Viscosity Solutions to Optimal Portfolio Allocation Problems in Models with Random Time Changes and Transaction Costs

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Abstract: We consider a risky asset whose instantaneous rate of return takes two different values and changes from one to the other one at random times which are neither known, nor directly observable. We study the optimal allocation strategy of traders who, in the presence of cost of transactions, invest in this risky asset or in a non risky asset according to their belief on the current state of the instantaneous rate of return.

We prove that the related trader’s value function is the unique viscosity solution of a system of HJB inequalities. We carefully prove the Dynamic Programming Principle and show results of numerical experiments.

Key words: Stochastic Control, HJB Inequalities, Viscosity Solutions, Dynamic Programming Principle, Portfolio Allocation, Transaction Costs

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1 Introduction

The practitioners use various rules to rebalance their portfolios. These rules usually come, either from fundamental economic principles, or from mathematical approaches derived from mathematical models, or technical analysis approaches. Technical analysis, which provides decision rules based on past prices behavior, avoids model specification and thus model risk (for a survey, see, e.g., Achelis [1]). Let us describe the technical analysis methodology in the framework of the detection of changes in stock returns.

There has been a considerable literature over the last three decades emphasizing predictability in stock returns. Researchers - such as Jegadeesh and Titman [9] and Lakonishok et al [12] have shown that stock returns are characterized by short term momentum or price continuation patterns. But the latter are only temporary in nature since stock returns also display long term reversals. The recent financial market crisis also clearly demonstrates that observed trends in stock or real estate prices eventually can be subject to sharp interruptions and thus detecting such interruptions or changes of regimes in modeling stock return distributions is of the highest importance. Speculative price bubbles, such as the one that was observed in the US equity housing markets until the subprime crisis or in the dot.com stock market segment at the turn of this Century further suggest that financial markets can be subject to market over-reactions driven by a subset of agents who can temporarily drive asset prices far away from their fundamental values. Once again, such tendencies will eventually correct and lead to sharp corrections that can be quite painful for agents who missed the “turning point”. In light of these historically observed price patterns and irrespectively of their rational or irrational origins, it is not surprising that technical or chartist methods have always been popular at least among a segment of the trading population and of investors who thereby attempt to identify how long they can ‘surf the wave’ and even more crucially at what time a given price trend or pattern is likely to reverse and thus commands a reversal of their specific transaction. While the origins of price regime changes are certainly debatable within the efficient market paradigm, the pursuit of scientific methods to potentially detect them is worthy of an academic analysis. This statement is reinforced by the fact that with access to intra - daily financial data and with the presence of a growing population of short term traders in pursuit of “quick trades”, chartist and scientific methods of price regime changes detection have gained renewed interest.

Chartist methodologies have not been intensively studied from a mathematical point of view. Pastukhov [16] has studied mathematical properties of volatility indicators. Shiryaev and Novikov [18] exhibit an optimal one-time rebalancing strategy in the Black-Scholes model when the drift term of the stock may change its value spontaneously at some random non-observable (hidden) time. Blanchet et al. [4] propose a framework allowing one to compare the performances obtained by various strategies derived from erroneously calibrated mathematical models and the performances obtained by technical analysis techniques, and compare such strategies when the exact model is a diffusion model.
with one and only one change of stock returns at a random time.

In this paper, motivated by the extension of the analysis made in [4] to models with a random number of changes of stock returns at random times and including transactions costs, we study the corresponding optimal allocation problem: obviously, the value function of this optimal allocation problem with a perfect model calibration is the benchmark for strategies derived from statistical procedures or misspecified mathematical models, as well as for the chartist strategies.

We therefore had to solve a stochastic control problem which, to the best of our knowledge, had not been solved in the literature so far. Related works actually concern other dynamics. For instance, Tang and Yong [20] study optimal switching and impulse controls. Brekke and Øksendal [6] consider optimal switching in an economic activity; Pham [17], Ly Vath and Pham [13] and Ly Vath et al. [14] obtained results on the optimal switching problems and the regularity of the related value functions for families of particular models which do not include our model.

Our paper is organized as follows. We first introduce our model and some notation in Section 2. In Section 3, we list a few useful estimates. In Section 4, we prove the continuity of the value functions of our optimal allocation problem. In Section 5, we rigorously prove the Dynamic Programming Principle. In Section 6 we prove that the value function is a viscosity solution to a system of Hamilton–Jacobi–Bellman (HJB) inequalities; uniqueness is proved in Section 7. Finally, in Section 8, we present numerical approximations of the value function, compare performances of several strategies, and briefly discuss misspecification issues.

2 Description of the Model and Notation

Consider a market with a deterministic short term rate $r$, a non risky asset with price process $S^0$, and a stock with price process $S^1$ whose instantaneous trend may only take two values $\mu_1$ and $\mu_2$ with $\mu_1 < r < \mu_2$. The changes of trend may occur at random times $\tau_n$ defined as follows:

$$\tau_0 = 0, \quad \tau_n := \nu_1 + \cdots + \nu_n,$$

where the time intervals $\nu_j$ between changes of trend are independent. The $\nu_{2n+1}$ (respectively, $\nu_{2n}$) are identically distributed; their common law is exponential with parameter $\lambda_1$ (respectively, $\lambda_2$). Thus the trend process is

$$\mu(\theta) := \begin{cases} 
\mu_1 & \text{if } \tau_{2n} \leq \theta < \tau_{2n+1}, \\
\mu_2 & \text{if } \tau_{2n+1} \leq \theta < \tau_{2n+2}.
\end{cases}$$

We suppose that the dynamics of the stock price obeys the stochastic differential equation

$$\frac{dS_\theta}{S_\theta} = \mu(\theta)d\theta + \sigma dB_\theta,$$
where \((B_\theta)\) is a Brownian motion under the historical probability and \(\sigma > 0\) is the constant and deterministic volatility of the stock.

Obviously the trader should totally rebalance his/her portfolio at each change of the trend. We actually consider that he/she will do it at certain decision times which should ideally be equal to \(\tau_n\). However the times \(\tau_n\) cannot be detected exactly and the trader’s strategy needs to be assumed progressively measurable w.r.t. the filtration generated by the observed prices, that is, the filtration \(\mathcal{F}^S := (\mathcal{F}_\theta^S, \ 0 \leq \theta \leq T)\) which is strictly smaller than the filtration generated by the Brownian motion and the \(\tau_n\)'s. This leads us to introduce the following definition of the admissible strategies.

**Definition 2.1.** Let \(T\) be the investment time period. Denote by \(\pi_\theta\) the proportion of wealth invested at time \(\theta\) in the risky asset and by \(U\) a given function (utility function).

Given any time \(t \in [0, T]\), an investment strategy \((\pi_\theta)\) over \([t, T]\) is said admissible if it is a piecewise constant càdlàg process taking values in the pair \(\{0; 1\}\) which is progressively measurable w.r.t the filtration \(\mathcal{F}^S\) and satisfies

\[\mathbb{E}[U(W_T^\pi)] < +\infty,\]

where \(W^\pi\) denotes the wealth process resulting from the strategy \(\pi\).

The set of such admissible strategies is denoted by \(A_t\).

We now introduce the Optional Projection process

\[F_\theta := \mathbb{P}(\mu(\theta) = \mu_1 \mid \mathcal{F}_\theta^S).\]

It is a classical result in filtering theory (see, e.g., Kurtz and Ocone [11]) that the process

\[
\mathcal{B}_\theta := \frac{1}{\sigma} \left( \log \frac{S_\theta}{S_0} - \int_0^\theta \left( \mu_1 F_s + \mu_2 (1 - F_s) - \frac{\sigma^2}{2} \right) ds \right)
\]

is a \(\mathcal{F}^S\) Brownian motion, and that

\[dF_\theta = (-\lambda_1 F_\theta + \lambda_2 (1 - F_\theta))d\theta + \frac{\mu_1 - \mu_2}{\sigma} F_\theta (1 - F_\theta)dB_\theta.\]  

Notice that Feller’s criterion ensures that the solution of the preceding SDE takes values in \([0, 1]\) when \(0 \leq F_0 \leq 1\). Equation (1) clearly yields

\[dS_\theta \over S_\theta = (\mu_1 F_\theta + \mu_2 (1 - F_\theta))d\theta + \sigma d\mathcal{B}_\theta,\]

from which \(\mathcal{F}^S = \mathcal{F}^{\mathcal{B}}\).

We consider the situation where the trader needs to face proportional transaction costs: given an amount \(W\) to transfer from the bank account to the stock,
the cost is $g_0W$; if $W$ is transferred from the stock to the bank account, then the cost is $g_{10}W$. In view of (3), we have, for all $\theta > 0$,

$$
\frac{dW^\pi_{\theta}}{W^\pi_{\theta-}} = (\pi_0(\mu_1 F_\theta + \mu_2 (1 - F_\theta) - r) + r) \, d\theta + \pi_\theta \sigma d\mathcal{B}_\theta - g_{01} I_{\Delta \pi_{\theta} = 1} - g_{10} I_{\Delta \pi_{\theta} = -1},
$$

where $\Delta \pi_{\theta} := \pi_{\theta} - \pi_{\theta-}$. In addition, if $(Z^\pi_{\theta})$ is the continuous part of the process $(\log(W^\pi_{\theta}))$, we have

$$
\frac{dZ^\pi_{\theta}}{} = \left(\pi_0(\mu_1 F_\theta + \mu_2 (1 - F_\theta) - \frac{\sigma^2}{2} - r) + r\right) \, d\theta + \pi_\theta \sigma d\mathcal{B}_\theta.
$$

We need to introduce some notation.

**Notation 2.2.** Given a time $t$ in $[0, T]$, $i \in \{0, 1\}$ and an admissible strategy $\pi$ in $A_t$, we denote by $(F^{t,f}_{\theta}, \pi^{t,z,f,\pi}, W^{t,x,f,i,\pi})$ the solution to (2), (5), and (4) respectively issued at time $t$ from $f \in [0, 1]$, $z \in \mathbb{R}$, and $x > 0$ if $\pi_t = i$ and $x(1 - g_{ij})$ if $\pi_t = j = 1 - i$.

For all $t \leq \theta \leq T$ we also set

$$
\xi^{t,i,\pi}_{\theta} = -\log(1 - g_{01} I_{\pi_t = i = 1}) - \log(1 - g_{10} I_{\pi_t = i = -1})
$$

$$
- \sum_{t < s \leq \theta} \left[\log(1 - g_{01}) I_{\Delta \pi_{s} = 1} + \log(1 - g_{10}) I_{\Delta \pi_{s} = -1}\right].
$$

Notice that $\xi^{t,i,\pi}_{\theta}$ is a positive process and that given a time $t$ in $[0, T]$, $i \in \{0, 1\}$ and an admissible strategy $\pi$ in $A_t$, the process

$$
W^{t,x,f,i,\pi}_{\theta} = x \exp(Z^{t,0,f,\pi}_{\theta} - \xi^{t,i,\pi}_{\theta})
$$

is the unique solution of (4) issued from $x > 0$ if $\pi_t = i$ and $x(1 - g_{ij})$ if $\pi_t = j = 1 - i$.

We consider a utility function $U$ which is, either the logarithmic utility function, or an element of the set $\mathcal{U}$ of the increasing and concave functions of class $\mathcal{C}^1((0, +\infty); \mathbb{R})$ which satisfy: $U(0) = 0$, and there exist real numbers $C > 0$ and $0 \leq \alpha \leq 1$ such that

$$
0 < U'(x) \leq C(1 + x^{-\alpha}) \text{ for all } x > 0.
$$

Notice that HARA utilities belong to the class $\mathcal{U}$.

For all $i \in \{0; 1\}$, $t \in (0, T]$, $x > 0$, $0 \leq f \leq 1$, we set

$$
\forall \pi \in A_t, \quad J^i(t, x, f, \pi) := \mathbb{E}[U(W^{t,x,f,i,\pi}_T)].
$$

Define the value functions as

$$
V^i(t, x, f) := \sup_{\pi \in A_t} J^i(t, x, f, \pi).
$$
We aim to show that the functions $V^i$ are continuous, satisfy the dynamic programming principle, and that the pair $(V^0, V^1)$ is the unique viscosity solution to a system of Hamilton-Jacobi-Bellman inequalities.

We end this section with an elementary inequality which we will often use in the sequel:

**Proposition 2.3.** Under the above assumptions on the utility function $U$, there exists $C > 0$ such that, for all real numbers $z$ and $\tilde{z}$ and all positive real numbers $x$, $\tilde{x}$, and $\zeta$,

$$
|U(xe^{z-\zeta}) - U(\tilde{x}e^{\tilde{z}-\zeta})| \\
\leq C \left( 1 + x^{-\alpha}e^{-\alpha z} + \tilde{x}^{-\alpha}e^{-\alpha \tilde{z}} \right) (|x - \tilde{x}| + (x + \tilde{x})|z - \tilde{z}|) (e^z + e^{\tilde{z}}),
$$

where $\alpha = 1$ if $U(x) = \log(x)$.

**Proof.** If $U(x) = \log(x)$, as $\log(u) \leq u - 1$ for all $u > 0$ we have

$$
\log \left( \frac{x}{\tilde{x}} e^{(z-\tilde{z})} \right) \leq \frac{x}{\tilde{x}} (e^{z-\tilde{z}}) - 1 \\
\leq \frac{1}{\tilde{x}e^{\tilde{z}}} ((x - \tilde{x}) e^{z} + \tilde{x} (e^z - e^{\tilde{z}})) \\
\leq \frac{1}{\tilde{x}e^{\tilde{z}}} (|x - \tilde{x}| e^z + \tilde{x} |z - \tilde{z}| (e^z + e^{\tilde{z}})),
$$

and, similarly,

$$
\log \left( \frac{\tilde{x}}{x} e^{(\tilde{z}-z)} \right) \leq \frac{1}{x e^{z}} (|x - \tilde{x}| e^{\tilde{z}} + x |z - \tilde{z}| (e^z + e^{\tilde{z}})).
$$

If the function $U$ belongs to the class $\mathcal{U}$, using the monotonicity of $U'$ and (8) we get

$$
|U(xe^{z-\zeta}) - U(\tilde{x}e^{\tilde{z}-\zeta})| \leq |xe^z - \tilde{x}e^{\tilde{z}}| e^{-\zeta} (U'(xe^{z-\zeta}) + U'(\tilde{x}e^{\tilde{z}-\zeta})) \\
\leq C |xe^z - \tilde{x}e^{\tilde{z}}| e^{-(1-\alpha)\zeta} \left( 2 + x^{-\alpha}e^{-\alpha z} + \tilde{x}^{-\alpha}e^{-\alpha \tilde{z}} \right).
$$

As $0 \leq \alpha \leq 1$ and $\zeta \geq 0$, the result follows from the obvious inequality

$$
|xe^z - \tilde{x}e^{\tilde{z}}| \leq |x - \tilde{x}|e^z + \tilde{x}|z - \tilde{z}|(e^z + e^{\tilde{z}}).
$$

\[ \square \]

### 2.1 Our main result

Consider the system

$$
\begin{align*}
\min \left\{ -\frac{\partial V^0}{\partial t} - \mathcal{L}^0 V^0; \ V^0(t,x,f) - V^1(t,x(1 - g_{01}),f) \right\} &= 0, \\
\min \left\{ -\frac{\partial V^1}{\partial t} - \mathcal{L}^1 V^1; \ V^1(t,x,f) - V^0(t,x(1 - g_{10}),f) \right\} &= 0,
\end{align*}
$$

(11)
with the boundary condition \( V^0(T, x, f) = V^1(T, x, f) = U(x) \), where

\[
\begin{align*}
\mathcal{L}^0 \phi(t, x, f) &:= \frac{\partial \phi}{\partial t}(t, x, f) + \left( - \lambda_1 f + \lambda_2 (1 - f) \right) \frac{\partial \phi}{\partial f}(t, x, f) \\
&\quad + \frac{1}{2} \left( \frac{\mu_1 - \mu_2}{\sigma} \right)^2 f^2 (1 - f)^2 \frac{\partial^2 \phi}{\partial f^2}(t, x, f),
\end{align*}
\]

and

\[
\begin{align*}
\mathcal{L}^1 \phi(t, x, f) &:= x(\mu_1 f + \mu_2 (1 - f) - r) \frac{\partial \phi}{\partial x}(t, x, f) + \left( - \lambda_1 f + \lambda_2 (1 - f) \right) \frac{\partial \phi}{\partial f}(t, x, f) \\
&\quad + \frac{1}{2} x^2 \sigma^2 \frac{\partial^2 \phi}{\partial x^2}(t, x, f) \\
&\quad + \frac{1}{2} \left( \frac{\mu_1 - \mu_2}{\sigma} \right)^2 f^2 (1 - f)^2 \frac{\partial^2 \phi}{\partial f^2}(t, x, f) \\
&\quad + x(\mu_1 - \mu_2) f (1 - f) \frac{\partial^2 \phi}{\partial x \partial f}(t, x, f).
\end{align*}
\]

Let us comment the system (11). If there would exist smooth value functions \( V^0 \) and \( V^1 \) and an optimal control \( \pi^* \), on the time intervals where \( \pi^* \) is equal to \( i \), the classical PDE would be satisfied:

\[
- \frac{\partial V^i}{\partial t} - \mathcal{L}^i V^i = 0;
\]

When \( \pi^* \) could switch from \( \pi^* = i \) to \( j = 1 - i \), we would have the boundary condition

\[
V^i(t, x, f) = V^j(t, x(1 - g_{ij}), f).
\]

In general, the value functions are not smooth and an optimal control does not exist. In Sec. 6, we rigorously prove that \( V^0, V^1 \) are viscosity solutions of the system (11).

Moreover, the system (11) combines the usual specificities of HJB equations for impulse and switching controls. The impulse part, due to the transaction costs, gives rise to the comparison between the PDE term and the boundary term. The switching part is due to the change of dynamics of the portfolio at each transaction. The originality of this system is that it is neither a classical switching nor classical impulse problem.

In addition, system (11) allows one to develop a numerical procedure to approximate numerically the value functions (see Sec. 8).

We define viscosity solutions for (11) as follows.

**Definition 2.4.** A pair of continuous functions \((V^0, V^1)\) from \([0, T] \times (0, +\infty) \times [0, 1]\) to \(\mathbb{R}\) is a viscosity upper solution to the system (11) if \( V^0(T, x, f) = V^1(T, x, f) = U(x) \) and if, for all \( i \neq j \) in \( \{0, 1\} \), all bounded function \( \phi \) of class \( C^{1,2}([0, T] \times \mathbb{R}_+ \times [0, 1]) \) with bounded derivatives, and all local minimum \((\hat{t}, \hat{x}, \hat{f})\) of \( V^i - \phi \), one has

\[
\min \left\{ - \frac{\partial \phi}{\partial t}(\hat{t}, \hat{x}, \hat{f}) - \mathcal{L}^i \phi(\hat{t}, \hat{x}, \hat{f}); \ V^1(\hat{t}, \hat{x}, \hat{f}) - V^j(\hat{t}, \hat{x}(1 - g_{ij}), \hat{f}) \right\} \geq 0.
\]
A viscosity lower solution is defined analogously: for all local maximum
\( (\hat{t}, \hat{x}, \hat{f}) \) of \( V^* - \phi \), for \( j \neq i \),
\[
\min \left\{ -\frac{\partial \phi}{\partial t}(\hat{t}, \hat{x}, \hat{f}) - L^i \phi(\hat{t}, \hat{x}, \hat{f}); \ V^j(\hat{t}, \hat{x}, \hat{f}) - V^j(\hat{t}, \hat{x}(1 - g_{ij}), \hat{f}) \right\} \leq 0.
\]

Finally, a viscosity solution is both an upper and lower viscosity solution.

**Theorem 2.5.** Let \( \mathcal{V}_0 \) be the class of functions \( \mathcal{V} \) which are continuous on \( [0, T] \times [0, +\infty) \times [0, 1] \) and satisfy: for all \( (t, f) \in [0, T] \times [0, 1] \), \( \mathcal{V}(t, 0, f) = 0 \) and there exists \( C > 0 \) such that
\[
|\mathcal{V}(t, x, f)| \leq C(1 + x^{-\alpha} + x) \text{ for all } (t, x, f) \in [0, T] \times (0, +\infty) \times [0, 1].
\]

Suppose that the utility function belongs to the class \( \mathcal{U} \) defined above. Then the pair of value functions \( (V^0, V^1) \) is the unique viscosity solution of (11) in \( \mathcal{V}_0 \), satisfying the boundary condition \( V^0(t, x, f) = V^1(t, x, f) = U(x) \) for all \( (x, f) \) in \( [0, +\infty) \times [0, 1] \).

If \( U \) is logarithmic, \( (V^0, V^1) \) is the unique viscosity solution of (11) in the set of function \( \{ \log(x) + \mathcal{V}(t, f) \} \) where \( \mathcal{V} \) is continuous on \( [0, T] \times [0, 1] \).

**Remark 2.6.** The theorem 2.5 allows one to use the numerical solution of the system of inequalities (11) in order to construct Markov allocation strategies \( \hat{\pi} \) such that \( J^*(t, x, f, \hat{\pi}) \) is close to \( V^*(t, x, f) \). To implement such a strategy, the investor needs to estimate \( F_t \) at each time \( t \) from the observation of the prices \( (S_\theta; \ \theta \leq t) \). From (2) and (3), we have for some smooth functions \( \alpha_1 \) and \( \alpha_2 \),
\[
dF_\theta = \alpha_1(F_\theta) d\theta + \alpha_2(F_\theta) \frac{dS_\theta}{S_\theta}.
\]

One can discretize this equation by using, e.g., the Euler scheme. In [15] the authors construct an approximation of \( F \) based on filtering theory which is more accurate than the Euler approximation.

### 3 Preliminary estimates

**Notation 3.1.** In this section we are given arbitrary times \( 0 \leq s \leq t \leq \hat{t} \leq T \), admissible strategies \( \pi \), \( \mathcal{F}_s^\mathcal{S} \)-measurable random variables \( F \) and \( \hat{F} \) taking values in \([0, 1]\), and \( \mathcal{F}_s^\mathcal{S} \)-measurable random variables \( Z, \hat{Z} \).

It must be understood that the various constants \( C \) in the inequalities below only depend on some of the constants \( m, \mu_1, \mu_2, \lambda_1, \lambda_2, \sigma \) and \( T \).

Elementary calculations show that we have, for all integer \( m \geq 1 \) and all time \( t \leq \theta \leq T \),
\[
E[Z_t^{\theta, 0, F, \pi} | 2m \leq C(\theta - t)^m, \quad E[Z_T^{\hat{F}, \hat{F}, \pi} - Z_T^{\hat{F}, \hat{F}, \pi} | 2m \leq C \left( E[Z - \hat{Z} | 2m + E[F - \hat{F}] | 2m + |\hat{t} - t|^m \right). \quad (12)
\]

\[
E[Z_T^{\hat{F}, \hat{F}, \pi} - Z_T^{\hat{F}, \hat{F}, \pi} | 2m \leq C \left( E[Z - \hat{Z} | 2m + E[F - \hat{F}] | 2m + |\hat{t} - t|^m \right). \quad (13)
\]
and, for all \( t \leq \theta \leq \theta^* \leq T, \)
\[
\mathbb{E}[F^t,F_\theta - F^t,F] \leq \mathbb{E}|F - \hat{F}|^m, \tag{14}
\]
and, for all \( \beta > 0, \)
\[
\mathbb{E}[F^t,F_\theta - F^t,F] \leq C(\theta^* - \theta)^m. \tag{15}
\]
As well, one has, for all \( \beta > 0, \)
\[
\mathbb{E} \exp \left( \beta \int_{t}^{\theta} \pi_s \sigma dB_s \right) \leq C \exp \left( \frac{\beta^2 \sigma^2}{2} (\theta - t) \right),
\]
and, for all \( \mathcal{F}_t^S \)-measurable random variables \( Z \) such that \( \mathbb{E}\exp(\beta Z) < \infty, \)
\[
\mathbb{E}\exp(\beta Z^i_{\theta}^\pi) \leq \mathbb{E}\exp(\beta Z) \exp(C(\theta - t)). \tag{16}
\]
We end this part by the following result which is easy to deduce from (7), (10), (13) and (16).

**Proposition 3.2.** There exists \( C > 0 \) such that, for all \( 0 \leq t \leq \hat{t} \leq T, \) all \((x, \hat{x}, f, \hat{f}) \in [0, +\infty)^2 \times [0,1]^2, \) for all admissible controls \( \pi \in \mathcal{A}_t \) such that \( \pi_0 = \pi_i \) for \( \theta \in [t, \hat{t}], \)
\[
|J^i(\hat{t}, \hat{x}, \hat{f}, \pi) - J^i(t, x, f, \pi)| \leq C(1 + x^{-\alpha} + \hat{x}^{-\alpha}) \left( |\hat{x} - x| + (x + \hat{x})(|\hat{f} - f| + |t - \hat{t}|^{1/2}) \right).
\]

## 4 Continuity of the Value Functions

**Theorem 4.1.** There exists \( C > 0 \) such that, for all \( i \in \{0,1\}, \) \( 0 \leq t \leq \hat{t} \leq T, \) \( x \) and \( \hat{x} \) in \((0, +\infty), \) \( f \) and \( \hat{f} \) in \([0,1], \) one has
\[
|V^i(\hat{t}, \hat{x}, \hat{f}) - V^i(t, x, f)| \leq C(1 + x^{-\alpha} + \hat{x}^{-\alpha}) \left( |\hat{x} - x| + (x + \hat{x})(|\hat{f} - f| + |t - \hat{t}|^{1/2}) \right). \]

**Proof.** We successively consider \( |V^i(\hat{t}, \hat{x}, \hat{f}) - V^i(\hat{t}, \hat{x}, f)| \) and \( |V^i(\hat{t}, \hat{x}, f) - V^i(t, x, f)|. \)

For the first term, choose a control \( \pi \in \mathcal{A}_{\hat{t}}. \) In view of proposition 3.2:
\[
|J^i(\hat{t}, \hat{x}, \hat{f}, \pi) - J^i(\hat{t}, \hat{x}, f, \pi)| \leq C(1 + x^{-\alpha} + \hat{x}^{-\alpha}) \left( |\hat{x} - x| + (x + \hat{x})(|\hat{f} - f|) \right).
\]

Taking the supremum over all admissible control \( \pi \in \mathcal{A}_{\hat{t}} \) yields the desired inequality. We now examine \( V^i(\hat{t}, x, f) - V^i(t, x, f). \) For all admissible control \( \hat{\pi} \in \mathcal{A}_{\hat{t}}, \) define the new admissible control \( \pi \in \mathcal{A}_t \) on \([t, T]\) by \( \pi := \hat{\pi} \) on \([t, \hat{t}]\) and \( \pi = i \) on \([\hat{t}, T]. \) Then \( \xi_T^{i,\hat{t},\hat{\pi}} = \xi_T^{i,\hat{t},\pi}. \) Therefore, using (10) again, we have
\[
J^i(\hat{t}, x, f, \hat{\pi}) - V^i(t, x, f) \leq J^i(\hat{t}, x, f, \hat{\pi}) - J^i(t, x, f, \pi) \leq Cx \mathbb{E}[1 + x^{-\alpha} \exp(-\alpha Z_t^{0,0,f,\hat{\pi}}) + x^{-\alpha} \exp(-\alpha Z_t^{0,0,f,\pi})]
\]
\[
Z_t^{0,0,f,\hat{\pi}} - Z_t^{0,0,f,\pi} \leq Cx \left( 1 + x^{-\alpha} \left( \mathbb{E}[Z_t^{0,0,f,\hat{\pi}} - Z_t^{0,0,f,\pi}]^2 \right)^{1/2} \right).
\]
As \( \pi = \hat{\pi} \) on \( [\hat{t}, T] \), the inequalities (12) and (15) and (13) imply
\[
\mathbb{E}|Z^t_{i, 0, f, \pi} - Z^T_{i, 0, f, \pi}|^2 \leq C \left( \mathbb{E}|Z^t_{i, 0, f, \pi}|^2 + \mathbb{E}|Z^T_{i, 0, f, \hat{\pi}} - Z^T_{i, 0, f, \pi}|^2 \right)
\leq C \left( \mathbb{E}|Z^t_{i, 0, f, \pi}|^2 + \mathbb{E}|F^t_{i, \hat{\pi}} - f|^2 \right)
\leq C(t - t).
\]
Taking the supremum over all admissible controls \( \hat{\pi} \in A_i \), we obtain
\[
V^i(\hat{t}, x, f) - V^i(t, x, f) \leq Cx(1 + x^{-a})|\hat{t} - t|^{1/2}.
\]
We finally consider \( V^i(t, x, f) - V^i(\hat{t}, x, f) \). Fix \( \varepsilon > 0 \) and an admissible control \( \pi \in A_i \) such that \( V^i(t, x, f) \leq J^i(t, x, f, \pi) + \varepsilon \). Then, for all admissible controls \( \hat{\pi} \in A_i \), one has
\[
V^i(t, x, f) - V^i(\hat{t}, x, f) \leq J^i(t, x, f, \pi) - J^i(\hat{t}, x, f, \hat{\pi}) + \varepsilon.
\]
We now aim to choose an admissible control \( \hat{\pi} \in A_i \) on \( [\hat{t}, T] \) which is close to \( \pi \) and satisfies \( \xi^i_{\hat{t}, \pi} = \xi^i_{\hat{t}, \hat{\pi}} \). The difficulty comes from the possible jumps of \( \pi \) before \( \hat{t} \). This leads us to choose
\[
\hat{\pi}_s := \begin{cases} 
\pi_{as+b} & \text{if } \hat{t} \leq s < \hat{t} + \varepsilon, \\
\pi_s & \text{if } s \geq \hat{t} + \varepsilon,
\end{cases}
\]
where \( a := (\hat{t} + \varepsilon - t)/\varepsilon, b := (\hat{t} + \varepsilon)(t - \hat{t})/\varepsilon. \) Notice that \( af + b = t \), and \( a(\hat{t} + \varepsilon) + b = \hat{t} + \varepsilon \). As \( as + b \leq s \) for all \( \hat{t} \leq s \leq \hat{t} + \varepsilon \), the control \( \hat{\pi} \) is progressively measurable. In addition, the costs at time \( T \) due to the jumps of \( \pi \) and \( \hat{\pi} \) are equal, and thus \( \xi^i_{\hat{t}, \pi} = \xi^i_{\hat{t}, \hat{\pi}}. \) As one also has \( \hat{\pi} = \pi \) on \( (\hat{t} + \varepsilon, T) \), in view of (10) and (16) it comes
\[
V^i(t, x, f) - V^i(\hat{t}, x, f) \leq \varepsilon + |J^i(t, x, f, \pi) - J^i(\hat{t}, x, f, \hat{\pi})| \\
\leq \varepsilon + Cx(1 + x^{-a}) \left( \mathbb{E}|Z^t_{i, 0, f, \pi} - Z^T_{i, 0, f, \pi}|^2 \right)^{1/2}.
\]
Using again inequalities (12), (15) and (13),
\[
\mathbb{E}|Z^t_{i, 0, f, \pi} - Z^T_{i, 0, f, \pi}|^2 \\
\leq C \left( \mathbb{E}|Z^t_{i, 0, f, \pi}|^2 + \mathbb{E}|Z^t_{i, 0, f, \pi}|^2 + \mathbb{E}|Z^T_{i, 0, f, \pi} - Z^T_{i, 0, f, \pi}|^2 \right)
\leq C \left( \mathbb{E}|Z^t_{i, 0, f, \pi}|^2 + \mathbb{E}|Z^t_{i, 0, f, \pi}|^2 + \mathbb{E}|F^t_{i, \hat{\pi}} - f|^2 \right)
\leq C \left( \hat{t} + \varepsilon - t + \hat{t} + \varepsilon - \hat{t} + \mathbb{E}|F^t_{i, \hat{\pi}} - f|^2 + \mathbb{E}|F^t_{i, \hat{\pi}} - f|^2 \right)
\leq C((\hat{t} - t) + \varepsilon).
Therefore
\[ V^i(t, x, f) - V^i(\hat{t}, x, f) \leq \varepsilon + Cx(1 + x^{-\alpha})(\varepsilon^{1/2} + (\hat{t} - t)^{1/2}), \]
and the desired result follows. \hfill \square

**Corollary 4.2.** For all \( \beta \geq \alpha \), \( 0 \leq s \leq t \leq T \), and \( i, x, f \), for all admissible control \( \pi \in \mathcal{A}_t \), one has
\[ \mathbb{E} \left[ V^i(t, W^t_{s,x,f,i,\pi}, F^t_{s,f}) - V^i(s, x, F^t_{s,i,\pi}) \right] < C(t - s)^{\beta/2}. \]

We conclude this section by showing that the functions \( V^0 \) and \( V^1 \) are continuous.

**Corollary 4.3.** If the utility function belongs to the class \( U \), then \( V^0 \) and \( V^1 \) are continuous on \([0, T] \times [0, +\infty) \times [0, 1]\).

**Proof.** As \( U \) is increasing and concave, for all \( i, t, x, f \) and all admissible controls \( \pi \) one has, in view of (16),
\[ V^i(t, x, f) = \sup_{\pi \in \mathcal{A}_t} \mathbb{E}[U(W^t_{x,f,i,\pi})] \]
\[ \leq U(x \sup_{\pi \in \mathcal{A}_t} \mathbb{E}(Z^t_{x,f,i,\pi})) \]
\[ \leq U(x \sup_{\pi \in \mathcal{A}_t} \mathbb{E}(Z^t_{x,f,i,\pi})) \]
\[ \leq U(Cx), \]
where \( C \) is a constant independent of \( t, x, f, \pi \). As \( U \) is continuous at 0, \( V^i(t, x, f) \) tends to 0 with \( x \), and the convergence is uniform w.r.t. \( t, f \). In addition, \( W^t_{x,f,i,\pi} = 0 \) for all \( t, f, \pi \), which ends the proof. \hfill \square

## 5 The Dynamic Programming Principle

**Proposition 5.1.** For all bounded continuous functions \( \varphi \) on \( \mathbb{R}^+ \), all stopping times \( \tau \) such that \( t \leq \tau \leq T \), all \( x > 0, 0 \leq f \leq 1 \), \( \pi \in \mathcal{A}_t \), one has
\[ \mathbb{E} \left[ \varphi(W^t_{x,f,i,\pi}) \mid \mathcal{F}^\tau \right] = \mathbb{E} \left[ \varphi(W^\tau_{x,f,i,\pi}) \mid \mathcal{F}^\tau \right], \mathbb{P}_\tau - \text{a.s.} \]

**Proof.** We first remark that the filtration \( \mathcal{F}^\tau \) is continuous (\( \mathcal{F}^\tau = \mathcal{F}^B \)). So, we can approximate the stopping time \( \tau \) by a sequence of stopping times \( \tau_n \) with countably many values. Furthermore, the equality (7) shows that for all \( \pi \in \mathcal{A}_t \), all \( i \in \{0, 1\} \), all Borel subset \( A \) of the set of càdlàg trajectories from \([t, T] \) to \([0, +\infty) \times [0, 1] \), the mapping \((t, x, f, i, \pi, \pi, [t, \tau]) \mapsto \mathbb{P} \left( (W^t_{x,f,i,\pi}, F^t_{x,i,\pi}) \in A \right) \) is measurable. Then, we apply Theorem 6.1.2 of [19] and obtain the result for each \( \tau_n \). Letting \( n \) go to infinity ends the proof. \hfill \square
The Dynamic Programming Principle is a key step to establish the relationship between value functions and Hamilton–Jacobi–Bellman equations. This principle in the framework of processes with jumps has often been just assumed in the literature. A rigorous proof can be found in Ishikawa [8] for a model with jumps which substantially differs from ours. We thus carefully prove the following theorem.

**Theorem 5.2 (Dynamic Programming Principle).** Let \( \mathcal{I}_{t,T} \) denote the set of all \( \mathcal{F}^S \) stopping times taking values in \([t,T]\). For all \( 0 \leq t \leq T \) and \( x > 0 \), \( 0 < f < 1 \), \( i \in \{0; 1\} \), one has

\[
V^i(t, x, f) = \sup_{\pi \in \mathcal{A}_t} \inf_{\tau \in \mathcal{I}_{t,T}} \mathbb{E}[\pi] V^i(\tau, W^t_{\tau}, x, f, i, \pi, F^t_{\tau}].
\]

**Proof.** The proof is divided in two parts: we first prove the upper bound \( V^i(t, x, f) \leq \sup_{\pi \in \mathcal{A}_t} \inf_{\tau \in \mathcal{I}_{t,T}} \mathbb{E}[\pi] V^i(\tau, W^t_{\tau}, x, f, i, \pi, F^t_{\tau}] \), and then the lower bound \( V^i(t, x, f) \geq \sup_{\pi \in \mathcal{A}_t} \sup_{\tau \in \mathcal{I}_{t,T}} \mathbb{E}[\pi] V^i(\tau, W^t_{\tau}, x, f, i, \pi, F^t_{\tau}] \) (which is less immediate to get than the upper bound).

**The upper bound**

In view of proposition 5.1 one has: for all admissible control \( \pi \) in \( \mathcal{A}_t \), and all stopping times \( \tau \in \mathcal{I}_{t,T} \)

\[
J^i(t, x, f, \pi) = \mathbb{E}[U(W^t_{\tau}, x, f, i, \pi)]
\]

\[= \mathbb{E}[\mathbb{E}[U(W^t_{\tau}, x, f, i, \pi) | \mathcal{F}^S_{t, \tau}]]
\]

\[= \mathbb{E}U(W^t_{\tau}, x, f, i, \pi, \pi | [\tau, T])
\]

\[= \sum_{k \in \{0, 1\}} \mathbb{E}U(W^t_{\tau}, x, f, i, k, \pi | [\tau, T]) I_{\pi_k = k}
\]

\[= \sum_{k \in \{0, 1\}} \mathbb{E}[J^k(\tau, W^t_{\tau}, x, f, i, \pi, F^t_{\tau} | [\tau, T]) I_{\pi_k = k}]
\]

\[\leq \sum_{k \in \{0, 1\}} \mathbb{E}[V^k(\tau, W^t_{\tau}, x, f, i, \pi, F^t_{\tau}) I_{\pi_k = k}]
\]

\[\leq \mathbb{E}[\pi] V^i(\tau, W^t_{\tau}, x, f, i, \pi, F^t_{\tau}].
\]

It then remains to take the supremum over all stopping times \( \tau \in \mathcal{I}_{t,T} \) and the infimum over all admissible controls \( \pi \in \mathcal{A}_t \) to obtain:

\[V^i(t, x, f) \leq \sup_{\pi \in \mathcal{A}_t} \inf_{\tau \in \mathcal{I}_{t,T}} \mathbb{E}[\pi] V^i(\tau, W^t_{\tau}, x, f, i, \pi, F^t_{\tau}].
\]
The lower bound

In view of theorem 4.1, for all $\varepsilon > 0$, one can find a countable partition of $[t, T] \times (0, +\infty) \times [0, 1]$ with Borel subsets $\mathcal{B}_p$ such that, for all $p$, all $(t, x, f)$ and $(\hat{t}, \hat{x}, \hat{f})$ in $\mathcal{B}_p$, all $i$,

$$|V^i(t, x, f) - V^i(\hat{t}, \hat{x}, \hat{f})| \leq \varepsilon. \quad (17)$$

In addition, if $t \leq \hat{t}$, thanks to Proposition 3.2, for all $i$, for all admissible controls $\pi$ in $\mathcal{A}_t$ such that $\pi_\theta = i$ for $\theta \in [t, \hat{t})$, one has also for all $p$, all $(t, x, f)$ and $(\hat{t}, \hat{x}, \hat{f})$ in $\mathcal{B}_p$,

$$|J^i(t, x, f, \pi) - J^i(\hat{t}, \hat{x}, \hat{f}, \pi)| \leq \varepsilon. \quad (18)$$

We now set

$$\rho := \sup_{\pi \in \mathcal{A}_p} \sup_{\tau \in T_{t,T}} \mathbb{E}[V^{\pi_{\tau}}(\tau, W^{\pi_{\tau}, i, \pi}_{\tau}, F^{\pi_{\tau}}_{\tau})].$$

Choose $\pi$ in $\mathcal{A}_t$ and $\tau$ in $T_{t,T}$ such that

$$\rho \leq \varepsilon + \mathbb{E}[V^{\pi_{\tau}}(\tau, W^{\pi_{\tau}, i, \pi}_{\tau}, F^{\pi_{\tau}}_{\tau})].$$

By definition of the partition $\{\mathcal{B}_p\}$,

$$\rho \leq \varepsilon + \mathbb{E} \left[ \sum_{p=0}^{\infty} V^{\pi_{\tau}}(\tau, W^{\pi_{\tau}, i, \pi}_{\tau}, F^{\pi_{\tau}}_{\tau}) \mathbb{1}_{(\tau, W^{\pi_{\tau}, i, \pi}_{\tau}, F^{\pi_{\tau}}_{\tau}) \in \mathcal{B}_p} \right].$$

Now, for all $p$, choose a triple $(t_p, x_p, f_p)$ in the closure $\overline{\mathcal{B}}_p$ of $\mathcal{B}_p$, where $t_p$ is the largest time in the trace of $\overline{\mathcal{B}}_p$ in $[t, T]$. In view of (17) one thus has

$$\rho \leq 2\varepsilon + \mathbb{E} \left[ \sum_{p=0}^{\infty} V^{\pi_{\tau}}(t_p, x_p, f_p) \mathbb{1}_{(t_p, x_p, f_p) \in \mathcal{B}_p} \right].$$

For all $p$, choose a control $\pi^{p,i}$ in $\mathcal{A}_p$ such that $V^i(t_p, x_p, f_p) \leq \varepsilon + J^i(t_p, x_p, f_p, \pi^{p,i})$. Then

$$\rho \leq 3\varepsilon + \mathbb{E} \left[ \sum_{p=0}^{\infty} J^{\pi_{\tau}}(t_p, x_p, f_p, \pi^{p,i}_{\tau}) \mathbb{1}_{(t_p, x_p, f_p, \pi^{p,i}_{\tau}) \in \mathcal{B}_p} \right].$$

Next, $p(\omega)$ being the integer s.t. $(\tau, W^{\pi_{\tau}, i, \pi}_{\tau}, F^{\pi_{\tau}}_{\tau})(\omega) \in \mathcal{B}_{p(\omega)}$, define the control $\hat{\pi}$ in $\mathcal{A}_t$ by

$$\hat{\pi}_s(\omega) := \begin{cases} 
\pi_s(\omega) & \text{if } t \leq s \leq \tau(\omega), \\
\pi_{\tau}(\omega) & \text{if } \tau(\omega) \leq s < t_{p(\omega)}, \\
\pi_{p(\omega), j}(\omega) & \text{if } s \geq t_{p(\omega)} \text{ and } \tau(\omega) = j.
\end{cases}$$

From now on, we write $\hat{\pi}$ for $\hat{\pi}|_{[t_p, T]}$ and $\hat{\pi}|_{[T, T]}$. We have

$$\rho \leq 3\varepsilon + \mathbb{E} \left[ \sum_{p=0}^{\infty} J^{\pi_{\tau}}(t_p, x_p, f_p, \hat{\pi}) \mathbb{1}_{(t_p, x_p, f_p, \hat{\pi}) \in \mathcal{B}_p} \right].$$
As the control \( \dot{\pi} \) is constant on \( [\tau, t_p) \), the inequality (18) leads to

\[
\rho \leq 4\varepsilon + \mathbb{E} \left[ \sum_{p=0}^{\infty} J^{\pi^p} (\tau, W^t_{\tau}, x_{\pi^p}, i, \pi^p) I(\tau, W^t_{\tau}, x_{\pi^p}, i, \pi^p, \pi^p) \right]
\]

\[
\leq 4\varepsilon + \mathbb{E} [J^{\pi^\tau}(\tau, W^t_{\tau}, x_{i, \pi^\tau}, F^t_{\tau}, \dot{\pi})]
\]

\[
= 4\varepsilon + \mathbb{E} [U(\pi^\tau, W^t_{\tau}, x_{i, \pi^\tau}, F^t_{\tau}, \dot{\pi})]
\]

Therefore, in view of proposition 5.1,

\[
\rho \leq 4\varepsilon + \mathbb{E} [U(\pi^\tau, W^t_{\tau}, x_{i, \pi^\tau}, F^t_{\tau}, \dot{\pi})] = 4\varepsilon + V^i (t, x, f),
\]

It now remains to let \( \varepsilon \) tend to 0. \( \square \)

6 Existence of a viscosity solution

6.1 Existence of a viscosity upper solution

In this section we show that the pair \((V^0, V^1)\) of value functions is a upper viscosity solution on \([0, T] \times (0, +\infty) \times [0, 1]\) of the system (11).

Fix \( i \in \{0; 1\} \). Let \((\dot{\hat{t}}, \dot{\hat{x}}, \dot{\hat{f}})\) be a local minimum of \( V^i - \varphi \), where \( \varphi \) is a function of class \( C^{1,2} \) defined on a neighborhood \([\hat{t}, \hat{t} + \varepsilon] \times B_\varepsilon(\hat{t}, \hat{x}, \hat{f})\), where

\[
B_\varepsilon := [\hat{x}(1 + \varepsilon)^{-1}, \hat{x}(1 + \varepsilon)] \times [\hat{f} - \varepsilon, \hat{f} + \varepsilon] \cap [0, 1],
\]

and such that \( V^i(\hat{t}, \hat{x}, \hat{f}) = \varphi(\hat{t}, \hat{x}, \hat{f}) \).

For all controls \( \pi \in A_i \), we have \( W^i_{\hat{t}}(1 - g_{ij}) \hat{x}, \hat{f}, \pi \leq W^i_{\hat{t}} \hat{x}, \hat{f}, \pi \). Then, for all \( \theta \geq \hat{t} \), we also have \( W^i_{\theta}(1 - g_{ij}) \hat{x}, \hat{f}, \pi \leq W^i_{\theta} \hat{x}, \hat{f}, \pi \). Therefore

\[
\mathbb{E} U(W^i_{\theta}(1 - g_{ij}) \hat{x}, \hat{f}, \pi) \leq \mathbb{E} U(W^i_{\theta} \hat{x}, \hat{f}, \pi),
\]

\[
V^i(\hat{t}, \hat{x}(1 - g_{ij}), \hat{f}) \leq V^i(\hat{t}, \hat{x}, \hat{f}).
\]

We now aim to prove

\[
- \left( \frac{\partial \varphi}{\partial \xi} + \mathcal{L}_i \varphi \right)(\hat{t}, \hat{x}, \hat{f}) \geq 0.
\]

Fix \( 0 < h < \varepsilon \) and choose a control \( \pi \) which takes the value \( i \) on the time interval \([\hat{t}, \hat{t} + h]\). The Dynamic Programming Principle with the constant stopping time
\( \hat{t} + h \) leads to

\[
\varphi(\hat{t}, \hat{x}, \hat{f}) = V^i(\hat{t}, \hat{x}, \hat{f}) \\
\geq E[V^i(\hat{t} + h, W^i_{\hat{t}+h}, F^i_{\hat{t}+h})] \\
= E[V^i(\hat{t} + h, W^i_{\hat{t}+h}, F^i_{\hat{t}+h})\mathbb{I}_{(W^i_{\hat{t}+h}, F^i_{\hat{t}+h}) \in \mathcal{B}_s}] \\
+ \mathbb{E}[V^i(\hat{t} + h, W^i_{\hat{t}+h}, F^i_{\hat{t}+h})\mathbb{I}_{(W^i_{\hat{t}+h}, F^i_{\hat{t}+h}) \notin \mathcal{B}_s}] \\
\geq E[\varphi (\hat{t} + h, W^i_{\hat{t}+h}, F^i_{\hat{t}+h})] + \mathbb{E}[(V_i - \varphi)(\hat{t} + h, W^i_{\hat{t}+h}, F^i_{\hat{t}+h})] \\
\mathbb{E}[(W^i_{\hat{t}+h}, F^i_{\hat{t}+h}) \notin \mathcal{B}_s].
\]

By Itô’s formula to \( \varphi(t, W_t, F_t) \),

\[
E[\varphi (\hat{t} + h, W^i_{\hat{t}+h}, F^i_{\hat{t}+h})] = \varphi (\hat{t}, \hat{x}, \hat{f}) + E \int_{\hat{t}}^{\hat{t}+h} \left( \frac{\partial \varphi}{\partial t} + \mathcal{L}^i \varphi \right) (s, W^i_s, F^i_s) ds,
\]

since \( \pi = i \) on \([\hat{t}, \hat{t} + h] \). Therefore

\[
0 \geq \frac{1}{h} E \left[ \int_{\hat{t}}^{\hat{t}+h} \left( \frac{\partial \varphi}{\partial t} + \mathcal{L}^i \varphi \right) (s, W^i_s, F^i_s) ds \right] \\
+ \frac{1}{h} \mathbb{E} \left[ (V_i - \varphi)(\hat{t} + h, W^i_{\hat{t}+h}, F^i_{\hat{t}+h})\mathbb{I}_{(W^i_{\hat{t}+h}, F^i_{\hat{t}+h}) \notin \mathcal{B}_s} \right]. \tag{19}
\]

When \( h \) tends to 0, the first term of the right-hand side tends to \( \left( \frac{\partial \varphi}{\partial t} + \mathcal{L}^i \varphi \right) (\hat{t}, \hat{x}, \hat{f}) \).

For the second term, in view of corollary 4.2,

\[
\left| \mathbb{E} \left[ (V_i - \varphi)(\hat{t} + h, W^i_{\hat{t}+h}, F^i_{\hat{t}+h})\mathbb{I}_{(W^i_{\hat{t}+h}, F^i_{\hat{t}+h}) \notin \mathcal{B}_s} \right] \right| \\
\leq \left( \mathbb{E} \left[ (V_i - \varphi)(\hat{t} + h, W^i_{\hat{t}+h}, F^i_{\hat{t}+h})\mathbb{I}_{(W^i_{\hat{t}+h}, F^i_{\hat{t}+h}) \notin \mathcal{B}_s} \right]^2 \right)^{1/2} \left[ \mathbb{P}((W^i_{\hat{t}+h}, F^i_{\hat{t}+h}) \notin \mathcal{B}_s) \right]^{1/2} \\
\leq C \mathbb{P}((W^i_{\hat{t}+h}, F^i_{\hat{t}+h}) \notin \mathcal{B}_s)^{1/2}.
\]

One now uses the inequalities (12) and (15): as \( W \) is continuous on \([\hat{t}, \hat{t} + h] \),

\[
\mathbb{P}((W^i_{\hat{t}+h}, F^i_{\hat{t}+h}) \notin \mathcal{B}_s) \leq \mathbb{P}(|\log(\hat{x}^{-1}W^i_{\hat{t}+h}) > \log(1 + \varepsilon)| + \mathbb{P}(|\hat{f}^i - \hat{f}| > \varepsilon) \\
+ \mathbb{P}(|\log(\hat{x}^{-1}W^i_{\hat{t}+h}) < - \log(1 + \varepsilon)|) \\
\leq \frac{\mathbb{E}|Z_{t+h}^{0,\varepsilon} |^8}{(\log(1 + \varepsilon))^8} + \frac{\mathbb{E}|\hat{f}^i - \hat{f}|^8}{\varepsilon^8} \\
\leq C h^4.
\]

Then, the second term in (19) is of order \( h \) and it remains to let \( h \) tend to 0.
6.2 Existence of a viscosity lower solution

Suppose that \((V^0, V^1)\) is not a viscosity lower solution. Then there exist \(i \in \{0, 1\}\), a smooth function \(\varphi\) with bounded derivatives, \(\epsilon > 0\), a local maximum \((\hat{t}, \hat{x}, \hat{f})\) of \(V^i - \varphi\) on \([\hat{t}, \hat{t} + \epsilon] \times B_\epsilon\), and \(\gamma > 0\) such that, for all \(t, x, f\) in \([\hat{t}, \hat{t} + \epsilon] \times B_\epsilon\),

\[
\frac{\partial \varphi}{\partial t} + \mathcal{L}^i \varphi(t, x, f) \geq \gamma,
\]

\[
V^i(t, x, f) - V^i(t, x(1 - g_{ij}), f) \geq \gamma, \quad j \neq i.
\]

We aim to exhibit a contradiction.

As \((\hat{t}, \hat{x}, \hat{f})\) of \(V_i(t, x, f)\) belongs to \([\hat{t}, \hat{t} + \epsilon] \times B_\epsilon\), one can substitute \(L_i^\tau\) to \(\mathcal{L}^\tau\) in the preceding inequality, from which

\[
V^i(\hat{t}, \hat{x}, \hat{f}) \geq \mathbb{E} \left[ V^i(\tau, W_{\tau, t}^{i, \pi}, F_\tau) + \gamma(\tau - \hat{t}) \right].
\]

For all \((i, t, x, f), \; j = 1 - i\) and for all controls \(\pi\) as above, either one has \(\pi_\tau = i\) and \(W^{i, \pi}_{\tau, t} = W^{i, \pi}_{\tau, t}\), or \(\pi_\tau \neq i\) and \(W^{i, \pi}_{\tau, t} = (1 - g_{ij})W^{i, \pi}_{\tau, t}\). In addition \(V^i(t, x, f) \geq V^j(t, (1 - g_{ij})x, f)\), hence

\[
V^i(\tau, W_{\tau, t}^{i, \pi}, F_\tau) \geq V^{\pi_\tau}(\tau, W_{\tau, t}^{i, \pi}, F_\tau).
\]

The Dynamic Programming Principle then implies

\[
V^i(\hat{t}, \hat{x}, \hat{f}) = \sup_{\pi \in A_i} \inf_{\tilde{\tau} \in I_i, \tau} \mathbb{E} V^{\pi_\tau}(\tilde{\tau}, W_{\tilde{\tau}, t}^{i, \pi}, F_{\tilde{\tau}}).
\]
Thus, for all $h > 0$ one can find a control $\pi^h$ such that, for all stopping times $\tilde{\tau}$ in $\mathcal{T}_{t,T}$, one has

$$h \geq V^i(\tilde{t}, \tilde{x}, \tilde{f}) - EV^{\pi^h}(\tilde{\tau}, W^{\pi^h}_{\tilde{\tau}^-}, F_{\tilde{\tau}}^-)$$

We now define $\tau^h$ and $\tau^h_2$ in an obvious way, and we obtain

$$h \geq E \left[ V^i(\tau, W^{\pi^h}_{\tau^-}, F_{\tau^-}) - V^{\pi^h}(\tau, W^{\pi^h}_{\tau^-}, F_{\tau^-}) + \gamma(\tau^h - \hat{t}) \right]$$

We now define a sequence $(h_n)$ which decreases to 0, and we distinguish two cases.

On the one hand, if there exists $0 < \beta < 1$ such that, for all $n$, $P(\tau^{h_n} = \tau^h_2) \geq \beta$, then, for all $n$, $h_n \geq \gamma \beta$, and we have exhibited the desired contradiction.

On the other hand, there exists a subsequence of $(h_n)$, still denoted by $(h_n)$, such that $P(\tau^{h_n} = \tau^h_2) \leq 1/n$, then, for all $n$,

$$E(\tau^{h_n} - \hat{t}) \geq \frac{\varepsilon}{2} P(\tau^{h_n} = \tau^h_2 + \frac{\varepsilon}{2})$$

Denote by $(W^{\pi^h}_{\tau^-})$ the wealth process corresponding to the constant regime $\pi_t \equiv i$ and

$$\mathcal{E}^i := \left\{ \inf\{t \geq \hat{t}; (W^{\pi^h}_{\tau^-}, F_t) \notin B_x > \hat{t} + \frac{\varepsilon}{2} \} \right\}.$$ 

In view of (20) we have

$$h_n \geq \frac{\gamma \varepsilon}{2} P(\tau^{h_n} = \hat{t} + \frac{\varepsilon}{2})$$

$$\geq \frac{\gamma \varepsilon}{2} P \left( \left[ \tau^{h_n} > \hat{t} + \frac{\varepsilon}{2} \right] \cap \left[ \tau^h_2 > \hat{t} + \frac{\varepsilon}{2} \right] \right)$$

$$= \frac{\gamma \varepsilon}{2} P \left( \mathcal{E}^i \cap \left[ \tau^h_2 > \hat{t} + \frac{\varepsilon}{2} \right] \right),$$

Notice that the event $\mathcal{E}^i$ does not depend on $n$ and that, by hypothesis,

$$P(\tau^{h_n} > \hat{t} + \frac{\varepsilon}{2}) \geq 1 - \frac{1}{n}.$$ 

Therefore, letting $n$ tend to 0 we get $0 \geq P(\mathcal{E}^i)$, which again provides the desired contradiction.

## 7 Uniqueness of the Viscosity Solution

The aim of this section is to prove the uniqueness of a viscosity solution of HJB system (11), that is the uniqueness part of Theorem 2.5.

For technical reasons we need to distinguish the logarithmic utility case from the other cases.
7.1 Logarithmic utility function

In the logarithmic utility case we have

$$V_i(t, x, f) = \log x + \sup_{\pi \in A_t} \mathbb{E}[Z_{t,0, f, \pi} - \xi_T^{i, \pi}].$$

Set

$$\overline{V}^i(t, f) = \sup_{\pi \in A_t} \mathbb{E}[Z_{t,0, f, \pi} - \xi_T^{i, \pi}].$$

In the preceding section we have shown that the functions $\overline{V}^i$ are viscosity solutions of the system

$$\begin{align*}
\min \left\{ -\frac{\partial \overline{V}^0}{\partial t} + \mathcal{L}^0 \overline{V}^0; \overline{V}^0(t, f) - \log(1 - g_{10}) - \overline{V}^1(t, f) \right\} &= 0, \\
\min \left\{ -\frac{\partial \overline{V}^1}{\partial t} + \mathcal{L}^1 \overline{V}^1; \overline{V}^1(t, f) - \log(1 - g_{10}) - \overline{V}^0(t, f) \right\} &= 0,
\end{align*}
$$

with boundary condition $\overline{V}^0(T, f) = \overline{V}^1(T, f) = 0$, where

$$\mathcal{L}^0 \varphi(t, f) = r + (\mu_1 f + \mu_2 (1 - f)) \frac{\partial \varphi}{\partial f}(t, f) + \frac{1}{2} \left( \frac{\mu_1 - \mu_2}{\sigma} \right)^2 f^2 (1 - f)^2 \frac{\partial^2 \varphi}{\partial f^2}(t, f),$$

and

$$\mathcal{L}^1 \varphi(t, f) = \mu_1 f + \mu_2 (1 - f) - \frac{\sigma^2}{2} + (\mu_1 f + \mu_2 (1 - f)) \frac{\partial \varphi}{\partial f}(t, f)$$

$$+ \frac{1}{2} \left( \frac{\mu_1 - \mu_2}{\sigma} \right)^2 f^2 (1 - f)^2 \frac{\partial^2 \varphi}{\partial f^2}(t, f).$$

Suppose that $(\Upsilon^0, \Upsilon^1)$ and $(\psi^0, \psi^1)$ are two distinct viscosity solutions of (21) on $[0, T] \times [0, 1]$ with boundary condition $\Upsilon^i(T, f) = \psi^i(T, f) = 0$ for all $f \in [0, 1]$ and all $i \in \{0, 1\}$. Then there would exist $(i, \hat{t}, \hat{f})$ in $\{0, 1\} \times [0, T[ \times [0, 1]$ such that

$$\eta := \Upsilon^i(\hat{t}, \hat{f}) - \psi^i(\hat{t}, \hat{f}) > 0.$$

Set

$$C^* := \max_{(t, f) \in [0, T] \times [0, 1]} (|\Upsilon^1(t, f)| + |\Upsilon^2(t, f)| + |\psi^1(t, f)| + |\psi^2(t, f)|),$$

and, for all $\varepsilon > 0$,

$$\Phi^i(t, f, f') := (1 + \gamma) \Upsilon^i(t, f) - \psi^i(t, f') - \frac{1}{2\varepsilon} (|f - f'|^2) + \beta t - \lambda t,$$

where

$$0 < \lambda < \frac{\eta T}{4(T - \hat{t})}, \quad 0 < \beta < \frac{\eta}{4(T - \hat{t})}, \quad 0 < \gamma < \frac{\eta}{4C^*}.$$
Thus $\Phi^i(t, f, f')$ tends to $-\infty$ uniformly in $f, f'$ when $t$ tends to 0, from which there exists $i_\varepsilon, t_\varepsilon, f_\varepsilon, f'_\varepsilon$ in $\{0; 1\} \times [0, T] \times [0, 1]^2$ such that

$$\Phi^{i_\varepsilon}(t_\varepsilon, f_\varepsilon, f'_\varepsilon) = \sup_{(i,t,f,f') \in \{0,1\} \times [0,T] \times [0,1]^2} \Phi^i(t, f, f').$$ (22)

7.1.1 Auxiliary lemmae

**Lemma 7.1.** For all $\varepsilon > 0$ one has $0 < t_\varepsilon < T$.

**Proof.** Suppose that $t_\varepsilon = T$. Then we would have

$$\Phi^{i_\varepsilon}(T, f_\varepsilon, f'_\varepsilon) = -\frac{1}{2\varepsilon} (|f_\varepsilon - f'_\varepsilon|^2) + \beta T - \frac{\lambda}{T} \geq \Phi^i(\hat{t}, \hat{f}, \hat{f}) = \eta + \gamma \Upsilon^i(\hat{t}, \hat{f}) + \beta \hat{t} - \frac{\lambda}{T},$$

from which

$$0 \geq \eta - \gamma C^* + \beta(\hat{t} - T) - \lambda \left(\frac{1}{\hat{t}} - \frac{1}{T}\right) \geq \frac{\eta}{4} > 0,$$

which implies a contradiction. \(\square\)

**Proposition 7.2.** The function $\Upsilon^{i_\varepsilon}$ is a viscosity lower solution of

$$-\frac{\partial \Upsilon}{\partial t}(t, f) - \overline{L}^{i_\varepsilon} \Upsilon(t, f) = 0,$$

and the function $\psi^{i_\varepsilon}$ is a viscosity upper solution of

$$-\frac{\partial \psi}{\partial t}(t, f) - \overline{L}^{i_\varepsilon} \psi(t, f) = 0,$$

where $i_\varepsilon$ is defined by (22).

**Proof.** In view of (21), it suffices to prove that, for all $\varepsilon > 0$,

$$\Upsilon^{i_\varepsilon}(t_\varepsilon, f_\varepsilon) > \log(1 - g_{i_\varepsilon, j_\varepsilon}) + \Upsilon^{j_\varepsilon}(t_\varepsilon, f_\varepsilon),$$

where $i_\varepsilon + j_\varepsilon = 1$. We have

$$\Phi^{j_\varepsilon}(t_\varepsilon, f_\varepsilon, f'_\varepsilon) \leq \Phi^{i_\varepsilon}(t_\varepsilon, f_\varepsilon, f'_\varepsilon).$$

As $\Upsilon$ and $\psi$ are viscosity solutions of (21) we also have

$$\Upsilon^{i_\varepsilon}(t_\varepsilon, f_\varepsilon) \geq \log(1 - g_{i_\varepsilon, j_\varepsilon}) + \Upsilon^{j_\varepsilon}(t_\varepsilon, f_\varepsilon),$$

$$\psi^{i_\varepsilon}(t_\varepsilon, f'_\varepsilon) \geq \log(1 - g_{i_\varepsilon, j_\varepsilon}) + \psi^{j_\varepsilon}(t_\varepsilon, f'_\varepsilon).$$

Therefore, if the desired result were not true we would have

$$\log(1 - g_{i_\varepsilon, j_\varepsilon}) \leq \log(1 - g_{i_\varepsilon, j_\varepsilon})(1 + \gamma),$$

which is impossible. \(\square\)
7.1.2 Application of Ishii’s lemma

We are in a position to apply Ishii’s lemma (see theorem 8.3 in [7] for this lemma and the definition of the sets \(\overline{P}_{2}((1 + \gamma)\Upsilon_i)(t, f)\) and \(\overline{P}_{2}^{-\psi_i}(t, f')\) below).

Consider the function \(\Psi\) defined on \([0, T] \times [0, 1]^2\) as follows:

\[
\Psi(t, f, f') = \frac{1}{2\varepsilon} (|f - f'|^2) - \beta t + \lambda t.
\]

Notice that \(\Phi_i(t, f, f') = (1 + \gamma)\Upsilon_i(t, f) - \psi_i(t, f') - \Psi(t, f, f').\) For all \(\varepsilon > 0,\) Ishii’s lemma implies that there exist two real numbers \(d\) and \(d'\) and two positive numbers \(X\) and \(X'\) such that

\[
\begin{align*}
&\left( d, \frac{\partial \Psi}{\partial f}(t_\varepsilon, f_\varepsilon, f'_\varepsilon), X \right) \in \overline{P}_{2}((1 + \gamma)\Upsilon_i)(t_\varepsilon, f_\varepsilon), \\
&\left( d', \frac{\partial \Psi}{\partial f'}(t_\varepsilon, f_\varepsilon, f'_\varepsilon), X' \right) \in \overline{P}_{2}^{-\psi_i}(t_\varepsilon, f'_\varepsilon), \\
&d + d' = \frac{\partial \Psi}{\partial t}(t_\varepsilon, f_\varepsilon, f'_\varepsilon), \\
&-\left( \frac{1}{\varepsilon} + \|A\| \right) I \leq \begin{pmatrix} X & 0 \\ 0 & X' \end{pmatrix} \leq A + \varepsilon A^2,
\end{align*}
\]

where \(A\) is the Hessian matrix of \(\Psi\) in \((f, f'),\) that is,

\[
A = \begin{pmatrix}
\frac{1}{\varepsilon} & -\frac{1}{\varepsilon} \\
-\frac{1}{\varepsilon} & \frac{1}{\varepsilon}
\end{pmatrix}.
\]

We now use the proposition 7.2. We have:

\[
-d - \left( \varepsilon (\mu_1 f_\varepsilon + \mu_2 (1 - f_\varepsilon) - r - \frac{\sigma^2}{2}) + r \right) - (-\lambda_1 f_\varepsilon + \lambda_2 (1 - f_\varepsilon)) \frac{\partial \Psi}{\partial f}(t_\varepsilon, f_\varepsilon, f'_\varepsilon)
\]

\[
-\frac{1}{2} \left( \frac{\mu_1 - \mu_2}{\sigma} \right)^2 f'_\varepsilon^2 (1 - f_\varepsilon)^2 X
\]

\[
\leq 0
\]

\[
\leq d' - \left( \varepsilon (\mu_1 f'_\varepsilon + \mu_2 (1 - f'_\varepsilon) - r - \frac{\sigma^2}{2}) + r \right) \\
+ (-\lambda_1 f'_\varepsilon + \lambda_2 (1 - f'_\varepsilon)) \frac{\partial \Psi}{\partial f'}(t_\varepsilon, f_\varepsilon, f'_\varepsilon) + \frac{1}{2} \left( \frac{\mu_1 - \mu_2}{\sigma} \right)^2 (f'_\varepsilon)^2 (1 - f'_\varepsilon)^2 X'.
\]
In view of the condition on $d + d'$ we deduce
\[
\beta + \frac{\lambda}{\varepsilon^2} \leq i\varepsilon(\mu_1 - \mu_2)(f_\varepsilon - f'_\varepsilon)
\]
\[
+ (-\lambda_1 f_\varepsilon + \lambda_2(1 - f_\varepsilon)) \frac{1}{\varepsilon}(f_\varepsilon - f'_\varepsilon) + (-\lambda_1 f'_\varepsilon + \lambda_2(1 - f'_\varepsilon)) \frac{1}{\varepsilon}(f'_\varepsilon - f_\varepsilon)
\]
\[
+ \left(\frac{\mu_1 - \mu_2}{\sigma} f_\varepsilon(1 - f_\varepsilon)\right)^t \begin{pmatrix} X & 0 \\ 0 & X' \end{pmatrix} \left(\frac{\mu_1 - \mu_2}{\sigma} f_\varepsilon(1 - f_\varepsilon)\right).
\]

Notice that
\[
A + \varepsilon A^2 = \begin{pmatrix} \frac{3}{\varepsilon} & -\frac{3}{\varepsilon} \\ -\frac{3}{\varepsilon} & \frac{3}{\varepsilon} \end{pmatrix}
\]

Therefore
\[
\beta + \frac{\lambda}{\varepsilon^2} \leq i\varepsilon(\mu_1 f_\varepsilon + \mu_2(1 - f_\varepsilon) - r - \frac{\sigma^2}{2}) + r + i\varepsilon(\mu_1 f'_\varepsilon + \mu_2(1 - f'_\varepsilon) - r - \frac{\sigma^2}{2}) + r
\]
\[
+ (-\lambda_1 f_\varepsilon + \lambda_2(1 - f_\varepsilon)) \frac{1}{\varepsilon}(f_\varepsilon - f'_\varepsilon) + (-\lambda_1 f'_\varepsilon + \lambda_2(1 - f'_\varepsilon)) \frac{1}{\varepsilon}(f'_\varepsilon - f_\varepsilon)
\]
\[
+ \frac{3}{\varepsilon} \left(\frac{\mu_1 - \mu_2}{\sigma}\right)^2 (f_\varepsilon(1 - f_\varepsilon) - f'_\varepsilon(1 - f'_\varepsilon))^2
\]
\[
=: K_1 + \frac{K_2}{\varepsilon}.
\]

We first estimate $K_2$:
\[
K_2 \leq - (\lambda_1 + \lambda_2)(f_\varepsilon - f'_\varepsilon)^2 + 3 \left(\frac{\mu_1 - \mu_2}{\sigma}\right)^2 (f_\varepsilon - f'_\varepsilon)^2(1 + f_\varepsilon + f'_\varepsilon)^2
\]
\[
\leq 27 \left(\frac{\mu_1 - \mu_2}{\sigma}\right)^2 (f_\varepsilon - f'_\varepsilon)^2,
\]
from which, owing to the lemma 7.3 below,
\[
\frac{K_2}{\varepsilon} \xrightarrow{\varepsilon \to 0} 0.
\]

**Lemma 7.3.** One has
\[
\lim_{\varepsilon \to 0} \frac{(f_\varepsilon - f'_\varepsilon)^2}{\varepsilon} = 0.
\]

**Proof.** For all $i, t, f, f'$ in $\{0; 1\} \times [0, T] \times [0, 1]^2$, by definition of the constant $C^*$ one has
\[
\Phi^i(t, f, f') \leq 2C^* - \frac{1}{2\varepsilon}(|f - f'|^2) + \beta T.
\]

Set
\[
H := \Phi^i(\hat{t}, \hat{f}, \hat{f}) = \eta + \gamma Y^i(\hat{t}, \hat{f}) + \beta \hat{t} - \frac{\lambda}{\hat{t}}.
\]
If $|f - f'|^2 \geq 2\varepsilon(2C^* + \beta T - H + 1)$, one has $\Phi^i(t, f, f') \leq H - 1$; therefore

$$|f_\varepsilon - f'_\varepsilon|^2 \leq 2\varepsilon(2C^* + \beta T - H + 1).$$

In particular, $|f_\varepsilon - f'_\varepsilon|$ tends to 0 with $\varepsilon$. One also has

$$2\Phi^i(t, f_\varepsilon, f'_\varepsilon) \geq \Phi^i(t, f_\varepsilon, f_\varepsilon) + \Phi^i(t, f'_\varepsilon, f'_\varepsilon),$$

and thus

$$\frac{1}{2\varepsilon}(f_\varepsilon - f'_\varepsilon)^2 \leq \Upsilon^i(t, f_\varepsilon) - \Upsilon^i(t, f'_\varepsilon) + \psi^i(t, f'_\varepsilon) - \psi^i(t, f_\varepsilon).$$

The right-hand side tends to 0 with $\varepsilon$ since $\Upsilon$ and $\psi$ are uniformly continuous on $[0, 1]$, and since $|f_\varepsilon - f'_\varepsilon|$ tends to 0 with $\varepsilon$.

It remains to estimate $K_1$. Notice that

$$K_1 = i_\varepsilon(\mu_1 - \mu_2)(f_\varepsilon - f'_\varepsilon),$$

and thus tends to 0 with $\varepsilon$. When $\varepsilon$ tends to 0, one obtains $\beta \leq 0$. We thus have exhibited a contradiction, and proven the uniqueness of the viscosity solution.

### 7.2 Utility function in the class $\mathcal{U}$

In this subsection, we consider the case where the utility function $U$ belongs to the class $\mathcal{U}$.

Suppose that $\Upsilon$ and $\psi$ are two viscosity solutions of (11) in $\mathcal{V}_\alpha$. Then there would exist $(i, t, \dot{x}, f)$ in $\{0, 1\} \times [0, T] \times [0, +\infty) \times [0, 1]$ such that

$$\eta := \Upsilon^i(t, \dot{x}, f) - \psi^i(t, \dot{x}, f) > 0.$$

As $\Upsilon$ and $\psi$ are null and continuous at $x = 0$, there also would exist $m > 0$ such that, for all $i, t, f$ and $x, x' \leq m$,

$$|\Upsilon^i(t, x, f) - \psi^i(t, x', f')| < \frac{\eta}{5}.$$  \hfill (23)

In addition, as $\Upsilon$ and $\psi$ are in $\mathcal{V}_\alpha$, there exists $C > 0$ such that, for all $i, t, x, f$,

$$|\Upsilon^i(t, x, f)| + |\psi^i(t, x, f)| \leq C(1 + x).$$  \hfill (24)

Define the functions $\Phi^0$ and $\Phi^1$ on $[0, T] \times [0, +\infty)^2 \times [0, 1]^2$ by

$$\Phi^i(t, x, x', f, f') := \Upsilon^i(t, x, f) - \psi^i(t, x', f') - \nu e^{-D\tau}(x^2 + x'^2) - \frac{1}{2\varepsilon}(|x - x'|^2 + |f - f'|^2) + \beta t - \frac{\lambda}{\varepsilon},$$

where, $C$ being as in (24),

$$\varepsilon > 0, \ D > (|\mu_1| + |\mu_2| + r + 3\sigma^2); \ 0 < \lambda < \frac{\eta T}{5(T - t)}; \ 0 < \beta < \frac{\eta}{5(T - t)}; \ 0 < \nu < \min \left\{ C e^{\alpha D}, \frac{\eta e D t}{10 x^2} \right\}.$$  \hfill (25)
Set
\[ H := \Phi^i(\hat{t}, \hat{x}, \hat{t}, \hat{f}) = \eta - 2\nu e^{-DT}\hat{x}^2 + \beta\hat{t} - \frac{\lambda}{\hat{t}}. \]

### 7.2.1 Auxiliary lemmae

**Lemma 7.4.** Set
\[ M := \frac{e^{DT}}{2\nu} (C + \sqrt{C^2 + 4\nu e^{-DT}(2C + \beta T + C^2\nu^{-1}e^{DT} + 1 - H)}). \]

If \( x > M \) or \( x' > M \) then, for all \( i, t, x', f, f' \),
\[ \Phi^i(t, x', f, f') \leq H - 1 < \Phi^i(\hat{t}, \hat{x}, \hat{t}, \hat{f}). \]

**Proof.** For all \( i, t, x', f, f' \) one has
\[ \Phi^i(t, x', f, f') \leq 2C + \beta T + C(x + x') - \nu e^{-DT}(x^2 + x'^2). \]

On the one hand, if \( x' \leq C\nu^{-1}e^{DT} \) then
\[ \Phi^i(t, x', f, f') \leq 2C + \beta T + C^2\nu^{-1}e^{DT} + Cx - \nu e^{-DT}x^2. \]

Therefore \( \Phi^i(t, x, x', f, f') \leq H - 1 \) when \( x \geq M \).

On the other hand, if \( x' > C\nu^{-1}e^{DT} \), then \( Cx' - \nu e^{-DT}x'^2 < 0 \) and
\[ \Phi^i(t, x', f, f') \leq 2C + \beta T + Cx - \nu e^{-DT}x^2 \leq 2C + \beta T + C^2\nu^{-1}e^{DT} + Cx - \nu e^{-DT}x^2. \]

Therefore we also have \( \Phi^i(t, x, x', f, f') \leq H - 1 \) when \( x \geq M \). \( \square \)

As \( \Phi^0 \) and \( \Phi^1 \) are continuous, we deduce that there exists \( (i_e, t_e, x_e, x'_e, f_e, f'_e) \) in \( \{0, 1\} \times [0, T] \times [0, M]^2 \times [0, 1]^2 \) such that
\[ \Phi^{i_e}(t_e, x_e, x'_e, f_e, f'_e) \geq \Phi^i(t, x, x', f, f') \]
for all \( i, t, x, x', f, f' \) in \( \{0, 1\} \times [0, T] \times [0, +\infty)^2 \times [0, 1]^2 \).

**Lemma 7.5.** One has
\[ |x_e - x'_e|^2 + |f_e - f'_e|^2 \leq 2\varepsilon(2C(1 + M) + \beta T - H + 1). \]

**Proof.** The constant \( C \) being defined as in (24), for all \( i, t, x, x', f, f' \) in \( \{0, 1\} \times [0, T] \times [0, M]^2 \times [0, 1]^2 \),
\[ \Phi^i(t, x', f, f') \leq 2C + 2CM - \frac{1}{2\varepsilon}(|x - x'|^2 + |f - f'|^2) + \beta T. \]

Therefore, if
\[ |x - x'|^2 + |f - f'|^2 \geq 2\varepsilon(2C(1 + M) + \beta T - H + 1) \]
then \( \Phi^i(t, x, x', f, f') \leq H - 1 \), from which the result follows. \( \square \)
The preceding lemma implies that \(|x_\varepsilon - x'_\varepsilon|^2 + |f_\varepsilon - f'_\varepsilon|^2|\) tends to 0 with \(\varepsilon\).

We now prove that \(x_\varepsilon\) and \(x'_\varepsilon\) cannot be in a small neighborhood of 0.

**Lemma 7.6.** For all \(\varepsilon\) small enough, \(x_\varepsilon \geq \frac{m}{2}\) and \(x'_\varepsilon \geq \frac{m}{2}\), where \(m\) is defined by (23)

**Proof.** Suppose that \(x_\varepsilon < \frac{m}{2}\) or \(x'_\varepsilon < \frac{m}{2}\). In view of the preceding lemma, there exists \(\varepsilon_0 > 0\) such that, for all \(0 < \varepsilon < \varepsilon_0\), \(|x_\varepsilon - x'_\varepsilon| < \frac{m}{2}\). Therefore \(x_\varepsilon \leq m\) and \(x'_\varepsilon \leq m\). By definition of \(m\), \(\eta\), \(\beta\) and \(\lambda\), one thus has

\[
\Phi^{i\varepsilon}(t, x_\varepsilon, x'_\varepsilon, f_\varepsilon, f'_\varepsilon) = \Upsilon^{i\varepsilon}(t, x_\varepsilon, f_\varepsilon) - \psi^{i\varepsilon}(t_\varepsilon, x'_\varepsilon, f'_\varepsilon) - \nu e^{-Dt_\varepsilon}(x_\varepsilon^2 + (x'_\varepsilon)^2)
\]

\[
- \frac{1}{2\varepsilon}(|x_\varepsilon - x'_\varepsilon|^2 + |f_\varepsilon - f'_\varepsilon|^2) + \beta t_\varepsilon - \frac{\lambda}{t_\varepsilon}
\]

\[
\leq \frac{\eta}{5} + \Phi^{i\varepsilon}(t, \hat{x}, \hat{x}, \hat{f}) - \eta + 2\nu e^{-D\varepsilon\hat{x}^2} - \beta(\hat{t} - T) + \frac{\lambda}{T} - \frac{\lambda}{T}
\]

\[
\leq \Phi^{i\varepsilon}(t, \hat{x}, \hat{x}, \hat{f}) - \frac{\eta}{5},
\]

which is impossible.

\[\square\]

**Lemma 7.7.** One has \(0 < t_\varepsilon < T\).

**Proof.** In view of the definition of \(\Phi\) and the preceding lemma, for all \(i, t, x, x', f, f'\) in \(\{0; 1\} \times [0, T] \times [m/2, M]^2 \times [0, 1]^2\) one has

\[
\Phi^{i}(t, x, x', f, f') \leq 2C + 2CM + \beta T - \frac{\lambda}{T}
\]

Therefore, if \(t \leq \lambda(2C(1 + M) + \beta T - H + 1)^{-1}\), then \(\Phi^{i}(t, x, x', f, f') \leq H - 1\), which implies that \(t_\varepsilon > 0\). Suppose that \(t_\varepsilon = T\). Then

\[
\Phi^{i\varepsilon}(T, x_\varepsilon, x'_\varepsilon, f_\varepsilon, f'_\varepsilon) = U(x_\varepsilon) - U(x'_\varepsilon) - \nu e^{-D\varepsilon T}(x_\varepsilon^2 + (x'_\varepsilon)^2)
\]

\[
- \frac{1}{2\varepsilon}(|x_\varepsilon - x'_\varepsilon|^2 + |f_\varepsilon - f'_\varepsilon|^2) + \beta T - \frac{\lambda}{T}
\]

\[
\geq \Phi^{i\varepsilon}(t, \hat{x}, \hat{x}, \hat{f})
\]

\[
= \eta - 2\nu e^{-D\varepsilon\hat{x}^2} + \beta \hat{t} - \frac{\lambda}{T}.
\]

Our choice of \(\nu\), \(\beta\), and \(\lambda\) (25) implies

\[
U'(m/2)|x_\varepsilon - x'_\varepsilon| \geq U(x_\varepsilon) - U(x'_\varepsilon)
\]

\[
\geq \eta - 2\nu e^{-D\varepsilon\hat{x}^2} + \beta (\hat{t} - T) - \lambda \left(\frac{1}{T} - \frac{1}{T}\right)
\]

\[
\geq \frac{2\eta}{5}.
\]

Thanks to lemma 7.5, we now choose \(\varepsilon\) small enough in order that \(U'(m/2)|x_\varepsilon - x'_\varepsilon| \leq \frac{\eta}{5}\) and obtain a contradiction.

\[\square\]
Lemma 7.8. One has

\[ \Upsilon^{i_e}(t_e, x_e, f_e) > \Upsilon^{j_e}(t_e, x_e(1 - g_{i_e,j_e}), f_e). \]

Proof. We already know:

\[ \Upsilon^{i_e}(t_e, x_e, f_e) \geq \Upsilon^{j_e}(t_e, x_e(1 - g_{i_e,j_e}), f_e), \]

\[ \psi^{i_e}(t_e, x'_e, f'_e) \geq \psi^{j_e}(t_e, x'_e(1 - g_{i_e,j_e}), f'_e), \]

where \( i_e + j_e = 1 \). By definition of \((i_e, t_e, x_e, x'_e, f_e, f'_e)\), we have

\[ \Phi^{j_e}(t_e, x_e(1 - g_{i_e,j_e}), x'_e(1 - g_{i_e,j_e}), f_e, f'_e) < \Phi^{i_e}(t_e, x_e, x'_e, f_e, f'_e). \]

Suppose that the desired result does not hold true. Then we would have

\[ 0 \leq ve^{-Dt_e}(x_e^2 + (x'_e)^2)((1 - g_{i_e,j_e})^2 - 1) + \frac{1}{2\epsilon}|x_e - x'_e|^2((1 - g_{i_e,j_e})^2 - 1) < 0, \]

which is impossible since \( x_e > 0, x'_e > 0, g_{10} \) and \( g_{01} \) are in \((0, 1)\). \( \square \)

Notice that the preceding lemma implies that

\[ \left\{ \begin{array}{l}
- \frac{\partial \Upsilon^{i_e}}{\partial t}(t_e, x_e, f_e) - L_i \Upsilon^{i_e}(t_e, x_e, f_e) = 0,
- \frac{\partial \psi^{i_e}}{\partial t}(t_e, x'_e, f'_e) - L_i \psi^{i_e}(t_e, x'_e, f'_e) \geq 0.
\end{array} \right. \]

### 7.2.2 Application of Ishii’s lemma

Define the function \( \Psi \) on \([0, T] \times [0, +\infty)^2 \times [0, 1]^2 \) as follows:

\[ \Psi(t, x, x', f, f') = ve^{-D(t)(x^2 + x'^2)} + \frac{1}{2\epsilon}|x - x'|^2 + |f - f'|^2 - \beta t + \frac{\lambda}{t}. \]

Notice that \( \Phi'(t, x, x', f, f') = \Upsilon'(t, x, f) - \psi'(t, x', f') - \Psi(t, x, x', f, f') \). For all \( \epsilon > 0 \), Ishii’s lemma implies that there exist two real numbers \( d \) and \( d' \) and two symmetric matrices \( X \) and \( X' \) such that

\[ \left( d, \left( \frac{\partial \Psi}{\partial x} \right) \left( t_e, x_e, x'_e, f_e, f'_e \right), X \right) \in \overline{P} \Upsilon^{i_e}(t_e, x_e, f_e), \]

\[ - \left( d', \left( \frac{\partial \Psi}{\partial x'} \right) \left( t_e, x_e, x'_e, f_e, f'_e \right), X' \right) \in \overline{P} \psi^{i_e}(t_e, x'_e, f'_e), \]

\[ d + d' = \frac{\partial \Psi}{\partial t}(t_e, x_e, x'_e, f_e, f'_e), \]

\[ - \left( \frac{1}{\epsilon} + \|A\| \right) I \leq \left( \begin{array}{cc} X & 0 \\ 0 & X' \end{array} \right) \leq A + \epsilon A^2, \]

\[ 25 \]
where $A$ is the Hessian matrix of $\Phi$ in $x, x', f, f'$, that is,

$$
A := \begin{pmatrix}
  a + \frac{1}{\varepsilon} & 0 & -\frac{1}{\varepsilon} & 0 \\
  0 & 1 & 0 & -\frac{1}{\varepsilon} \\
  -\frac{1}{\varepsilon} & 0 & a + \frac{1}{\varepsilon} & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 \\
  0 & 0 & a & 0 \\
  0 & 0 & 0 & 0
\end{pmatrix} + \frac{1}{\varepsilon} \begin{pmatrix}
  1 & 0 & -1 & 0 \\
  0 & 1 & 0 & -1 \\
  -1 & 0 & 1 & 0 \\
  0 & -1 & 0 & 1
\end{pmatrix},
$$

where $a := 2\nu e^{-D\varepsilon}$.

We now use that we deal with viscosity solutions and get

$$
-d - x\varepsilon (i\varepsilon (\mu_1 f + \mu_2 (1 - f)) - r) + r) \frac{\partial \Psi}{\partial x}(t\varepsilon, x\varepsilon, x', f, f')
$$

$$
- (-\lambda_1 f + \lambda_2 (1 - f)) \frac{\partial \Psi}{\partial f}(t\varepsilon, x\varepsilon, x', f, f') - \frac{i\varepsilon}{2} x^2 \sigma^2 X_{11}
$$

$$
- \frac{1}{2} \left( \frac{\mu_1 - \mu_2}{\sigma} \right)^2 f^2 (1 - f)^2 X_{22}
$$

$$
- i\varepsilon x\varepsilon (\mu_1 - \mu_2) f (1 - f) X_{12}
$$

$$
\leq 0
$$

$$
\leq d' + x'\varepsilon (i\varepsilon (\mu_1 f' + \mu_2 (1 - f')) - r) + r) \frac{\partial \Psi}{\partial x'}(t\varepsilon, x\varepsilon, x', f, f')
$$

$$
+ (-\lambda_1 f' + \lambda_2 (1 - f')) \frac{\partial \Psi}{\partial f'}(t\varepsilon, x\varepsilon, x', f, f') + \frac{i\varepsilon}{2} (x')^2 \sigma^2 X_{11}^1
$$

$$
+ \frac{1}{2} \left( \frac{\mu_1 - \mu_2}{\sigma} \right)^2 (f')^2 (1 - f')^2 X_{22} + i\varepsilon x'\varepsilon (\mu_1 - \mu_2) f' (1 - f') X_{12}.
$$

In view of the condition on $d + d'$ we deduce

$$
D e^{-D\varepsilon} \nu (x^2 + (x')^2) + \beta + \lambda \frac{\varepsilon}{\varepsilon}
$$

$$
\leq x\varepsilon (i\varepsilon (\mu_1 f + \mu_2 (1 - f)) - r) + r) \left( 2\nu e^{-D\varepsilon} x + \frac{1}{\varepsilon} (x - x') \right)
$$

$$
+ x'\varepsilon (i\varepsilon (\mu_1 f' + \mu_2 (1 - f')) - r) + r) \left( 2\nu e^{-D\varepsilon} x' + \frac{1}{\varepsilon} (x' - x) \right)
$$

$$
+ (-\lambda_1 f + \lambda_2 (1 - f)) \left( \frac{1}{\varepsilon} (f - f') \right) + (-\lambda_1 f' + \lambda_2 (1 - f')) \left( \frac{1}{\varepsilon} (f' - f) \right)
$$

$$
+ \left( \begin{pmatrix}
  \frac{i\varepsilon}{\sqrt{2}} x\varepsilon \sigma \\
  \frac{\mu_1 - \mu_2}{\sigma} f (1 - f) \\
  \frac{i\varepsilon}{\sqrt{2}} x'\varepsilon \sigma \\
  \frac{\mu_1 - \mu_2}{\sigma} f' (1 - f')
\end{pmatrix} \right)^t \begin{pmatrix}
  X & 0 \\
  0 & X'
\end{pmatrix} \begin{pmatrix}
  \frac{i\varepsilon}{\sqrt{2}} x\varepsilon \sigma \\
  \frac{\mu_1 - \mu_2}{\sigma} f (1 - f) \\
  \frac{i\varepsilon}{\sqrt{2}} x'\varepsilon \sigma \\
  \frac{\mu_1 - \mu_2}{\sigma} f' (1 - f')
\end{pmatrix}.
$$
Notice that
\[
A + \varepsilon A^2 = \begin{pmatrix}
3a + \varepsilon a^2 + \frac{3}{\varepsilon} & 0 & -2a - \frac{3}{\varepsilon} & 0 \\
0 & \frac{3}{\varepsilon} & 0 & -\frac{3}{\varepsilon} \\
-2a - \frac{3}{\varepsilon} & 0 & 3a + \varepsilon a^2 + \frac{3}{\varepsilon} & 0 \\
0 & -\frac{3}{\varepsilon} & 0 & \frac{3}{\varepsilon}
\end{pmatrix}.
\]

In addition,
\[
\beta + \frac{\lambda t}{\varepsilon} \leq \frac{1}{\varepsilon^2} \left[(x_\varepsilon - x_\varepsilon')x_\varepsilon (i_\varepsilon (\mu_1 f_\varepsilon + \mu_2 (1 - f_\varepsilon)) - r) + r - x_\varepsilon' (i_\varepsilon (\mu_1 f_\varepsilon' + \mu_2 (1 - f_\varepsilon') - r) + r) + 3i_\varepsilon \frac{\sigma^2}{2} (x_\varepsilon - x_\varepsilon')^2 \\
-(\lambda_1 + \lambda_2) (f_\varepsilon - f_\varepsilon')^2 + 3 \left(\frac{\mu_1 - \mu_2}{\sigma}\right)^2 (f_\varepsilon (1 - f_\varepsilon) - f_\varepsilon' (1 - f_\varepsilon'))^2 \right] \\
+ \varepsilon i_\varepsilon a^2 \frac{\sigma^2}{2} (x_\varepsilon^2 + (x_\varepsilon')^2) \\
+ \nu \left(-De^{-Dt_\varepsilon} (x_\varepsilon^2 + (x_\varepsilon')^2) + 2e^{-Dt_\varepsilon} \right) \left(x_\varepsilon^2 (i_\varepsilon (\mu_1 f_\varepsilon + \mu_2 (1 - f_\varepsilon)) - r) + r + (x_\varepsilon')^2 (i_\varepsilon (\mu_1 f_\varepsilon' + \mu_2 (1 - f_\varepsilon') - r) + r)\right)\right) \\
+ i_\varepsilon a \frac{\sigma^2}{2} (3x_\varepsilon^2 + 3(x_\varepsilon')^2 - 4x_\varepsilon x_\varepsilon') \\
= \frac{K_1}{\varepsilon} + \varepsilon K_2 + K_3.
\]

We now estimate $K_3$. As $a > 0$,
\[
K_3 \leq e^{-Dt_\varepsilon} \nu (x_\varepsilon^2 (-D + i_\varepsilon (\mu_1 f_\varepsilon + \mu_2 (1 - f_\varepsilon)) - r + 3\sigma^2) + r) \\
+(x_\varepsilon')^2 (-D + i_\varepsilon (\mu_1 f_\varepsilon' + \mu_2 (1 - f_\varepsilon') - r + 3\sigma^2) + r)) \\
\leq e^{-Dt_\varepsilon} \nu (x_\varepsilon^2 + (x_\varepsilon')^2) (-D + |\mu_1| + |\mu_2| + r + 3\sigma^2).
\]

As $D > |\mu_1| + |\mu_2| + r + 3\sigma$, we obtain $K_3 \leq 0$.

Notice that
\[
K_2 \leq 4i_\varepsilon \sigma^2 \nu^2 M^2,
\]
from which, since $M$ and $\nu$ do not depend on $\varepsilon$,
\[
\varepsilon K_2 \xrightarrow{\varepsilon \to 0} 0.
\]
It remains to estimate $K_1$. One has

$$K_1 \leq (x_\varepsilon - x'_\varepsilon)((x_\varepsilon - x'_\varepsilon)( |\mu_1| + |\mu_2| + r) + x'_\varepsilon(\mu_1 - \mu_2)(f_\varepsilon - f'_\varepsilon))$$

$$+ 3i\varepsilon \frac{\sigma^2}{2}(x_\varepsilon - x'_\varepsilon)^2 + 12 \left(\frac{\mu_1 - \mu_2}{\sigma}\right)^2 (f_\varepsilon - f'_\varepsilon)^2$$

$$\leq (x_\varepsilon - x'_\varepsilon)^2 \left(3i\varepsilon \frac{\sigma^2}{2} + |\mu_1| + |\mu_2| + r\right) + x'_\varepsilon(\mu_1 - \mu_2)(f_\varepsilon - f'_\varepsilon)(x_\varepsilon - x'_\varepsilon)$$

$$+ 12 \left(\frac{\mu_1 - \mu_2}{\sigma}\right)^2 (f_\varepsilon - f'_\varepsilon)^2$$

$$\leq (x_\varepsilon - x'_\varepsilon)^2 \left(3i\varepsilon \frac{\sigma^2}{2} + |\mu_1| + |\mu_2| + r\right)$$

$$+ \frac{M}{2} |\mu_1 - \mu_2|((f_\varepsilon - f'_\varepsilon)^2 + (x_\varepsilon - x'_\varepsilon)^2) + 12 \left(\frac{\mu_1 - \mu_2}{\sigma}\right)^2 (f_\varepsilon - f'_\varepsilon)^2$$

$$\leq (x_\varepsilon - x'_\varepsilon)^2 \left(3i\varepsilon \frac{\sigma^2}{2} + |\mu_1| + |\mu_2| + r + \frac{M}{2} |\mu_1 - \mu_2|\right)$$

$$+ 12 \left(\frac{\mu_1 - \mu_2}{\sigma} + \frac{M}{2} |\mu_1 - \mu_2|\right)^2 (f_\varepsilon - f'_\varepsilon)^2.$$

Thus, in view of the lemma 7.9 below,

$$\frac{K_1}{\varepsilon} \xrightarrow{\varepsilon \to 0} 0.$$

As $\varepsilon$ tends to 0, we obtain $\beta \leq 0$, which exhibits a contradiction.

**Lemma 7.9.** One has

$$\lim_{\varepsilon \to 0} \frac{(x_\varepsilon - x'_\varepsilon)^2 + (f_\varepsilon - f'_\varepsilon)^2}{\varepsilon} = 0.$$

**Proof.** Observe that

$$2\Phi^{1s}(t_\varepsilon, x_\varepsilon, x'_\varepsilon, f_\varepsilon, f'_\varepsilon) \geq \Phi^{1s}(t_\varepsilon, x_\varepsilon, x'_\varepsilon, f_\varepsilon, f'_\varepsilon) + \Phi^{1s}(t_\varepsilon, x'_\varepsilon, f'_\varepsilon).$$

As the function $\Upsilon$ and $\psi$ are in $\mathcal{V}_\alpha$, in view of lemma 7.5,

$$\frac{1}{2\varepsilon} \left( (x_\varepsilon - x'_\varepsilon)^2 + (f_\varepsilon - f'_\varepsilon)^2 \right)$$

$$\leq \Upsilon^{1s}(t_\varepsilon, x_\varepsilon, f_\varepsilon) - \Upsilon^{1s}(t_\varepsilon, x'_\varepsilon, f'_\varepsilon) + \psi^{1s}(t_\varepsilon, x'_\varepsilon, f'_\varepsilon) - \psi^{1s}(t_\varepsilon, x_\varepsilon, f_\varepsilon)$$

$$\leq K(1 + x_{\varepsilon}^{-\alpha} + (x'_\varepsilon)^{-\alpha})(|x_\varepsilon - x'_\varepsilon| + (x_\varepsilon + x'_\varepsilon)|f_\varepsilon - f'_\varepsilon|)$$

$$\leq K(1 + 2(m/2)^{-\alpha} + 2M)2\varepsilon(2C(1 + M) + \beta T - H + 1).$$

The desired result follows. ☐
8 Numerical illustrations

8.1 Numerical scheme

The characterization of \((V^0, V^1)\) as the unique solution of (11) enables us to propose a numerical scheme to approximate the value functions. We only consider here the special case where the utility is a power function: \(U(x) = x^\alpha\) with \(0 < \alpha < 1\). Hence, for all \(t, x, f, i\) and \(\pi\) one has \(U(W^t, x, f, \pi, \pi) = U(x)U(W^t, f, i, \pi)\), and therefore, for \(i \in \{0, 1\}, V^i(t, x, f) = U(x)V^i(t, 1, f)\). With a slight abuse of notation, set \(V^i(t, f) = V^i(t, 1, f)\). The system (11) can now be simplified as follows:

\[
\begin{align*}
\min \left\{ - \frac{\partial U^0}{\partial f}(t, f) - \mathcal{L}^0 V^0(t, f); \ V^0(t, f) - (1-g_0^0)\alpha V^1(t, f) \right\} &= 0 \\
\min \left\{ - \frac{\partial U^1}{\partial f}(t, f) - \mathcal{L}^1 V^1(t, f); \ V^1(t, f) - (1-g_1^0)\alpha V^0(t, f) \right\} &= 0
\end{align*}
\]

with the boundary conditions \(V^0(T, f) = 1\) and \(V^1(T, f) = 1\) for all \(0 \leq f \leq 1\). Here, we have for all \(0 \leq t \leq T, 0 \leq f \leq 1\) and \(i \in \{0, 1\}\)

\[\mathcal{L}^i \varphi(t, f) = c(f, i)\varphi(t, f) + b(f, i)\frac{\partial \varphi}{\partial f}(t, f) + a(f)\frac{\partial^2 \varphi}{\partial f^2}(t, f),\]

with

\[
\begin{align*}
a(f) &= \frac{1}{2} \left( \frac{\mu_1 - \mu_2}{\sigma} \right)^2 f^2 (1 - f)^2, \\
b(f, 0) &= -\lambda_1 f + \lambda_2 (1 - f), \\
b(f, 1) &= -\lambda_1 f + \lambda_2 (1 - f) + \alpha (\mu_1 - \mu_2) f (1 - f), \\
c(f, 0) &= \alpha r, \\
c(f, 1) &= \alpha \left( \mu_1 f + \mu_2 (1 - f) - (1 - \alpha)\frac{\sigma^2}{2} \right).
\end{align*}
\]

For a time discretization step \(\delta t\) and a space discretization step \(\delta f\), set:

\[
S^i \varphi(t, f) = \frac{\varphi(t, f) - \varphi(t - \delta t, f)}{\delta t} + \hat{\mathcal{L}}^i \varphi(t, f)
\]

\[
= \frac{\varphi(t, f) - \varphi(t - \delta t, f)}{\delta t} + c(f, i)\varphi(t, f) + b(f, i)^+ \frac{\varphi(t, f + \delta f) - \varphi(t, f)}{\delta f} \frac{\varphi(t, f) - \varphi(t, f - \delta f)}{\delta f} + a(f) \varphi(t, f + \delta f) - 2 \varphi(t, f) + \varphi(t, f - \delta f) \frac{\varphi(t, f + \delta f) - \varphi(t, f - \delta f)}{\delta f^2},
\]

where \(x^+ = \frac{|x|+x}{2}\) and \(x^- = \frac{|x|+x}{2}\). Note that the first-order term in \(f\) depends on the sign of \(b\), and the second-order term \(a(f)\) is positive except at the boundaries \(f = 0\) and \(f = 1\).

We construct the approximation \((\hat{V}^0, \hat{V}^1)\) of the value functions \((V^0, V^1)\) as follows:
• Set $\hat{V}^0(T, \cdot) = \hat{V}^1(T, \cdot) = 1$.

• At each time step:
  
  - set $\hat{V}^0(t, \cdot) = \hat{V}^0(t, \cdot) = \hat{V}^1(t, \cdot)$.
  
  - compute $\hat{V}^i(t - \delta t, f)$ solution of $S^i\hat{V}^i(t, f) = 0$.
  
  - set $\hat{V}^0(t - \delta t, f) = \max\{\hat{V}^0(t - \delta t, f); (1 - g_{10})^a\hat{V}^1(t - \delta t, f)\}$, and $\hat{V}^1(t - \delta t, f) = \max\{\hat{V}^1(t - \delta t, f); (1 - g_{10})^a\hat{V}^0(t - \delta t, f)\}$.

One can easily show by induction the following result.

**Lemma 8.1.** $(\hat{V}^0, \hat{V}^1)$ is the unique solution of the system

\[
\begin{cases}
\min \{-S^0\varphi^0(t, f); \varphi^0(t - \delta t, f) - (1 - g_{10})^a\varphi^1(t - \delta t, f)\} = 0, \\
\min \{-S^1\varphi^1(t, f); \varphi^1(t - \delta t, f) - (1 - g_{10})^a\varphi^0(t - \delta t, f)\} = 0, \\
\hat{V}^0(T, \cdot) = \hat{V}^1(T, \cdot) = 1.
\end{cases}
\]

**Remark 8.2.** So far, we have not proven the convergence of this scheme to the value functions $(\hat{V}^0, \hat{V}^1)$. Key results in that direction are in [10, 2, 5].

### 8.2 Approximate value function

We implemented the above-mentioned numerical scheme for $U(x) = \sqrt{x}$ ($\alpha = 1/2$), $T = 3$, $\mu_1 = -0.2$, $\mu_2 = 0.21$, $\lambda_1 = \lambda_2 = 2$, $\sigma = 0.15$, $g_{10} = g_{10} = 0.01$, $r = 0$ and the discretization steps $\delta_t = 10^{-6}$ and $\delta_f = 10^{-3}$.

Figure 1 shows the approximate value function $\hat{V}^0$ as a function of time and $f$. Note that here $\mu_2 > \mu_1$, hence the value function is larger when $f$ is close to 0, which means when $\mu(t)$ is likely to equal to $\mu_2$. Theorem 4.1 shows that the value functions are Lipschitz-continuous in $f$ and Hölder continuous with index $1/2$ in time. Figure 2 is a zoom of Figure 1 for $2.5 \leq t \leq 3$. Figure 3 shows $\hat{V}^0(t, 0.05)$ for $0 \leq t \leq 3$. It exhibits that the time derivative is discontinuous. Figure 4 shows $\hat{V}^0(t, f)$ for $t = 2.9$ (highest curve), $t = 2.91$, $t = 2.92$, $t = 2.93$, $t = 2.94$, $t = 2.95$ (flat curve) respectively.

### 8.3 Efficient strategy

As mentioned above, here $\mu_2 > \mu_1$, hence the investor should invest in the stock when $\mu(t) = \mu_2$, i.e. when $f$ is close to 0, and sell when $\mu(t) = \mu_1$, i.e. when $f$ is close to 1. One has to decide what close to means. We propose the following so called efficient strategy. It corresponds to the discrete Dynamic Programming Principle for $(\hat{V}^0, \hat{V}^1)$.

• Compute $(\hat{V}^0, \hat{V}^1)$ for all $t$ and $f$ in the discretization grid.

• At time $t$ in the grid, compute an estimate $\hat{F}_t$ of $F_t$ from the observation of the stock (using classical filtering theory) (see [15]).

• Compare $\hat{V}^0(t, \hat{F}_t)$ and $\hat{V}^1(t, \hat{F}_t)$: buy if $\hat{V}^0(t, \hat{F}_t) = (1 - g_{10})^a\hat{V}^1(t, \hat{F}_t)$, sell if $\hat{V}^1(t, \hat{F}_t) = (1 - g_{10})^a\hat{V}^0(t, \hat{F}_t)$.

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Note that to apply this strategy, one only needs to know the areas where $\hat{V}^0 = (1 - g_{01})^\alpha \hat{V}^1$ or $\hat{V}^1 = (1 - g_{10})^\alpha \hat{V}^0$. It is not necessary to keep in memory the values of $\hat{V}^0$ and $\hat{V}^1$ at each point of the grid. Figure 5 illustrates this strategy. It shows the buying area where $\hat{V}^0 = (1 - g_{01})^\alpha \hat{V}^1$ (lower area), which means that $F$ is close enough to 0 to buy the stock and the selling area where $\hat{V}^1 = (1 - g_{10})^\alpha \hat{V}^0$ (upper area), where $F$ is close enough to 1 to sell the stock. The last area is a no-transaction zone: it means that the investor has to keep his/her position. This area is due to the transaction costs. On the Figure 5, we plot also the process $\hat{F}_t$ estimated from the stock. At time $t = 0$, $\hat{F}_0 \approx 0.2$: the investor buys the stock. At time $t = 0.64$, $\hat{F}$ enters the selling zone, so he/she invests in the bond. At time $t = 1.24$, the process $\hat{F}$ reenters the buying zone, etc.

Note that all transactions should stop at a certain time before the time horizon $T$. This is due to the transaction costs: there is not enough time left to regain the price of the transaction. Far from the horizon, we can see that, approximately, $\hat{F}_t$ is small enough to buy when $\hat{F}_t \leq 0.3$ and is large enough to sell when $\hat{F}_t \geq 0.7$.

By Monte Carlo simulations, we can evaluate the expectation of the utility of the wealth when the efficient strategy is run, and compare the result to the approximate value function. Table 1 shows the results for $10^5$ Monte Carlo simulations. One can see that this strategy is close to be optimal.
Figure 2: Zoom of $\hat{V}^0$ close to the horizon

<table>
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<th>$F_0$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
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<tr>
<td>$\hat{V}^0$</td>
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<td>1.057</td>
<td>1.053</td>
<td>1.049</td>
<td>1.045</td>
<td>1.043</td>
<td>1.041</td>
<td>1.039</td>
<td>1.038</td>
<td>1.037</td>
</tr>
<tr>
<td>$\tilde{V}^0$</td>
<td>1.061</td>
<td>1.056</td>
<td>1.052</td>
<td>1.049</td>
<td>1.045</td>
<td>1.043</td>
<td>1.040</td>
<td>1.039</td>
<td>1.038</td>
<td>1.037</td>
</tr>
</tbody>
</table>

Table 1: Optimality of the efficient strategy

8.4 Misspecifications

Our last result illustrates the critical effect of calibration. We compare two strategies: the first one, called misspecified strategy is the efficient strategy with miscalibrated coefficients: the stock follows the set of parameters mentioned above: $\mu_1 = -0.2$, $\mu_2 = 0.21$, $\lambda_1 = \lambda_2 = 2$, $\sigma = 0.15$; but the agent computes the approximation of the value functions and the approximation of $F_t$ with a set of different (misspecified) coefficients. The second strategy is a classical allocation procedure issued from technical analysis which does not require any knowledge of the dynamics of the stock or parameter estimation: the moving average strategy (with windowing size $\delta = 0.8$). The trader estimates the moving average of the prices

$$M_t^{(\delta)} = \frac{1}{\delta} \int_{t-\delta}^{t} S_u du,$$

and he/she decides to invest in the risky asset $S$ if $S_t \geq M_t^{(\delta)}$ and to invest in the non risky asset $S^0$ otherwise. So, his/her strategy is $\pi_t = \mathbb{1}_{S_t \geq M_t^{(\delta)}}$. See [3]
As benchmarks, we use the efficient strategy (upper curve) with the right parameters and the buy and hold strategy (lower curve). For each strategy, we compute the utility of the corresponding wealth at each time and run $10^5$ Monte Carlo simulations to estimate its expectation. Here, $g_{01} = g_{10} = 0.005$. Figure 6 shows the results for the set of misspecified parameters: $\mu_1 = -0.2$, $\mu_2 = 0.21$, $\sigma = 0.3$, $\lambda_1 = 0.5$, $\lambda_2 = 1$. Here the misscalibration mainly concerns the mean times of change of trend and the volatility. One can see that the miscalibrated strategy is still better than the moving average one. Figure 7 shows another set of miscalibrated parameters: $\mu_1 = -0.3$, $\mu_2 = 0.17$, $\sigma = 0.3$, $\lambda_1 = 2$, $\lambda_2 = 2$. Here the misscalibration mainly concerns the trends and the volatility. The moving average strategy is now better than the miscalibrated one. Further mathematical studies are necessary to understand these misspecification effects.
Figure 4: Regularity of $\tilde{V}^0(t, \cdot)$ for $t = 2.9$ (highest curve), $t = 2.91$, $t = 2.92$, $t = 2.93$, $t = 2.94$, $t = 2.95$ (flat curve) respectively.

Figure 5: The efficient strategy
Figure 6: Comparison of strategies 1

Figure 7: Comparison of strategies 2
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References


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