a closed ball of radius \( n \) in \( \mathbb{R}^d \). Then \( \Pi^S(C \cap \overline{B(0, n)}) \) is compact-valued as \( C \) is closed-valued. Since \( C \) is predictable and \( \Pi^S(\omega, t)x \) with \( x \in \mathbb{R}^d \) is a Carathéodory function, \( \overline{\Pi^S C} \) is predictable by Proposition 2.7. By the same argument, \( \Pi^S\left(C \cap \overline{B(0, n)}\right) = \bigcup_{n=1}^{\infty} \Pi^S(C \cap \overline{B(0, n)}) \) is predictable since \( C \cap \overline{B(0, n)} \) is, and then so is \( \Pi^S C = \bigcup_{n=1}^{\infty} \Pi^S(C \cap \overline{B(0, n)}) \) as a countable union of predictable correspondences; see Proposition 2.8. Then Proposition 2.6 implies that \( \Pi^S C \) and \( \Pi^S(C \cap \overline{B(0, n)}) \) have predictable graph and hence so does \( \Pi^S C \). Therefore \( \text{gr}(\overline{\Pi^S C}) \cap (\text{gr}(\Pi^S C))^c \) is \( \mathcal{P} \otimes B(\mathbb{R}^d) \)-measurable, and so by Proposition 2.5,

\[
\tilde{F} = \{(\omega, t) \in \tilde{\Omega} \mid \Pi^S(\omega, t)C(\omega, t) \text{ is not closed}\}
\]

\[
= \{(\omega, t) \in \tilde{\Omega} \mid \Pi^S(\omega, t)C(\omega, t) \setminus \Pi^S(\omega, t)C(\omega, t) \neq \emptyset\}
\]

\[
= \pi_{\overline{\Omega}}\left(\text{gr}(\overline{\Pi^S C}) \cap (\text{gr}(\Pi^S C))^c\right)
\]

is indeed \( \mathcal{P}_{P_B} \)-measurable. Thus there exists a predictable set \( F \subseteq \tilde{F} \) with \( P_B(F) > 0 \).

Now fix some \( C \)-valued integrand \( \tilde{\psi} \in \mathcal{L}(S) \) and define the correspondence \( C' \) by

\[
C'(\omega, t) = \begin{cases} 
\Pi^S(\omega, t)C(\omega, t) \setminus \Pi^S(\omega, t)C(\omega, t) & \text{for } (\omega, t) \in F, \\
\tilde{\psi}(\omega, t) & \text{else}.
\end{cases}
\]

Then \( C' \) has non-empty values and predictable graph and therefore admits a \( P_B \)-a.e. measurable selector \( \psi \) by Proposition 2.4. By possibly subtracting a predictable \( P_B \)-nullset
from $F$, we can without loss of generality assume that $\vartheta$ takes values in $C'$. Moreover, the predictable sets $F_n := F \cap \{|\vartheta| \leq n\}$ increase to $F$ and so we can, by shrinking $F$ to some $F_n$ if necessary, assume that $\vartheta$ is uniformly bounded in $(\omega, t)$ on $F$. Let $\{\psi^n | m \in \mathbb{N}\}$ be a Castaing representation of $\vartheta$. Then $\Pi^S \vartheta = \{\Pi^S \vartheta^m | m \in \mathbb{N}\}$, and because $\vartheta \in \Pi^S \vartheta$, we can find for each $n \in \mathbb{N}$ a predictable process $\psi^n$ such that $\Pi^S(\omega, t) \psi^n(\omega, t) \in \vartheta(\omega, t) + B(0, \frac{1}{n})$ on $F$ and $\psi^n = \tilde{\psi}$ on $F^c$. Note that on $F$, we have $\vartheta \in \Pi^S \vartheta \subseteq \Pi^S \mathbb{R}^d$ and therefore $\Pi^S \vartheta = \vartheta$; so $\Pi^S \vartheta = 1_{F^c} \vartheta + 1_F \Pi^S \tilde{\psi}$ and this shows that $\Pi^S \psi^n \to \Pi^S \vartheta$ uniformly in $(\omega, t)$ by construction. Since $\Pi^S \vartheta \in \mathcal{L}(S)$ because $\vartheta$ is bounded on $F$, we thus first get $\Pi^S \psi^n \in \mathcal{L}(S)$, hence $\psi^n \in \mathcal{L}(S)$, and then also that $\psi^n \cdot S \to \vartheta \cdot S$ in $\mathcal{S}(P)$ by dominated convergence. But now $\{\Pi^S \vartheta\} \cap \Pi^S R = \emptyset$ on $F$ shows by Lemma 5.3 that there exists no $C$-valued integrand $\psi \in [\vartheta]$ and therefore $[\vartheta] \notin [C]$. This completes the proof.

6 Related work

We have already explained how our results generalise most of the existing literature on optimisation problems under constraints. In this section, we discuss the relation of our results to the work of Karatzas and Kardaras [15].

We start by introducing the terminology of [15]. As in [13], Theorem II.2.34, the semimartingale $S$ has the canonical representation

$$S = S^c + \tilde{A} + [x 1_{\{|x| \leq 1\}}] * (\mu - \eta) + [x 1_{\{|x| > 1\}}] * \mu$$

with the jump measure $\mu$ of $S$ and its predictable compensator $\eta$. Then the triplet $(b, c, \nu)$ of predictable characteristics of $S$ consists of a predictable $\mathbb{R}^d$-valued process $b$, a predictable nonnegative-definite matrix-valued process $c$ and a predictable process $\nu$ with values in the set of Lévy measures such that (compare with (4.1))

$$\tilde{A} = b \cdot B, \quad [S^c, S^c] = c \cdot B \quad \text{and} \quad \eta = \nu \cdot B,$$

where $\tilde{B} := \sum_{i=1}^d \left( [S^c, S^c]^i + \text{Var}(\tilde{A}^i) + [x 1_{\{|x| \leq 1\}}] * \eta \right)$. Note that for this discussion, we can and do choose $B := \tilde{B}$ in (4.1). For a given $S$ with triplet $(b, c, \nu)$, the linear subspace of null investments $\mathfrak{N}$ is given by the predictable correspondence

$$\mathfrak{N}(\omega, t) := \{ z \in \mathbb{R}^d | z^\top c = 0, \nu(\omega, t)(\{ x | z^\top x \neq 0 \}) = 0 \text{ and } z^\top b(\omega, t) = 0 \}$$

(see Definition 3.6 in [15]). As in Definition 3.7 in [15], a correspondence $C : \overline{\Omega} \to 2^{\mathbb{R}^d}$ is said to impose predictable closed convex constraints if

0) $\mathfrak{N}(\omega, t) \subseteq C(\omega, t)$ for all $(\omega, t) \in \overline{\Omega},$

1) $C(\omega, t)$ is a closed and convex set for all $(\omega, t) \in \overline{\Omega}$, and

2) $C$ is predictable.
To avoid confusion, we call constraints with 0), 1) and 2) KK-constraints in the sequel.

In the comment following their Theorem 4.4 on p. 467 in [15], Karatzas and Kardaras (KK) remark that \( C \cdot S \) is closed in \( S(P) \) if \( C \) describes KK-constraints. For comparison, our Theorem 4.4 starts with \( C \cdot S \) is then closed in \( S(P) \) if and only if \( \Pi^S C \) is closed \( P_B \text{-a.e.} \). So we do not need convexity of \( C \), and our condition on \( C \cdot S \) is not only sufficient, but also necessary.

Before explaining things in more detail, we make the simple but important observation that

\[
0) \text{ plus } 1) \text{ imply that } C + \mathfrak{N} = C \quad \text{(for all } (\omega, t) \in \Omega). \tag{6.1}
\]

Indeed, each \( \mathfrak{N}(\omega, t) \) is a linear subspace, hence contains 0, and so \( C \subseteq C + \mathfrak{N} \). Conversely, \( \frac{1}{\varepsilon}z \in \mathfrak{N} \subseteq C \) for every \( z \in \mathfrak{N} \) and \( \varepsilon > 0 \) due to 0; so for every \( c \in C \), \( (1 - \varepsilon)c + z \in C \) by convexity and hence \( c + z = \lim_{\varepsilon \downarrow 0} (1 - \varepsilon)c + z \) is in \( C \) by closedness, giving \( C + \mathfrak{N} \subseteq C \).

As a matter of fact, KK say, but do not explicitly prove, that \( C \cdot S \) is closed in \( S(P) \). However, the clear hint they give suggests the following reasoning. Let \( (\vartheta^n) \) be a sequence in \( C \) such that \( (\vartheta^n \cdot S) \) converges to \( X \) in \( S(P) \). By the proof of Theorem V.4 in [19], there exist \( \tilde{\vartheta}^n \in [\vartheta^n] \) and \( \vartheta \in \mathcal{L}(S) \) such that \( \vartheta \cdot S = X \) and \( \tilde{\vartheta}^n \to \vartheta \text{ } P_B \text{-a.e.} \). From the description of \( \mathfrak{N} \) in Section 3.3 in [15], \( \tilde{\vartheta}^n \in [\vartheta^n] \) translates into \( \tilde{\vartheta}^n - \vartheta^n \in \mathfrak{N} P_B \text{-a.e.} \) or \( \tilde{\vartheta}^n \in \vartheta^n + \mathfrak{N} P_B \text{-a.e.} \). Because each \( \vartheta^n \) has values in \( C \), (6.1) thus shows that each \( \vartheta^n \) can be chosen to be \( C \)-valued, and by the closedness of \( C \), the same is then true for the limit \( \vartheta \) of \( (\vartheta^n) \). Hence we are done.

In order to relate the KK result to our work, we now observe that

\[
0) \text{ plus } 1) \text{ imply that } \Pi^S C \text{ is closed } P_B \text{-a.e.}
\]

To see this, we start with the fact that the null investments \( \mathfrak{N} \) and \( [0]^S \) are linked by

\[
[0]^S = \{ \varphi \mid \varphi \mathbb{R}^d \text{-valued predictable with } \varphi \in \mathfrak{N} P_B \text{-a.e.} \}; \tag{6.2}
\]

see Section 3.3 in [15]. Recalling that \( \Pi^S \) is the projection on the orthogonal complement of \( [0]^S \), we see from (6.2) that the column vectors of \( \Pi^S \) are \( P_B \text{-a.e.} \) generating system of \( \mathfrak{N}^\perp \) so that the projection of \( \vartheta \in \mathcal{L}(S) \) on the predictable range of \( S \) can be alternatively defined \( P_B \text{-a.e.} \) as a measurable selector of the closed-valued predictable correspondence \( \{ \vartheta + \mathfrak{N} \} \cap \mathfrak{N}^\perp \) or \( P_B \text{-a.e.} \) as the pointwise projection \( \Pi^{\mathfrak{N}(\omega,t)} \vartheta(\omega,t) \) in \( \mathbb{R}^d \) of \( \vartheta(\omega,t) \) on \( \mathfrak{N}(\omega, t) \), which is always a predictable process. This yields \( \Pi^S C = \{ C + \mathfrak{N} \} \cap \mathfrak{N}^\perp P_B \text{-a.e.;} \) but by (6.1), \( C + \mathfrak{N} = C \text{ due to } 0) \text{ and } 1) \), and so \( \Pi^S C \) is \( P_B \text{-a.e.} \) closed since \( C \) and \( \mathfrak{N}^\perp \) are.

In the KK notation, we could reformulate our Theorem 4.4 as saying that for a predictable and closed-valued \( C \), the space \( C \cdot S \) is closed in \( S(P) \) if and only if \( C + \mathfrak{N} \) is closed \( P_B \text{-a.e.} \). This is easily seen from the argument above showing that \( \Pi^S C = \{ C + \mathfrak{N} \} \cap \mathfrak{N}^\perp P_B \text{-a.e.} \) if \( C \) is also convex-valued, 0) is a simple and intuitive sufficient condition; it seems however more difficult to find an elegant formulation without convexity.

The difference between our constraints and the KK formulation in [15] is as follows. We fix a set \( C(\omega, t) \) of constraints and demand that the strategies should lie in \( C(\omega, t) \)
pointwise, so that \( \vartheta(\omega, t) \in C(\omega, t) \) for all \( (\omega, t) \). KK in contrast only stipulate that \( \vartheta(\omega, t) \in C(\omega, t) + \mathfrak{N}(\omega, t) \) or, equivalently, that \( [\vartheta] \in [C] \). At the level of wealth (given by the stochastic integral \( \vartheta \cdot S \)), this makes no difference since all \( \mathfrak{N} \)-valued processes have integral zero. But for practical checking and risk management, it is much simpler if one can just look at the strategy \( \vartheta \) and tick off whether or not it lies in \( C \). If \( S \) has complicated redundancy properties, it may be quite difficult to see whether one can bring \( \vartheta \) into \( C \) by adding something from \( \mathfrak{N} \). Of course, when discussing the closedness of the space of integrals \( \vartheta \cdot S \), we face the same level of difficulty when we have to check whether \( \Pi^S C \) is closed \( P_B \)-a.e. But for actually working with given strategies, we believe that our formulation of constraints is more natural and simpler to handle.

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