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A Credit Risk Model Incorporating Microstructural Dependencies and Stochastic Recovery

Matthias Jüttner

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A credit risk model incorporating microstructural dependencies and stochastic recovery*

Matthias P. Jüttner†

University of Zürich and Swiss Finance Institute

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†**Correspondence Information:** Swiss Banking Institute, Plattenstrasse 14, CH-8032 Zurich, Switzerland; `mailto: juettner@isb.uzh.ch`

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Abstract

A credit risk model for determining aggregated portfolio losses is suggested. Beside the common macrostructural dependencies between asset and recovery value, we incorporate possible inter-firm relations among the obligors of the portfolio. Through this channel we also establish related default probabilities and correlation between probability of default and loss given default. Proposing a recursive approximation we obtain explicit representations for the asset value. We compare the model with the IRB approach of Basel II with respect to the implied asset correlation and comparative statics of the conditional expected loss. Due to this analysis of the dynamics we realize that the regulatory framework neglects many important relations, e.g., among obligors and also between risk components. An adequate risk assessment is consequently not possible.

1 Introduction

Due to the subprime crisis financial institutions around the world have reported tremendous losses. Further consequences for all parts of the economy are entirely unforeseen. Central banks and governments around the world take steps against the crisis, decreasing interest rates, establishing spending programs, relaxing financial accounting methods or discussing more rigorous supervisory processes and regulations. In this paper we focus especially on aspects associated with capital requirements for credit risk. The key element is the computation of the distribution of aggregate credit portfolio losses. One of the crucial parameters is the expected loss (EL), which is defined as the product of three risk components: probability of default (PD), loss given default (LGD) and the exposure at default (EAD). The LGD describes the magnitude of likely loss on the exposure and is expressed as a percentage of the exposure.[1] Moreover, the loss is contingent upon the amount to which the bank was exposed to at the time of default, commonly expressed as EAD. Like in many credit risk models, also the rules of the Internal Rating-Based Approach (IRB approach) of the Basel Capital Accord are implicitly based on the assumption of independent LGD rates and default events. However, there is strong evidence of correlation between LGDs and default events. Relevant papers supporting this observation are for instance Frye (2000a, 2003), Düllmann and Trapp (2004) and Altman et al. (2005). Each of the components of the EL can be affected by systematic risk factors. For instance, the current decline of real estate prices has a positive impact on the LGD of loans which are secured by means of collaterals. Therefore, during an economic downturn the misfortune of banks may be twofold: a higher rate of default among their borrowers (see Keenan (2000)), and a lower rate of recovery on defaulted loans (see Schuermann (2004)). Additionally, firm interdependence, e.g., legal or business connections, can produce correlated PDs. The credit deterioration of a counterparty can trigger the credit deterioration of other counterparties through these inter-firm links. Beside common systematic factors, also the dependence between PD and LGD can be established by firm interdependence. The lower recovery rate of firm A implies ceteris paribus a higher PD of the counterparty firm B. In the case that firm A defaults, only a smaller fraction of the amount of outstanding receivables or debt can be reimbursed, which weakens the financial condition of firm B and therefore increases the PD of the firm. At the bottom line, PD and LGD are time-varying and are affected by common systematic factors. Inter-firm links can create correlated PDs and can also establish the dependence between the risk components.

We provide a new structural credit risk model, which accounts for all described dependencies and contains as a special case the IRB approach of Basel II. Assuming that the EAD is independent of the other components, we start by modeling the recovery rate and PD explicitly, where both depend on macroe-

[1] We use the terms "loss given default" and "recovery rate" interchangeably. It is clear that a higher recovery rate, value of the debt of an entity after it defaults (in percentages), implies a lower LGD of same size, vice versa.

conomic variables. The shared set of systematic factors allows to capture the empirical observation that both default and recovery are affected by the state of the economy. Moreover, the common systematic factors create correlation between the PDs and LGDs of the obligors in the credit portfolio itself. This basic setup is similar to Andersen and Sidenius (2004/05). However, there the fact is disregarded that on the one hand the connection between PD and LGD cannot only be established by the dependence on macrostructural factors but also by inter-firm connections. On the other hand, a lower recovery rate has a negative impact on the PD. By incorporating business dependencies we can model default contagion and counterparty risk explicitly. We integrate the recovery dynamics into the macrostructural setup following Egloff et al. (2007). In this setting, asset values depend as usual on a systematic and idiosyncratic risk factor. Here, the idiosyncratic factor can be described by a function which depends on firm dependencies and macroeconomic variables. Thus, macroeconomic shocks can have a direct impact on the asset values of a firm and an indirect impact through the inter-firm channel. We extend this modeling perspective also anticipating the possible defaults and corresponding losses of interlinked firms. Hence, we establish a link between PD and LGD, i.e., the dependencies serve as channel for defaults triggered by other than pure macroeconomic factors. A comparison of the introduced model with the IRB approach is provided. Both frameworks are analyzed with respect to the induced asset correlation and some comparative statics of the conditional expected loss, which is the core for determining the capital charge. Due to this sensitivity analysis we can show which terms additionally to Basel II are relevant for the dynamics of the conditional expected loss. The regulatory framework neglects many important dynamics. This result leads to a modified formula for determining the capital requirement.

The link between recovery rates and default rates has traditionally been disregarded by credit risk models. As most of them focused on default risk, the key element in current credit risk models for pricing credit derivatives is the mechanism generating the dependence between default times for instance with factor copulas such as Li (2000) or with direct interaction between default intensities such as Jarrow and Yu (2001).[2] Treating the recovery rate either as a constant parameter (e.g. Jarrow and Turnbull (1995) or Backhaus and Frey (2008)) or as a stochastic variable independent from the PD, the models ignore the fact that LGD is itself an important driver of the portfolio credit risk, because of its dependence on economic cycles. During the last years, new approaches explicitly modeling the relationship between PD and LGD have been developed. For instance, in the model of Frye (2000b) or Chabaane et al. (2004), PD and LGD both depend on the state of the macroeconomy, additional to idiosyncratic risk factors. Andersen and Sidenius (2004/05) also randomize recovery rates and tailor the modeling to the pricing of synthetic CDOs. The synchronized dependence on macroeconomic fundamentals can lead to a clustering in defaults, correlated LGDs and relations between the components, but

[2] A comparative analysis of factor models and CDO pricing models in general can be found in Andersen and Sidenius (2005) and Burtschell et al. (2008).

firm interdependencies are not anticipated by all of these models. Therefore our model contributes to the literature by capturing this intricated dependency structure, where we also establish connections between the risk components by the inter-firm link structure.

The paper is organized as follows. Section 2 is devoted to our model. We present the basic setting, introduce the notion of microstructure and illustrate the mechanism with an example. The crucial step is the integration of the firm interdependencies into the basic model. After deriving the main properties we approximate recursively the value dynamics. In section 3 we compare our model with the IRB approach of Basel II. We analyze the induced asset correlation of both models and provide a sensitivity analysis of the conditional expected loss. Section 4 concludes.

2 The model

2.1 Basic setup

In contrast to reduced form models, where the default times of the obligors are modeled by an stochastic intensity process, we introduce a structural model in line with Merton (1974) and Vasicek (1987). The details of our basic model are drawn from Andersen and Sidenius (2004/05). We assume that our economy consists of N firms, where $N^* \leq N$ are default-risky obligors in the credit portfolio of the considered bank. Default is triggered when the asset value A_i of a firm i falls below some fixed threshold γ_i.[3] The *asset value* A_i of firm i can be described by

$$A_i = a_i^T Z + \sqrt{1 - \|a_i\|^2}\epsilon_i \quad \text{for } i = 1, \ldots, N, \tag{1}$$

where a_i^T is a transposed real-valued d-dimensional vector with $\|a_i\| < 1$ and $a_i^j \geq 0$ for $j = 1, \ldots, d$ (deterministic factor loadings). The d-dimensional random variable $Z = (Z_1, \ldots, Z_d) \sim \mathcal{N}(0, \boldsymbol{I})$ refers to the vector of systematic factors (macroeconomic variables, e.g., factor prices), where $\boldsymbol{I}$ denotes the identity matrix. We assume that $Z_j \sim \mathcal{N}(0, 1)$ *iid* for $j = 1, \ldots, d$. The vector $\epsilon = (\epsilon_1, \ldots, \epsilon_d) \sim \mathcal{N}(0, \boldsymbol{I})$ describes the *iid* idiosyncratic shocks, in particular independent of Z. ϵ_i denotes the asset value shock of firm i. The parameter a_i controls how much the systematic factor affects firm i. If a firm has a low value of $\|a_i\|$ the connection to the state of the economy is little. Therefore the idiosyncratic factor may be far more important. Otherwise the asset value is highly cyclical. The vector Z is common for all firms in the economy. The weighting of Z and ϵ as well as the restrictions on a_i are necessary to obtain a standard normally distributed asset value.

[3]Here we assume that the asset value is the unique driver of default events. However, other effects such as liquidity shortage can drive defaults (see Cetin et al. (2004)).

Next, we model the recovery rates $R_i \in [0, 1]$, $i = 1, \ldots, N$. Conditioned on default, firms in the economy generate loss amounts l_i, which are assumed to be random and bounded, i.e., $l_i \in [0, l_i^{\max}]$ with $l_i^{\max} \in \mathbb{R}_+$. The loss amount can then be described by

$$l_i = l_i^{\max}(1 - R_i). \tag{2}$$

The paramter $l_i^{\max}$ represents the total loss (sum of all receiveables), which is realized at the default date. We assume that the random *recovery rate*[4] R_i of firm i depends on the macroeconomic situation and on an idiosyncratic component. It can be described by the following functional form

$$R_i = C_i(\mu_i + b_i^T Z + \vartheta_i) \quad \text{for } i = 1, \ldots, N, \tag{3}$$

where $C_i : \mathbb{R} \to [0, 1]$ are arbitrary functions and μ_i are constants for all i. Let ϑ_i, $i = 1, \ldots, N$, be a sequence of independent zero-mean r.v. with finite variances $\sigma^2_{\vartheta_i}$, independent of Z and the ϵ_is. The parameter μ_i may be interpreted as the quantity of recovery. The factor loading b_i controls the strength of the effect of the systematic factor on recovery and represents different kinds of collaterals. By retaining the factor-structure we have established a channel for the relationship between default and recovery. Therefore we take into account the empirical fact that both default and recovery are affected by the state of the economy. Let us remark that for an unsecured loan, the asset's value is the only collateral. For secured loans, the specification of the LGD is more involved since it requires to deal both with the values of the guarantee and the assets. For the sake of simplification we neglect the guarantee provided by the collateral, i.e., in default the value of the assets is the collateral. Therefore all other creditors (e.g. firms with accounts receivables) require a ratio of the collateral in relation to their receivables. In the following subsection 2.2 we get back to this issue.

For clarifying the introduced notation we note that the random variable l_i is conditioned on the default event, called the *loss given default*. The actual observed loss L_i, for $i = 1, \ldots, N$, contains two features. First of all we observe the loss only in case of default. Therefore the default event[5] $\mathbb{1}_{A_i \leq \gamma_i}$ can also be described as $\mathbb{1}_{L_i > 0}$, where the probability of default is given by $\mathbf{P}[\mathbb{1}_{L_i>0} = 1] = \mathbf{P}[A_i \leq \gamma_i]$. The LGD of obligor i can be written as

$$\text{LGD}_i := l_i = L_i | \{A_i \leq \gamma_i\}.$$

The *loss* is given by

$$L_i = \text{LGD}_i \, \mathbb{1}_{A_i \leq \gamma_i} = l_i \, \mathbb{1}_{A_i \leq \gamma_i}. \tag{4}$$

For the *expected loss* of obligor i we obtain

$$\begin{aligned} \mathbf{E}[L_i] &= \mathbf{E}[l_i \, \mathbb{1}_{A_i \leq \gamma_i}] \\ &= \mathbf{P}[A_i \leq \gamma_i] \, \mathbf{E}[l_i | A_i \leq \gamma_i]. \end{aligned} \tag{5}$$

[4]In contrast to the recovery rate, which is a fraction of the total loss, the recovery is expressed in absolute terms, here it is given by $\mu_i + b_i^T Z + \vartheta_i$.

[5]Here we assume, that in default the obligor is exposed to a loss for sure.

As usual we assume that the portfolio is large enough to average away sampling variation (law of large numbers). If the portfolio has enough loans, and if the loans are similar enough in exposure amount, the expected rates of default and LGD can be assumed to occur.

By conditioning on the common vector Z we obtain a conditional independence structure. Thus, given a state of the economy allows us to compute many expressions quite easily.

Assumption 2.1 *There exists a d-dimensional vector Z such that all components of the augmented vector $(A_1, \dots, A_N, l_1, \dots, l_N)$ are independent when conditioned on Z.*

To sum up, the general framework of our model for obligor i can be described:

$$\left.\begin{aligned} A_i &= a_i^T Z + \sqrt{1 - \|a_i\|^2}\epsilon_i \\ l_i &= l_i^{\max}(1 - C_i(\mu_i + b_i^T Z + \vartheta_i)) \end{aligned}\right\} \quad i = 1, \dots, N. \tag{6}$$

We have introduced the dynamics of the model on the individual obligor level. For the *portfolio*-level we use the following compact matrix form:

$$\begin{aligned}
\begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_N \end{pmatrix} &= \begin{pmatrix} a_1^1 & a_1^2 & \dots & a_1^d \\ a_2^1 & \dots & & \vdots \\ \vdots & & & \vdots \\ a_N^1 & \dots & \dots & a_N^d \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ \vdots \\ Z_d \end{pmatrix} \\
&+ \begin{pmatrix} \sqrt{1 - \|a_1\|^2} & 0 & \dots & \dots & 0 \\ 0 & \sqrt{1 - \|a_2\|^2} & 0 & \dots & 0 \\ \vdots & & & & \\ \vdots & & & & 0 \\ 0 & \dots & \dots & 0 & \sqrt{1 - \|a_N\|^2} \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \vdots \\ \vdots \\ \vdots \\ \epsilon_N \end{pmatrix} \\
&=: \mathbf{a}Z + \mathbf{D}(\sqrt{1 - \|a\|^2})\epsilon.
\end{aligned} \tag{7}$$

Analogously for the losses $l = (l_1, \dots, l_N)^T$, resp. $l^{\max}$, μ and ϑ, in case of default[6]:

$$l = \mathbf{D}(l^{\max})(\mathbf{1} - C(\mu + \mathbf{b}Z + \vartheta)),$$

where $C = (C_1, \dots, C_N)$ is a vector of arbitrary functions (compare individual level (3)).

[6] $\boldsymbol{D}(l^{\max})$ is the diagonal matrix with diagonal elements $l^{\max}$.

2.2 Integrating microstructure

The creditworthiness of a debtor is given by the rating, which depends on the asset value. Asset values as well as recovery rates are dependent on common systematic factors described by the vector Z (compare equation (6)). As already descibed in the Introduction, the interrelation between PD and LGD can also be established by anticipating firm interdependence. In this subsection we incorporate microstructural dependencies among firms, which enables us to capture the characterized effects. We start assuming $N^* = N$, i.e., we anticipate business relations of all firms in the economy, because the credit portfolio of a bank consists of these firms.

Example

We start with an example to illustrate our modeling approach. We consider a credit portfolio of three companies 1, 2 and 3. We introduce two matrices. The so-called "business matrix" $\mathbf{\Xi} = (\xi_{ij})_{i,j=1,\ldots,3}$ describes the business dependencies between two counterparties of the three obligors in our portfolio. Moreover, a firm does not self-affect itself, i.e., $\xi_{jj} = 0$. ξ_{12} denotes the strength of the counterparty risk effect of firm 2 on 1 regarding the asset value. The matrix entries can be determined by considering the income statements of the companies. We use the ratio of gross income, which was generated by the business relation of firm 1 to 2, and total gross income. If firm 1 has a gross income of 1000 CHF where 200 CHF were generated with firm 2, we have $\xi_{12} = 0.2$. As all firms are included in the credit portfolio, the rows of the matrix $\mathbf{\Xi}$ sum up to one. So the rest of the gross income 800 CHF is created with firm 3. In the business matrix

$$\mathbf{\Xi} = \begin{pmatrix} 0 & 0.2 & 0.8 \\ \xi_{21} & 0 & \xi_{23} \\ \xi_{31} & \xi_{32} & 0 \end{pmatrix}$$

the first row represents the counterparty risk on firm 1.

On the other hand if firm 2 defaults, i.e., the asset value is below a certain threshold, the company is not able to reimburse the total amount of outstanding receivables or debt. Firm 2 realizes a loss of $l_2 = l_2^{\max}(1 - R_2)$. Companies with trade account receivables have to write off parts of this amount. Therefore we introduce the matrix $\mathbf{\Theta} = (\theta)_{i,j=1,\ldots,3}$, where for instance θ_{12} denotes the ratio of the liabilities from firm 2 to firm 1 and the total liabilities. Assuming firm 2 defaults and realizes a net loss of 200 CHF (already the recovery incorporated) and firm 2 has liablities to firm 1 in the amount of 50 CHF, to firm 3 in the amount of 200 CHF and to the bank in the amount of 100 CHF respectively. Consequently

$$\mathbf{\Theta} = \begin{pmatrix} 0 & 0.14 & \theta_{13} \\ \theta_{21} & 0 & \theta_{23} \\ \theta_{31} & 0.57 & 0 \end{pmatrix}$$

with $\theta_{12} = 50/350 = 0.14$ and $\theta_{32} = 0.57$. Firms 1 and 3 have to write off 28 CHF, resp. 114 CHF. We introduce the row vector $(\theta_{01}, \ldots, \theta_{0N})$. For example θ_{02} is ratio of the liabilities (loan) from firm 2 to the bank, which is here $a_{02} = 0.29$. The extended liability matrix is denoted by $\tilde{\mathbf{\Theta}}$. We do not distinguish between secured and junior debt. All creditors have the same claim and write off the same fraction of their account receivables. ♦

To summarize formally, the *business matrix* $\mathbf{\Xi} = (\xi_{ij})_{i,j=1,\ldots,N}$ collects the business dependencies between two counterparties of all obligors in our portfolio. The analysis is restricted to positive business relations. Moreover, a firm does not self-affect itself, i.e., $\xi_{jj} = 0$ for $j = 1, \ldots, N$. A business dependency ξ_{ij} between obligor i and j can be defined as the ratio of the gross income, which was created by the business between i and j and the total gross income. Beside the business matrix we introduce the *liability matrix* $\mathbf{\Theta} = (\theta)_{i,j=1,\ldots,N}$, where θ_{ij} denotes the ratio of the liabilities from firm j to firm i and the total liabilities. It is clear that $\theta_{ii} = 0$ for all $i = 1, \ldots, N$.
The *residual risk* of each debtor, which is assessed by expert knowledge, is denoted by the vector $\eta = (\eta_i)_{i=1,\ldots,N}$. In the diagram obligor i has an counterparty risk effect on j (right arrow) of strength $\xi_{ji} - \theta_{ji}\mathbb{1}_{A_i \leq \gamma_i}$, vice versa:

$$\eta_i \underset{\xi_{ij}-\theta_{ij}\mathbb{1}_{A_j\leq\gamma_j}}{\overset{\xi_{ji}-\theta_{ji}\mathbb{1}_{A_i\leq\gamma_i}}{\rightleftharpoons}} \eta_j .$$

Definition 2.2 *A microstructure for a collection of counterparties $\mathcal{C} = \{1, \ldots, N\}$ is a directed weighted graph $\mathcal{G} = (\mathcal{C}, \mathcal{E}, \mathbf{\Xi}, \mathbf{\Theta}, \eta)$. The nodes correspond to the counterparties. A directed weighted edge $\mathcal{E}_{ij}$ from i to j indicates that the firm i has a counterparty risk effect on j of strength $\xi_{ji} - \theta_{ji}\mathbb{1}_{A_i\leq\gamma_i}$ given by the edge weights. The node weight η_i represents the residual risk of debtor i.*

Assumption 2.3 *The rating dynamics depends both on macrostructural as well as microstructural variables.*

To satisfy Assumption 2.3 we follow the approach of Egloff et al. (2007). We replace the idiosyncratic r.v. ϵ in equation (7) by an *idiosyncratic function* $\zeta(\mathcal{G}, Z, \epsilon, \vartheta)$. We have

$$A = \mathbf{a}Z + \mathbf{D}(\sqrt{1 - \|a\|^2})\zeta(\mathcal{G}, Z, \epsilon, \vartheta). \tag{8}$$

We specify the idiosyncratic function by assuming the following linear effect[7]:

$$\zeta(\mathcal{G}, Z, \epsilon, \vartheta) := \mathbf{\Xi}A - \mathbf{\Theta}L + \boldsymbol{D}(\eta)\epsilon, \tag{9}$$

where

$$L := (l_1\mathbb{1}_{\{A_1\leq\gamma_1\}}, \ldots, l_N\mathbb{1}_{\{A_N\leq\gamma_N\}})^T.$$

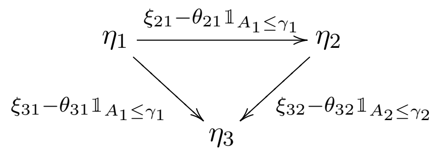
[7]Recall, $\boldsymbol{D}(\eta)$ is the diagonal matrix with diagonal elements η.

Since $\xi_{jj} = \theta_{jj} = 0$ for all j the first and second term of equation (9) describe the dependence to the asset values as well as recovery rates to the other obligors in the portfolio. The weighting matrix $\boldsymbol{\Xi}$ is an approximative sensitivity measure for the dependence of a firm on the asset value of all other firms. In contrast, the matrix $\boldsymbol{\Theta}$ describes exactly the loss fraction, which the firm has to write off in case of default of counterparty firms.

Plugging in the specific idiosyncratic effect into the original equation we obtain:

$$A = \mathbf{a}Z + \mathbf{D}(\sqrt{1 - \|a\|^2})\big[\boldsymbol{\Xi}A - \boldsymbol{\Theta}L + \boldsymbol{D}(\eta)\epsilon\big]. \tag{10}$$

Example (**continued**)
Again we consider a credit portfolio consisting of firms 1, 2 and 3. The microstructure of this portfolio can be illustrated by the following diagram, where we neglect the arrows in the opposite direction respectively.



The directed weighted edge from firm 2 to 3 indicates that firm 2 has an counterparty risk effect on 3 of strength $\xi_{32} - \theta_{32}\mathbb{1}_{A_2 \le \gamma_2}$. The node weight η_2 represents the residual risk of debtor 2. We obtain for the asset value of these firms the representation as in equation (10); the one of firm 2 can be described by

$$\begin{aligned} A_2 &= a_2^T Z + \sqrt{1 - \|a_2\|^2}\ \zeta_2(\mathcal{G}, Z, \epsilon_2, \vartheta_2) \\ &= a_2^T Z + \sqrt{1 - \|a_2\|^2}\ \Big[\xi_{21}A_1 + \xi_{23}A_3 \\ &\qquad - \theta_{21} l_1 \mathbb{1}_{\{A_1 \le \gamma_1\}} - \theta_{23} l_3 \mathbb{1}_{\{A_3 \le \gamma_3\}} + \eta_2\epsilon_2\Big]. \end{aligned} \tag{11}$$

The first term describes the usual influence of the macroeconomic factor variable weighted by the factor loading a_2. The second part integrates the firm dependency structure and own residual risk. If firm 2 has no business relation to firm 1 ($\xi_{21} = \theta_{21} = 0$), the decreasing asset value or even the default of firm 1 has no *direct* influence on the asset value of firm 2. Another scenario is that firm 2 will be indirectly influenced by the business relation to firm 3, who is influenced by the default of firm 1 (feedback-effect):

$$\eta_1 \xrightleftharpoons[\xi_{13} - \theta_{13}\mathbb{1}_{A_3 \le \gamma_3}]{\xi_{31} - \theta_{31}\mathbb{1}_{A_1 \le \gamma_1}} \eta_3 \xrightleftharpoons[\xi_{32} - \theta_{32}\mathbb{1}_{A_2 \le \gamma_2}]{\xi_{23} - \theta_{23}\mathbb{1}_{A_3 \le \gamma_3}} \eta_2.$$

Assuming that firm 3 exhibits a decreasing asset value, the third term $\xi_{23}A_3$ in the asset value equation (11) has an increasing negative impact on the value of firm 2. If the value falls even below the threshold γ_3, i.e., the firm defaults, firm 2 has to write off parts of the outstanding accounts receivable, that is $\theta_{23}l_3$. Note that the recovery process R_3, which is part of the LGD l_3 also depends on the macroeconomic variable Z and on the idiosyncratic part ϑ_3. Hence, by introducing the dependence structure, the asset value of firm 2 becomes also dependent on $a_3^T Z$, the idiosyncratic term ϵ_3, but also on R_3, which is influenced by $b_3^T Z$ and ϑ_3. ♦

Equation (10) describes the asset value structure on the portfolio level. For an arbitrary single firm i it is given by:

$$\begin{aligned} A_i &= \sum_{j=1}^{d} a_i^j Z_j + \sqrt{1-\|a_i\|^2}\eta_i\epsilon_i + \sqrt{1-\|a_i\|^2}\Big(\sum_{\substack{j=1\\ j\neq i}}^{N}\xi_{ij}\big(\sum_{j=1}^{d} a_i^j Z_j\big)\Big) \\ &\quad + \sqrt{1-\|a_i\|^2}\Big(\sum_{\substack{j=1\\ j\neq i}}^{N}\xi_{ij}\sqrt{1-\|a_i\|^2}\epsilon_i\Big) - \sqrt{1-\|a_i\|^2}\Big(\sum_{\substack{j=1\\ j\neq i}}^{N}\theta_{ij} l_j \mathbb{1}_{\{A_j\leq\gamma_j\}}\Big) \\ &= a_i^T Z + \sqrt{1-\|a_i\|^2}\Big(\eta_i\epsilon_i + \sum_{\substack{j=1\\ j\neq i}}^{N}[\xi_{ij}A_j - \theta_{ij} l_j \mathbb{1}_{\{A_j\leq\gamma_j\}}]\Big), \end{aligned} \tag{12}$$

where $l_j = l_j^{\max}(1 - C_j(\mu_j + b_j^T Z + \vartheta_j))$. This representation of the asset value of a single firm provides a rich structure for business dependencies. Note, the asset values of all other firms are of course themselves dependent on A_i. We assume that a solution to equation (10) exists.

2.3 Approximation

Unfortunately, we are not able to find an analytical solution for A. Alternatively, we propose the following recursive approximation, which was also used in Egloff et al. (2007).

For the first-order approximation we replace the asset value A on the right hand side of equation (10) by the pure macroeconomic asset value (see equation (1)), here denoted by $A^{(0)}$. Hence, we define the idiosyncratic function $\zeta^{(1)}$ of first-order in the following way:

$$\begin{aligned} \zeta^{(1)}(\mathcal{G}, Z, \epsilon, \vartheta) :&= \mathbf{\Xi}A^{(0)} + \mathbf{\Theta}L^{(0)} + \boldsymbol{D}(\eta)\epsilon \\ &= \mathbf{\Xi}\mathbf{a}Z + \Big[\mathbf{\Xi}\mathbf{D}(\sqrt{1-\|a\|^2}) + \boldsymbol{D}(\eta)\Big]\epsilon - \mathbf{\Theta}L^{(0)} \\ &=: \mathbf{R}_Z^{(1)} Z + \mathbf{R}_\epsilon^{(1)}\epsilon - \mathbf{R}_L^{(1)} L^{(0)}. \end{aligned}$$

Note, the observed loss L is also dependent on A. Analogously we use $L^{(0)} = (l_1\mathbb{1}_{\{A_1^{(0)}\leq\gamma_1\}}, \ldots, l_N\mathbb{1}_{\{A_N^{(0)}\leq\gamma_N\}})^T$.

The first-order approximated asset values of the N obligors are given by:

$$\begin{aligned}A^{(1)} &= \boldsymbol{a}\, Z + \mathbf{D}(\sqrt{1-\|a\|^2})\, \zeta^{(1)}(\mathcal{G}, Z, \epsilon, \vartheta)\\ &= \Big(\boldsymbol{a} + \mathbf{D}(\sqrt{1-\|a\|^2})\mathbf{R}_Z^{(1)}\Big) Z + \mathbf{D}(\sqrt{1-\|a\|^2})\, \mathbf{R}_\epsilon^{(1)}\epsilon\\ &\quad - \mathbf{D}(\sqrt{1-\|a\|^2})\, \boldsymbol{\Theta}\, L^{(0)}\\ &=: \boldsymbol{C}_Z^{(1)}\, Z + \boldsymbol{C}_\epsilon^{(1)}\, \epsilon - \boldsymbol{C}_L^{(1)} L^{(0)},\end{aligned} \tag{13}$$

where

$$\begin{aligned}\boldsymbol{C}_Z^{(1)} &= \boldsymbol{a} + \mathbf{D}(\sqrt{1-\|a\|^2})\mathbf{R}_Z^{(1)},\\ \boldsymbol{C}_\epsilon^{(1)} &= \mathbf{D}(\sqrt{1-\|a\|^2})\, \mathbf{R}_\epsilon^{(1)},\\ \boldsymbol{C}_L^{(1)} &= \mathbf{D}(\sqrt{1-\|a\|^2})\, \boldsymbol{\Theta} \quad .\end{aligned}$$

We do not report the approximated dynamics for a *single* firm. It is given analogously to equation (12).
The general recursion formula for the approximation of order n is given by

$$\begin{cases} A^{(0)} & = \mathbf{a}Z + \mathbf{D}(\sqrt{1-\|a\|^2})\epsilon \\ A^{(n)} & = \mathbf{a}Z + \mathbf{D}(\sqrt{1-\|a\|^2})\ \zeta^{(n)}(\mathcal{G}, Z, \epsilon, \vartheta) \end{cases}$$

with $\zeta^{(n)}(\mathcal{G}, Z, \epsilon, \vartheta) = \boldsymbol{\Xi} A^{(n-1)} - \boldsymbol{\Theta} L^{(n-1)} + \boldsymbol{D}(\eta)\epsilon$. Now we are able to solve the asset value system, because we separated the interwoven value streams by defining the approximated asset value recursively.
Nevertheless, in the following we just consider approximated values of first-order, because the structure of higher order terms is difficult to handle theoretically and the contribution to the content is low. However, it is clear that at this approximation level feedback effects cannot be captured, because the influencing value streams in the observed loss are independent of the business dependency structure. In our context the *true* asset value is given by $A^{(\infty)}$. All stages of mutual influences are captured. Independent of the order of the approximation the obtained expression is static, i.e., it describes a firm dependent on the current status of inter-firm linkages. The firm dependencies change of course in time. This has an effect on the asset value with integrated microstructure.

2.4 Properties of the model

In the following subsection we study our model more closely. Although we introduced the framework in a very general way, we use more specific assumptions for obtaining concrete results. Here we follow the model assumptions of Andersen and Sidenius (2004/05), who introduced a Gaussian recovery model. Let Φ and Φ_2 denote the cumulative Gaussian distribution function, resp. the bivariate Gaussian cdf. All proofs can be found in the Appendix.

Assumption 2.4 *(i) Set $C_i := \Phi$, for all i, the standard cumulative Gaussian distribution function.*
(ii) Let $Z_j \sim \mathcal{N}(0,1)$ for $j = 1,\dots,d$, $\epsilon_i \sim \mathcal{N}(0,1)$ and $\vartheta_i \sim \mathcal{N}(0,\sigma^2_{\vartheta_i})$ for $i = 1,\dots,N$, all iid. Set $Y_i := \mu_i + b^{i,T} Z + \vartheta_i$. Then $\mathbf{Var}[Y_i] = b^{i,T} b^i + \sigma^2_{\vartheta_i} =: \sigma^2_i$.

Proposition 2.5 *Under assumption 2.4 we have:*
(i) For $k \in [0,1]$:

$$\mathbf{P}[R_i \le k] = \Phi\left(\frac{\Phi^{-1}(k) - \mu_i}{\sigma_i}\right). \tag{14}$$

(ii)

$$\mathbf{E}[R_i] = \Phi\left(\frac{\mu_i}{\sqrt{1+\sigma_i^2}}\right) \quad \text{and} \quad \mathbf{E}[R_i|Z=z] = \Phi\left(\frac{\mu_i + b_i^T z}{\sqrt{1+\sigma^2_{\vartheta_i}}}\right). \tag{15}$$

(iii)

$$\mathbf{E}[R_i|A_i \le \gamma_i] = \frac{1}{\mathbf{P}[A_i \le \gamma_i]} \int_{-\infty}^{\gamma_i} \Phi\Big(\frac{\mu_i + \rho_i \sigma_i x}{\sqrt{1+\sigma_i^2(1-\rho_i^2)}}\Big)\varphi(x)\,dx \tag{16}$$

(iv)

$$\mathbf{P}[R_i \mathbb{1}_{A_i \le \gamma_i} \le k] = 1 - \Phi(\gamma_i) + \Phi_2\left(\frac{\Phi^{-1}(k) - \mu_i}{\sigma_i}, \gamma_i; \frac{a^{i,T} b^i}{\sigma_i}\right). \tag{17}$$

In contrast to equation (14), where the cdf of the pure recovery rate is expressed, equation (17) is the distribution function of the observed recovery rate. Here we see the effects of the modeling approach: first of all, a loss or recovery is not observed in case of no default[8], i.e., the first two terms $1 - \Phi(\gamma_i)$ describe the probability $\mathbf{P}[R_i \mathbb{1}_{A_i \le \gamma_i} = 0]$, where the value of the random variable R_i is obviously irrelevant. Consequently the last term is $\mathbf{P}[0 < R_i \mathbb{1}_{A_i \le \gamma_i} \le k]$, which is the joint distribution of the recovery and asset value variable (see the proof). This can be expressed by the bivariate normal distribution. The correlation parameter describes the dependence between recovery and asset value. Equation (16) describes quite intuitively the expected recovery rate conditioned on the default event. Again, recovery and asset value are not independent. This is expressed in the integrand, where the correlation ρ_i with respect to the macroeconomic variable Z between Y_i and A_i appears in the numerator. The asset value A_i is assumed to be standard normally distributed, i.e., we integrate with respect to different value scenarios ($\varphi(x)$ is the pdf), where the firm defaults (up to the threshold γ_i).
Considering again our modeling approach, e.g. equation (13), it is clear that the sole anticipation of losses triggered by defaults of related firms (and the normality assumption of the other r.v.) leads to a negative expected asset value

[8]The "no default" scenario is here based on the macroeconomic setup, i.e., without integrated microstructure.

$\mathbf{E}[A^{(1)}] = -\boldsymbol{C}_L^{(1)}\mathbf{E}[L^{(0)}]$. The term $\mathbf{E}[L^{(0)}]$ can be interpreted as the historical LGD average.[9] The aim of our work is to introduce a model which can capture the described interactions between risk components, since these can have tremendous effects especially in recessive business cycles. Therefore we adjust the original asset value to retain the long-term average, but still including the correlation structure.

Definition 2.6 *The cost-adjusted asset value is given by*

$$\tilde{A}^{(1)} = A^{(1)} + \boldsymbol{C}_L^{(1)}\mathbf{E}[L^{(0)}] \tag{18}$$

Proposition 2.7 *The asset value $\tilde{A}^{(1)}$ is given by equation* (18). *Then we have (i)*

$$\mathbf{E}[\tilde{A}^{(1)}] = 0, \tag{19}$$

(ii)

$$\begin{aligned}\mathbf{Var}[\tilde{A}^{(1)}] = \boldsymbol{C}_Z^{(1)}\boldsymbol{C}_Z^{(1)T} + \boldsymbol{C}_\epsilon^{(1)}\boldsymbol{C}_\epsilon^{(1)T} - \boldsymbol{C}_L^{(1)}\mathbf{Var}[L^{(0)}]\boldsymbol{C}_L^{(1)T}\\ - \boldsymbol{C}_Z^{(1)}\,\mathbf{Cov}[Z, L^{(0)}]\boldsymbol{C}_L^{(1)T} - \boldsymbol{C}_L^{(1)}\,\mathbf{Cov}[L^{(0)}, Z]\,\boldsymbol{C}_Z^{(1)T}\\ - \boldsymbol{C}_\epsilon^{(1)}\,\mathbf{Cov}[\epsilon, L^{(0)}]\,\boldsymbol{C}_L^{(1)T} - \boldsymbol{C}_L^{(1)}\,\mathbf{Cov}[L^{(0)}, \epsilon]\,\boldsymbol{C}_\epsilon^{(1)T}.\end{aligned} \tag{20}$$

We observe a quite intricated structure. As already pointed out in 2.1, the observed loss $L^{(0)}$ is conditioned on the event "default", i.e., is dependent on the condition of the companies and therefore correlated with the variables Z and ϵ (the recovery is also dependent on the macroeconomic situation and an extra idiosyncratic term ϑ).
In the following proposition we calculate all expressions which are necessary to obtain the expected value, the variance and covariances explicitly, again resting upon the Gaussian recovery setup.

Proposition 2.8 *Assuming 2.4 and recalling $l_i := l_i^{max}(1 - \Phi(\mu_i + b_i^T Z + \vartheta_i))$ we have for $i = 1, \ldots, N$:*
(i)

$$\mathbf{E}[L_i^{(0)}] = \mathbf{E}[l_i \mathbb{1}_{\{A_i^{(0)} \le \gamma_i^{(0)}\}}] = \Phi(\gamma_i^{(0)})\, l_i^{max}\,(1 - \mathbf{E}[R_i | A_i^{(0)} \le \gamma_i^{(0)}]),$$

where an expression for the last term can be found in Proposition 2.5 (iv).

(ii)

$$\begin{aligned}\mathbf{Var}[l_i \mathbb{1}_{\{A_i^{(0)} \le \gamma_i^{(0)}\}}] = (l_i^{max})^2 \Big(\Phi(\gamma_i^{(0)}) - 2\Phi(\gamma_i^{(0)})\mathbf{E}[R_i | A_i^{(0)} \le \gamma_i^{(0)}] + \mathbf{E}[R_i^2 \mathbb{1}_{\{A_i^{(0)} \le \gamma_i^{(0)}\}}]\Big)\\ - \mathbf{E}[l_i \mathbb{1}_{\{A_i^{(0)} \le \gamma_i^{(0)}\}}]^2,\end{aligned}$$

[9] Under the Basel II foundation approach it is possible to choose a constant LGD of 45%.

with

$$\mathbf{E}[R_i^2 \mathbb{1}_{\{A_i^{(0)} \leq \gamma_i^{(0)}\}}] = [\Phi(\sigma_i \gamma_i^{(0)} + \mu_i)^2 \Phi(\gamma_i^{(0)})] - 2\sigma_i \int_{-\infty}^{\gamma_i^{(0)}} \Phi(\sigma_i x + \mu_i)\varphi(\sigma_i x + \mu_i)\varphi(x)dx,$$

where $\sigma_i^2 = b^{i,T} b^i + \sigma_{\vartheta_i}^2$.
(iii) For the covariance term $\mathbf{Cov}[Z, L^{(0)}]$ *we consider w.l.o.g.:*

$$\mathbf{Cov}[Z_1, l_1 \mathbb{1}_{\{A_1^{(0)} \leq \gamma_1^{(0)}\}}] = l_1^{max} \Big(\Phi(\gamma_1^{(0)}) \mathbf{E}[Z_1 | A_1^{(0)} \leq \gamma_1^{(0)}] - \mathbf{E}[Z_1 R_1 \mathbb{1}_{\{A_1^{(0)} \leq \gamma_1^{(0)}\}}] \Big),$$

with

$$\mathbf{E}[Z_1 | A_1^{(0)} \leq \gamma_1^{(0)}] = -a_1^1 \varphi(\gamma_1^{(0)})$$

and

$$\mathbf{E}[Z_1 R_1 \mathbb{1}_{\{A_1^{(0)} \leq \gamma_1^{(0)}\}}] = \int_{-\infty}^{\infty} x \, \Phi\left(\frac{\mu_1 + b_1^T x}{\sqrt{1 + \sigma_{\vartheta_1}^2}} \right) \Phi\left(\frac{\gamma_1^{(0)} - a_1^T x}{\sqrt{1 - \|a_1\|^2}} \right) \varphi(x) \, dx.$$

The last equation (iii) of Proposition 2.8 is representative for all other dependencies between observed losses and the variables $Z_i, i = 1, \ldots, d$. The covariance between the idiosyncratic shock ϵ_i and $L^{(0)}$ can be obtained analogously.

3 Comparative analysis

3.1 Introduction

This section deals with the comparison of the introduced model with the Basel II framework and the stochastic recovery setting from Andersen & Sidenius (our basic setup). We analyze the models in terms of asset correlation and comparative statics of the conditional expected loss. As our model is an extension (integrated microstructure) of a stochastic recovery model, the Basel II framework is a special case of our model. The IRB approach uses among other things the so-called risk-bucketing system, firm-size and maturity adjustments. However, we work with a stylized setup focusing on one exposure class (e.g. corporates), a maturity of one year and neglect SMEs.[10]

3.2 Asset correlation

As already pointed out in Gordy (2003), the Basel II framework can be represented as an asymptotic single risk factor model, that is, the systematic risk

[10] Also the Basel committee is aware of the shortcoming that e.g. portfolio concentrations are not treated within the framework (Basel II, 2005b).

factor Z is one-dimensional. In our model this is the special case of $A^{(0)}$ for $d=1$. On the firm level (w.l.o.g. $i=1$) we obtain

$$\mathbf{Cov}[A_1^{(0)}, Z] = \mathbf{E}[(a_1 Z + \sqrt{1-a_1^2}\epsilon)Z] - \mathbf{E}[Z]\mathbf{E}[a_1 Z + \sqrt{1-a_1^2}\epsilon] = a_1,$$

which is equal to the correlation. Under the IRB approach this measure is set to $\mathbf{Corr}[A_1^{(0)}, Z] = 0.12(1+e^{-50\,PD_1})$. In contrast, we consider the portfolio level of the cost-adjusted first-approximation model:

$$\begin{aligned}
\mathbf{Cov}[\tilde{A}^{(1)}, Z] &= \mathbf{Cov}[\boldsymbol{a}Z + \sqrt{1-\|a\|^2}\zeta^{(1)} + \boldsymbol{C}_L^{(1)}\mathbf{E}[L^{(0)}], Z] \\
&= \boldsymbol{aI} + \boldsymbol{D}(\sqrt{1-\|a\|^2})\mathbf{Cov}[\zeta^{(1)}, Z] \\
&= \boldsymbol{aI} + \boldsymbol{D}(\sqrt{1-\|a\|^2})\Big(\boldsymbol{\Xi a} - \boldsymbol{\Theta}\mathbf{Cov}[L^{(0)}, Z]\Big) \\
&= \boldsymbol{aI} + \boldsymbol{D}(\sqrt{1-\|a\|^2})\boldsymbol{\Xi a} - \boldsymbol{D}(\sqrt{1-\|a\|^2})\boldsymbol{\Theta}\mathbf{Cov}[L^{(0)}, Z]
\end{aligned}$$

with

$$\mathbf{Cov}[L^{(0)}, Z] = \begin{pmatrix} \mathbf{Cov}[l_1 \mathbb{1}_{A_1^{(0)} \le \gamma_1^{(0)}}, Z_1] & \dots & \dots & \mathbf{Cov}[l_1 \mathbb{1}_{A_1^{(0)} \le \gamma_1^{(0)}}, Z_d] \\ \vdots & & \dots & \vdots \\ \vdots & & & \vdots \\ \mathbf{Cov}[l_N \mathbb{1}_{A_N^{(0)} \le \gamma_N^{(0)}}, Z_1] & \dots & \dots & \mathbf{Cov}[l_N \mathbb{1}_{A_N^{(0)} \le \gamma_N^{(0)}}, Z_d] \end{pmatrix}.$$

For a direct comparison with Basel II we consider again the special case $d=1$ on the firm level:

$$\begin{aligned}
\mathbf{Cov}[\tilde{A}_1^{(1)}, Z] &= a_1 + (\sqrt{1-a_1^2}, 0, \dots, 0)\,\boldsymbol{\Xi}\,(a_1, a_2, \dots, a_N)^T \\
&\quad - (\sqrt{1-a_1^2}, 0, \dots, 0)\boldsymbol{\Theta}\begin{pmatrix} \mathbf{Cov}[l_1 \mathbb{1}_{A_1^{(0)} \le \gamma_1^{(0)}}, Z] \\ \vdots \\ \mathbf{Cov}[l_N \mathbb{1}_{A_N^{(0)} \le \gamma_N^{(0)}}, Z] \end{pmatrix}.
\end{aligned} \tag{21}$$

Equation (21) is equal to the correlation a_1 in the basic setup (Basel II) if and only if the following condition holds:

$$(0, \xi_{12}, \dots, \xi_{1N})a^T = (0, \theta_{12}, \dots, \theta_{1N})\begin{pmatrix} \mathbf{Cov}[l_1 \mathbb{1}_{A_1^{(0)} \le \gamma_1^{(0)}}, Z] \\ \vdots \\ \mathbf{Cov}[l_N \mathbb{1}_{A_N^{(0)} \le \gamma_N^{(0)}}, Z] \end{pmatrix},$$

where we reported some expressions for the last term in Proposition 2.8 (iii). The extended setting induces the same correlation structure as the Basel II framework if no inter-firm linkages are established or if the relationship between asset value and Z of all aligned firms (factor loadings weighted by business dependency), is compensated by the correlation between LGD and Z of all other firms; this time weighted by the liability matrix.

3.3 Conditional expected loss

Here we derive the expected loss for the bank. Since the conditional expected loss is the core of the Basel II capital rule, we analyze models with constant recovery, stochastic recovery and the most general setup (at least the first approximation) with respect to the comparative statics of the conditional expected loss. Hereby we condition on the main driver Z and study the sensitivities of this variable.

Under the IRB approach banks estimate the risk components PD, LGD and EAD. We assume that the exposure at default $\mathrm{EAD}_i = 1$ for all $i = 1, \ldots, N$ is constant.[11]

Keep in mind, in our model two different dimensions are considered. The developed model anticipates the possible losses between the obligors, although we are interested in the loss distribution of the bank. As already remarked, at this point we do not account for different seniorities. The liabilities of N obligors to the creditor (the bank) are given by $\theta_0 = (\theta_{01}, \ldots, \theta_{0N})$. Therefore the total loss for the bank, denoted by $\mathbf{L}$, experienced by N obligors is given by:

$$\mathbf{L} = \sum_{i=1}^{N} \theta_{0i} L_i = \sum_{i=1}^{N} \theta_{0i} l_i \mathbb{1}_{A_i \leq \gamma_i}. \tag{22}$$

With equation (5) we obtain for the expected loss:

$$\mathbf{E}[\mathbf{L}] = \sum_{i=1}^{N} \theta_{0i} \mathbf{E}[l_i \mathbb{1}_{A_i \leq \gamma_i}] = \sum_{i=1}^{N} \theta_{0i}\, \mathbf{P}[A_i \leq \gamma_i]\, \mathbf{E}[l_i | A_i \leq \gamma_i]. \tag{23}$$

Basel II - constant recovery We just consider the expected loss of a single firm. In case of a constant LGD, the expression in equation (23) simplifies to $\mathbf{E}[L_i^{(0),c}] = \mathbf{P}[A_i^{(0)} \leq \gamma_i^{(0)}]\, l_i = \Phi(\gamma_i^{(0)})\, l_i$, where $L_i^{(0),c}$ denotes the loss of firm i in the basic setting with constant recovery. Using the tower property this expression can be rewritten:

$$\mathbf{E}[L_i^{(0),c}] = \mathbf{E}\big[\mathbf{E}[L_i^{(0),c} | Z = z]\big] =: \mathbf{E}[C_i^c(z)] = \mathbf{E}\big[\mathbf{P}[A_i^{(0)} \leq \gamma_i^{(0)} | Z = z]\big]\, l_i,$$

where $C_i^c(z)$ denotes the conditional expected loss function given Z. Plugging the 0.1^{th} percentile of Z into $C_i^c(z)$ yields the core of the Basel II capital rule.[12] Here we are interested in the sensitivities of $C_i^c(z)$. Due to the fact $\Phi^{-1}(PD_i) =$

[11] There is empirical evidence that this is not the case, see e.g. Allen and Saunders (2004).

[12] Here we would plug in $\Phi^{-1}(0.001)$, since we are interested in an appropriately conservative value of the single systematic risk factor. With our formulation of the asset value this means, values should stay above a certain threshold with probability 99.9%, i.e. $0.999 = 1 - \mathbf{P}[Z \leq z]$.

$\gamma_i^{(0)}$ we obtain

$$\frac{\partial}{\partial z}C_i^c(z) = l_i^{\max}\big(\frac{\partial}{\partial z}\mathbf{P}[A_i^{(0)} \le \gamma_i^{(0)}|Z] - R_i\frac{\partial}{\partial z}\mathbf{P}[A_i^{(0)} \le \gamma_i^{(0)}|Z = z]\big) \tag{24}$$
$$= l_i^{\max}(1 - R_i)\frac{\partial}{\partial z}\Phi\Big(\frac{\Phi^{-1}(PD_i) - a_i z}{\sqrt{1 - a_i^2}}\Big)$$
$$= l_i^{\max}(1 - R_i)\Big(\frac{-a_i}{\sqrt{1 - a_i^2}}\Big)\varphi\Big(\frac{\Phi^{-1}(PD_i) - a_i z}{\sqrt{1 - a_i^2}}\Big).$$

With improving macroeconomic conditions ($z \uparrow$) the conditional expected loss $C_i^c(z)$ decreases, where the factor $-a_i/(1 - a_i^2)^{1/2}$ describes the influence of the change in z on the PD. The LGD is constant and therefore not influenced.

Stochastic recovery In the basic setup introduced by equation (6) under Assumption 2.4, we use again the tower property. Since both asset value and LGD are dependent on Z we obtain conditional independence:

$$\mathbf{E}[L_i^{(0),s}] = \mathbf{E}\big[\mathbf{E}[L_i^{(0),s}|Z = z]\big] =: \mathbf{E}[C_i^s(z)]$$
$$= \mathbf{E}\big[l_i^{\max}(1 - \mathbf{E}[R_i|Z = z])\mathbf{P}[A_i^{(0)} \le \gamma_i^{(0)}|Z = z]\big],$$

where $\mathbf{E}[R_i|Z = z]$ is given by equation (15). We consider the comparative statics of $C_i^s(z)$:

$$\frac{\partial C_i^s(z)}{\partial z} = l_i^{\max}\Big[\frac{\partial}{\partial z}\mathbf{P}[A_i^{(0)} \le \gamma_i^{(0)}|Z = z] - \frac{\partial}{\partial z}\mathbf{E}[R_i|Z = z]\mathbf{P}[A_i^{(0)} \le \gamma_i^{(0)}|Z = z]$$
$$- \mathbf{E}[R_i|Z = z]\frac{\partial}{\partial z}\mathbf{P}[A_i^{(0)} \le \gamma_i^{(0)}|Z = z]\Big],$$

where

$$\frac{\partial}{\partial z}\mathbf{E}[R_i|Z = z] = \frac{b_i}{\sqrt{1 + \sigma_{\vartheta_i}^2}}\varphi\Big(\frac{\mu_i + b_i z}{\sqrt{1 + \sigma_{\vartheta_i}^2}}\Big).$$

Since the recovery is stochastic and dependent on Z we use the conditional expected value. Additional to the effect that a better macroeconomic environment induces lower default probabilities (equation (24)) we also observe in the second term an increasing recovery value which decreases $C_i^s(z)$. This new marginal effect is weighted by the PD, because it is only observed in the default case.

Integrated microstructure & Stochastic recovery Now we study our most general model (see Definition 2.6). $L_i^{(1)}$ denotes the loss of firm i in this general setting, i.e., integrated microstructure and stochastic recovery. For reasons of simplification we just consider a portfolio with two obligors ($N = 2$). Also in this setting the single components are independent conditional on the driver Z; therefore we use again the tower property. However, for the comparative static analysis we have to extend the conditional expected loss function by the additional random terms of the linked firm $\{\epsilon_2, \vartheta_2\}$ which pop up in the asset value of firm 1.[13] Otherwise, we are not able to separate the effects of changing macroeconomic conditions. We have:

$$\mathbf{E}[L_i^{(1)}] = \mathbf{E}\big[\mathbf{E}[L_i^{(1)}|\{Z=z\},\{\epsilon_2=\hat{\epsilon_2}\},\{\vartheta_2=\hat{\vartheta}_2\}]\big] =: C_1^{(1)}(z,\hat{\epsilon_2},\hat{\vartheta}_2).$$

The structure of the marginal effects of $C_1^{(1)}(z,\hat{\epsilon_2},\hat{\vartheta}_2)$ and the recovery function are unaffected by the new setting. The conditional default probability take the form:

$$\begin{aligned}
\mathbf{P}\big[\tilde{A}_1^{(1)} \le \gamma_1^{(1)}|Z,\epsilon_2,\vartheta_2] = \Phi\Big(&\frac{\gamma_1^{(1)} - a_1 z}{\eta_1\sqrt{1-a_1^2}} - \frac{\xi_{12}(a_2 z + \sqrt{1-a_2^2}\hat{\epsilon}_2)}{\eta_1} \\
&+ \frac{\theta_{12}\big(l_2^{\max}(1-\Phi(\mu_2+b_2 z+\hat{\vartheta}_2))\mathbb{1}_{\{a_2 z+\sqrt{1-a_2^2}\hat{\epsilon}_2\le\gamma_2^{(0)}\}}\big)}{\eta_1} \\
&- \frac{\theta_{12}\mathbf{E}[L_2^{(0)}]}{\eta_1}\Big) \\
&=: \Phi(k).
\end{aligned}$$

For the partial differentiation of the observed loss, we note that the indicator function has a discontinuity at $\gamma_2^{(0)}$. For the sake of mathematical correctness we use the extension of the derivative, so called generalized functions (see e.g. Gelfand et al. (1966)). Thereby δ is the Dirac-delta function.[14] We obtain:

$$\begin{aligned}
\frac{\partial}{\partial z}\Phi(k) = \varphi(k)\cdot\Big[\Big(\frac{-a_1}{\eta_1\sqrt{1-a_1^2}}\Big) - \Big(\frac{\xi_{12}a_2}{\eta_1}\Big) + \frac{\theta_{12}l_2^{\max}}{\eta_1}\Big(\frac{\partial}{\partial z}\mathbb{1}_{\{a_2 z+\sqrt{1-a_2^2}\hat{\epsilon}_2\le\gamma_2^{(0)}\}} \\
-\big(\varphi(\mu_2+b_2 z+\hat{\vartheta}_2)b_2\mathbb{1}_{A_2^{(0)}\le\gamma_2^{(0)}} + \Phi(\mu_2+b_2 z+\hat{\vartheta}_2)\frac{\partial}{\partial z}\mathbb{1}_{\{a_2 z+\sqrt{1-a_2^2}\hat{\epsilon}_2\le\gamma_2^{(0)}\}}\big)\Big)\Big],
\end{aligned}$$

with

$$\frac{\partial}{\partial z}\mathbb{1}_{\{A_2^{(0)}\le\gamma_2^{(0)}\}} = -\mathbb{1}_{\{\sqrt{1-a_2^2}\hat{\epsilon}_2\le\gamma_i^{(0)}-a_2 z\}}\delta_{\frac{\gamma_2^{(0)}-\sqrt{1-a_2^2}\hat{\epsilon}_2}{a_2}}(z), \qquad (25)$$

where

$$\delta_{\frac{\gamma_2^{(0)}-\sqrt{1-a_2^2}\hat{\epsilon}_2}{a_2}}(z) = \begin{cases} \infty & \text{, if } \dfrac{\gamma_2^{(0)}-\sqrt{1-a_2^2}\hat{\epsilon}_2}{a_2} = z \\ 0 & \text{, else.} \end{cases}$$

Additional to the marginal structure of the stochastic recovery case we have to modify the sensitivity of the conditional default probability. Except for the

[13]Recall the structure of the asset value: $\mathbf{P}\big[A_1^{(1)} \le \gamma_1^{(1)}|Z,\epsilon_2,\vartheta_2] = \mathbf{P}[a_1 z + \sqrt{1-a_1^2}\big(\eta_1\epsilon_1 + \xi_{12}(a_2 z+\sqrt{1-a_2^2}\hat{\epsilon}_2) - \theta_{12}(l_2^{\max}(1-\Phi(\mu_2+b_2 z+\hat{\vartheta}_2))\mathbb{1}_{a_2 z+\sqrt{1-a_2^2}\hat{\epsilon}_2\le\gamma_2^{(0)}})\big) \le \gamma_1^{(1)}\big]$.

[14]For details see the Appendix. Here we also discuss the d-dimensional case.

first term within the brackets, which describes the usual dependence between asset value of the firm itself and macro variable Z, all other expressions show up due to our new modeling approach. First, the dependence of asset value and Z persists here in two ways: the already described canonical way, but moreover through all other related firms (here just firm 2). An increase in the consumer confidence index has a more pronounced impact on the asset value of a producer of consumer goods than of pharmaceuticals. Nevertheless, in our approach the impact on the asset value of all aligned firms, independent of the sector, is anticipated with the specific weight (factor loading and Ξ). Both terms have in a natural way with increasing Z a negative impact on the conditional expected loss $C_1^{(1)}(z, \hat{\epsilon_2}, \hat{\vartheta_2})$.
The rest of the expression describes the influence of Z on the LGD of the linked firms. It is clear, that a change in Z has only an effect on the asset value and then PD of firm 1 if Z initiates the default of linked firms, here firm 2, i.e., $z = (\gamma_2^{(0)} - (1 - a_2^2)^{1/2}\hat{\epsilon}_2)/a_2$, otherwise not. A higher threshold $\gamma_2^{(0)}$, a lower value of $(1 - a_2^2)^{1/2}\hat{\epsilon}_2$ and a lower value of the factor loading a_2 imply that default can be initiated by Z earlier. In other words, a strong increase in Z is necessary to pass the threshold, i.e., not to default. The last two terms describe the sensitivities of the loss reducing effect of the recovery. Also the recoveries of the linked firms are influenced by a change in Z. The factor loading b_2 weights the marginal change of the recovery function, where the effect can only be observed in the default event, i.e., Z initiates the default.

3.4 Modified Basel II

After studying the marginal effects, we give some implications for the minimum capital requirements for the introduced framework. It is obvious that the capital rule for our microstructural model is not portfolio invariant in the sense that the conditional expected loss $C_1^{(1)}(z, \hat{\epsilon_2}, \hat{\vartheta_2})$ is dependent on the portfolio structure. In Basel II the capital charge depends only on the characteristics of creditor i and thus is invariant. Here, a change of interfirm-linkages would also change the capital charge. The formula for the calculation of the capital requirement of obligor 1 with integrated microstructure $K_1^{(1)}$ is given analogously, where we assume a 0.1^{th} percentile of Z:

$$K_1^{(1)} = C_1^{(1)}(\Phi^{-1}(0.001), \hat{\epsilon_2}, \hat{\vartheta_2}). \qquad (26)$$

An open issue is the determination of the firm-specific random variables ϵ_2 and ϑ_2. The same treatment as for the systematic factor (set a very conservative value) would imply a bad condition of each of the linked firms. In the worst case, the capital requirement for each firm in the credit portfolio is determined by assuming that all related companies default.

4 Conclusion

A thourough understanding of aggregate credit loss risk is of utmost importance for the management of financial institutions, the pricing of credit derivatives and regulatory authorities supervising the financial sector. The current subprime crisis highlights the relevance of careful risk assessment and accurate derivative pricing. To evaluate bank's credit risk, it is not adequate to scrutinize individual obligors and aggregate each risk exposure. The dependence between losses on positions is a significant factor, where the expected loss is mainly the product of PD and LGD. In that respect most of standard models applied in the industry focus on correlations which are due to the dependence between the asset values on the common macroeconomic environment neglecting the relationship between recovery and PD and firm interdependencies, especially within the portfolio. This ignorance of default contagion might underestimate the aggregate loss.

In response, we propose a model which captures the business interactions of firms in addition to cyclical correlation effects. Using a multi-factor model we present the asset value and recovery of a firm explicitly. By incorporating the firm interdependencies we establish an additional link between PD and LGD. With a recursive approximation we are able to achieve a closed form representation of the asset value, which captures the inter-firm connections. The comparison of the induced asset correlation between our model and the regulatory framework exhibits that important relations are not included in the Basel II framework. A sensitivity analysis of the conditional expected loss with respect to the systematic factor identifies the marginal effects which occur additionally to the Basel II setup. In particular, capital requirements are in contrast to Basel II not portfolio-invariant, since debtors' business relations are anticipated in the conditional expected loss function.

5 Appendix

Proof (**Proposition 2.5**)
Parts (i)-(iv) can be found in Proposition 3 in the work from Andersen and Sidenius (2005). Therefore part (v) is left to be shown.
Recall that $Y_i := \mu_i + b_i^T Z + \vartheta_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$ with $\sigma_i^2 = b_i^T b_i + \sigma_{\vartheta_i}^2$ and $Y_i|Z \sim \mathcal{N}(\mu_i + b_i^T Z, \sigma_{\vartheta_i}^2)$. Then we can write:

$$\begin{aligned}
\mathbf{P}[R_i \mathbb{1}_{A_i \le \gamma_i} \le k] &= 1 - \mathbf{P}[R_i \mathbb{1}_{A_i \le \gamma_i} > k] \\
&= 1 - \mathbf{P}[\{R_i > k\} \wedge \{\mathbb{1}_{A_i \le \gamma_i} = 1\}] \\
&= 1 - \mathbf{P}[\{Y_i > \Phi^{-1}(k)\} \wedge \{A_i \le \gamma_i\}] \\
&= \mathbf{P}[\{Y_i \le \Phi^{-1}(k)\} \wedge \{A_i \le \gamma_i\}] + 1 - \mathbf{P}[A_i \le \gamma_i] \\
&= 1 - \mathbf{P}[A_i \le \gamma_i] + \mathbf{P}[Y_i \le \Phi^{-1}(k)\ ,\ A_i \le \gamma_i].
\end{aligned}$$

The first part is trivial, because we have assumed that the asset value process is standard normal, i.e.

$$\mathbf{P}[A_i \le \gamma_i] = \Phi(\gamma_i).$$

For the second part we have to determine the joint distribution of Y_i and A_i. For this we need the correlation between Y_i and A_i. Starting with calculating the covariance:

$$\begin{aligned}
\mathbf{Cov}[Y_i, A_i] &= \mathbf{Cov}[\mu_i + b_i^T Z + \vartheta_i, a_i^T Z + \sqrt{1 - \|a_i\|^2}\epsilon_i] \\
&= a_i^1 b_i^1 \mathbf{E}[Z_1^2] + \ldots + a_i^d b_i^d \mathbf{E}[Z_d^2] \\
&= a_i^T b_i
\end{aligned}$$

Therefore we have for the correlation $\rho_{Y_i, A_i} = \frac{a^{i,T} b^i}{\sigma_i}$. All in all we obtain

$$\mathbf{P}[Y_i \le \Phi^{-1}(k)\ ,\ A_i \le \gamma_i] = \Phi_2\left(\frac{\Phi^{-1}(k) - \mu_i}{\sigma_i}, \gamma_i; \frac{a^{i,T} b^i}{\sigma_i}\right).$$

On the other hand using the tower property and conditional independence we can write:

$$\begin{aligned}
\mathbf{P}[Y_i \le \Phi^{-1}(k) =: d\ ,\ A_i \le \gamma_i] &= \mathbf{E}[\mathbb{1}_{Y_i \le d\ ,\ A_i \le \gamma_i}] \\
&= \mathbf{E}[\mathbf{E}[\mathbb{1}_{Y_i \le d\ ,\ A_i \le \gamma_i}|Z]] \\
&= \mathbf{E}[\mathbf{E}[\mathbb{1}_{Y_i \le d}|Z]\mathbf{E}[\mathbb{1}_{A_i \le \gamma_i}|Z]] \\
&= \mathbf{E}[\mathbf{P}[Y_i \le d|Z]\mathbf{P}[A_i \le \gamma_i|Z]] \\
&= \mathbf{E}\Big[\Phi\left(\frac{d - b_i^T Z - \mu_i}{\sigma_{\vartheta_i}}\right)\Phi\left(\frac{\gamma_i - a_i^T Z}{\sqrt{1 - \|a_i\|^2}}\right)\Big] \\
&= \int_{-\infty}^{\infty} \Phi\left(\frac{d - b_i^T x - \mu_i}{\sigma_{\vartheta_i}}\right)\Phi\left(\frac{\gamma_i - a_i^T x}{\sqrt{1 - \|a_i\|^2}}\right)\varphi(x)\, dx,
\end{aligned}$$

where φ denotes the density function of the standard normal distribution. ■

Proof (**Proposition 2.8**)
(i) We obtain:

$$\begin{aligned}\mathbf{E}[l_i \mathbb{1}_{\{A_i^{(0)} \leq \gamma_i\}}] &= \mathbf{P}[\tau_i^{(0)} \leq T]\, \mathbf{E}[l_i | \tau_i^{(0)} \leq T] \\ &= \Phi(\gamma_i)\, \mathbf{E}[l_i^{\max}(1 - R_i) | \tau_i^{(0)} \leq T] \\ &= \Phi(\gamma_i)\, l_i^{\max}\, (1 - \mathbf{E}[R_i | \tau_i^{(0)} \leq T]),\end{aligned}$$

where the last term can be expressed by the result from Proposition 2.5 (iv).
(ii)

$$\mathbf{Var}[l_i \mathbb{1}_{\{A_i^{(0)} \leq \gamma_i\}}] = \mathbf{E}[(l_i \mathbb{1}_{\{A_i^{(0)} \leq \gamma_i\}})^2] - \mathbf{E}[l_i \mathbb{1}_{\{A_i^{(0)} \leq \gamma_i\}}]^2,$$

where the second term is known (see first part). Therefore we calculate, using the definition $Y_i = \mu_i + b_i^T Z + \vartheta_i$ or $R_i = \Phi(Y_i)$:

$$\begin{aligned}\mathbf{E}[(l_i \mathbb{1}_{\{A_i^{(0)} \leq \gamma_i\}})^2] &= \mathbf{E}[l_i^2 \mathbb{1}_{\{A_i^{(0)} \leq \gamma_i\}}] \\ &= (l_i^{\max})^2\, \mathbf{E}[\big((1 - \Phi(Y_i))\big)^2 \mathbb{1}_{\{A_i^{(0)} \leq \gamma_i\}}] \\ &= (l_i^{\max})^2 \Big(\, \mathbf{E}[\mathbb{1}_{\{A_i^{(0)} \leq \gamma_i\}}] - 2\mathbf{E}[\Phi(Y_i) \mathbb{1}_{\{A_i^{(0)} \leq \gamma_i\}}] + \mathbf{E}[\Phi(Y_i)^2 \mathbb{1}_{\{A_i^{(0)} \leq \gamma_i\}}]\Big) \\ &= (l_i^{\max})^2 \Big(\Phi(\gamma_i) - 2\mathbf{P}[A_i^{(0)} \leq \gamma_i]\mathbf{E}[R_i | A_i^{(0)} \leq \gamma_i] + \mathbf{E}[R_i^2 \mathbb{1}_{\{A_i^{(0)} \leq \gamma_i\}}]\Big) \\ &= (l_i^{\max})^2 \Big(\Phi(\gamma_i) - 2\Phi(\gamma_i)\mathbf{E}[R_i | \tau_i^{(0)} \leq T] + \mathbf{E}[R_i^2 \mathbb{1}_{\{A_i^{(0)} \leq \gamma_i\}}]\Big),\end{aligned}$$

where only the last expression has to be studied more closely. For this we consider the random variable $k := \sigma_i x + \mu_i$ with $x \sim \mathcal{N}(0,1)$ and therefore $k \sim \mathcal{N}(\mu_i, \sigma_i^2)$, which has obviously the same distribution as Y_i (x has the same distribution as A). We obtain:

$$\begin{aligned}\mathbf{E}[R_i^2 \mathbb{1}_{\{A_i^{(0)} \leq \gamma_i\}}] &= \mathbf{E}[\Phi(\sigma_i x + \mu_i)^2 \mathbb{1}_{\{x \leq \gamma_i\}}] \\ &= \int_{-\infty}^{\gamma_i} \Phi(\sigma_i x + \mu_i)^2 \varphi(x)\, dx \\ &= [\Phi(\sigma_i x + \mu_i)^2 \Phi(x)]_{-\infty}^{\gamma_i} - 2\sigma_i \int_{-\infty}^{\gamma_i} \Phi(\sigma_i x + \mu_i)\varphi(\sigma_i x + \mu_i)\varphi(x)\, dx\end{aligned}$$

(iii)

$$\begin{aligned}\mathbf{Cov}[Z_1, l_1 \mathbb{1}_{\{A_1^{(0)} \leq \gamma_1\}}] &= \mathbf{E}\Big[Z_1 \big(l_1 \mathbb{1}_{\{A_1^{(0)} \leq \gamma_1\}} - \mathbf{E}[l_1 \mathbb{1}_{\{A_1^{(0)} \leq \gamma_1\}}]\big)\Big] \\ &= \mathbf{E}[Z_1 \big(l_1 \mathbb{1}_{\{A_1^{(0)} \leq \gamma_1\}}] - \mathbf{E}[Z_1]\mathbf{E}[l_1 \mathbb{1}_{\{A_1^{(0)} \leq \gamma_1\}}] \\ &= \mathbf{E}[Z_1 l_1 \mathbb{1}_{\{A_1^{(0)} \leq \gamma_1\}}] \\ &= l_1^{\max}\Big(\mathbf{E}[Z_1 \mathbb{1}_{\{A_1^{(0)} \leq \gamma_1\}}] - \mathbf{E}[Z_1 R_1 \mathbb{1}_{\{A_1^{(0)} \leq \gamma_1\}}]\Big).\end{aligned}$$

For the first expression within the parentheses we can write

$$\begin{aligned}\mathbf{E}[Z_1 \mathbb{1}_{\{A_1^{(0)} \leq \gamma_1\}}] &= \mathbf{P}[A_1^{(0)} \leq \gamma_1]\mathbf{E}[Z_1|A_1^{(0)} \leq \gamma_1] \\ &= \Phi(\gamma_1)\mathbf{E}[Z_1|A_1^{(0)} \leq \gamma_1].\end{aligned}$$

For simplification we use the notation A_1 and $A_1^{(0)}$ interchangeably. Note[15] that $Z_1|A_1 \sim \mathcal{N}(0 + \rho_{Z_1,A_1}\gamma_1, 1 - \rho^2_{Z_1,A_1})$ with $\rho_{Z_1,A_1} = a_1^1$. Therefore we have $\mathbf{E}[Z_1|A_1 = \gamma_1] = a_1^1\gamma_1$ and consequently

$$\mathbf{E}[Z_1|A_1 \leq \gamma_1] = \int_{-\infty}^{\gamma_1} a_1^1 x\varphi(x)\, dx = -a_1^1 \int_{-\infty}^{\gamma_1} \varphi'(x)\, dx = -a_1^1\varphi(\gamma_1).$$

For the second expression we have:

$$\begin{aligned}\mathbf{E}[Z_1 R_1 \mathbb{1}_{\{A_1 \leq \gamma_1\}}] &= \mathbf{E}\big[\mathbf{E}[Z_1|Z]\mathbf{E}[R_1|Z]\mathbf{P}[A_1 \leq \gamma_1|Z]\big] \\ &= \int_{-\infty}^{\infty} x\, \Phi\left(\frac{\mu_1 + b_1^T x}{\sqrt{1+\sigma^2_{\vartheta_1}}}\right) \Phi\left(\frac{\gamma_1 - a_1^T x}{\sqrt{1-\|a_1\|^2}}\right) \varphi(x)\, dx.\end{aligned}$$

■

Proof (**Equation** (25) **in subsection 3.3**)
We use the theory of generalized functions (for details see e.g. Delfand et al. (1966)) to obtain an expression for the first-order derivative of the indicator function $\mathbb{1}_{A_i^{(0)} \leq \gamma_i^0}$. We introduce on the one hand the set of test functions, i.e. for $\Omega \subset \mathbb{R}^{d+1}$ open,

$$\mathcal{D}(\Omega) := \{\phi \in \mathcal{C}^\infty(\Omega) | supp(\phi) \text{ is a compact subset of } \Omega\},$$

where $\mathcal{C}^\infty$ denotes the space of infinitely differentiable functions. On the other hand we define the set of distributions as

$$\mathcal{D}'(\Omega) := \{T : \mathcal{D}(\Omega) \longrightarrow \mathbb{C} : T \text{ is linear and continuous}\}.$$

We also define the Heaviside function $h : \mathbb{R}^{d+1} \to \{0, 1\}$:

$$h(x) := h(z_1, \ldots, z_d, \hat{\epsilon}) := \begin{cases} 1, & g(x) := a^T z + \sqrt{1 - \|a\|^2}\hat{\epsilon} \leq \gamma \\ 0, & \text{else} \end{cases}.$$

Therefore we introduce the plane $\mathcal{G} = \{x \in \mathbb{R}^{d+1} : g(x) \leq \gamma\}$ and the boundary $\partial\mathcal{G} = \{x \in \mathbb{R}^{d+1} : g(x) = 0\}$, where we note that the function h is discontinuous across $\partial\mathcal{G}$. We set $\tilde{x} = (z_2, \ldots, z_d, \hat{\epsilon})$ and define analogously: $h : \mathbb{R}^d \to \{0, 1\}$:

$$h(\tilde{x}) := h(z_2, \ldots, z_d, \hat{\epsilon}) := \begin{cases} 1, \; a^2 z_2 + \ldots + a^d z_d + \sqrt{1 - \|a\|^2}\hat{\epsilon} \leq \gamma - a^1 z_1 \\ 0, \; \text{else} \end{cases}$$

[15] In general, the conditional distributions in the bivariate normal case are given by $Y_1|Y_2 \sim \mathcal{N}(\mu_1 + \rho\sigma_1\sigma_2^{-1}(y_2 - \mu_2), \sigma_1^2(1 - \rho^2))$, where as usual μ_1 denotes the mean of Y_1, σ_1 the variance, y_2 the given realization and ρ the correlation between Y_1 and Y_2.

and $\mathcal{H} := \{x \in \mathbb{R}^d : \tilde{g}(x) \leq \gamma - a^1 z_1\}$. Due to the fact that the test functions have a compact support $\phi(-\infty, z_2, \ldots, z_d, \epsilon) = \ldots = \phi(z_1, z_2, \ldots, z_d, -\infty) = 0$. We apply the common differentiation rules in the distribution context and obtain w.l.o.g. for the partial derivative of z_1, where we can use the integral representation (regular distribution), because $h \in L^{1,\,\mathrm{loc}}(\mathbb{R}^{d+1})$:

$$\begin{aligned}
\left\langle \frac{\partial h(x)}{\partial z_1} \, , \, \phi(x) \right\rangle &= - \left\langle h(x) \, , \, \frac{\partial \phi(x)}{\partial z_1} \right\rangle \\
&= - \int_{\mathcal{G}} \frac{\partial \phi(x)}{\partial z_1} \, dx \\
&= - \int_{\mathcal{H}} \int_{-\infty}^{\frac{\gamma - \tilde{g}(\tilde{x})}{a^1}} \frac{\partial \phi(x)}{\partial z_1} \, dz_1 d\tilde{x} \\
&= - \int_{\mathcal{H}} [\phi(z_1, \tilde{x})]_{-\infty}^{\frac{\gamma - \tilde{g}(\tilde{x})}{a^1}} \, d\tilde{x} \\
&= - \int_{\mathcal{H}} \phi \left(\frac{\gamma - \tilde{g}(\tilde{x})}{a^1}, \tilde{x} \right) \, d\tilde{x} \\
&= - \int_{\mathbb{R}^d} h(\tilde{x}) \phi \left(\frac{\gamma - \tilde{g}(\tilde{x})}{a^1}, \tilde{x} \right) \, d\tilde{x} \\
&= - \int_{\mathbb{R}^d} h(\tilde{x}) \int_{\mathbb{R}} \delta\Big(z_1 - \frac{\gamma - \tilde{g}(\tilde{x})}{a^1} \Big) \phi(z_1, \tilde{x}) \, dz_1 d\tilde{x} \\
&= - \int_{\mathbb{R}^{d+1}} h(\tilde{x}) \, \delta\Big(z_1 - \frac{\gamma - \tilde{g}(\tilde{x})}{a^1} \Big) \phi(x) \, dx,
\end{aligned}$$

where we use the well known relation

$$\int_{-\infty}^{\infty} \phi(x) \delta(x - a) \, dx = \phi(a).$$

Therefore we obtain for the partial differentiation (in distributional sense):

$$\frac{\partial h(x)}{\partial z_1} = -h(\tilde{x}) \, \delta\Big(z_1 - \frac{\gamma - \tilde{g}(\tilde{x})}{a^1} \Big) =: -h(\tilde{x}) \, \delta_{\frac{\gamma - \tilde{g}(\tilde{x})}{a^1}}(z_1).$$

■

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