How Far Apart Can Two Riskless Interest Rates Be? (One Moves, the Other One Does Not)

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First version: June 2009
Current version: June 2009

This research has been carried out within the NCCR FINRISK project on “Equilibrium Asset Pricing”
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June 30, 2009

Abstract

In the presence of transactions costs, no matter how small, arbitrage activity does not necessarily render equal all riskless rates of return. When two such rates follow stochastic processes, it is not optimal immediately to arbitrage out any discrepancy that arises between them. The reason is that immediate arbitrage would induce a definite expenditure of transactions costs whereas, without arbitrage intervention, there exists some, perhaps sufficient, probability that these two interest rates will come back together without any costs having been incurred. Hence, one can surmise that at equilibrium the financial market will permit the coexistence of two riskless rates that are not equal to each other. For analogous reasons, randomly fluctuating expected rates of return on risky assets will be allowed to differ even

*We thank for their comments Blaise Allaz, David Bates, George Constantinides, François Degeorge, Julian Franks, Mark Garman, Philippe Henrotte, David Hsieh, Pete Kyle, Hayne Leland, David Parsley, Jean-Charles Rochet, Hans Stoll, Lars Svensson, and especially Bertrand Jacquillat and Jean-Luc Vila whose questions led to the correction of an error.

†Delgado is from Universidad del Pacífico
‡Dumas is from University of Lausanne, Swiss Finance Institute, NBER and CEPR. Financial support from the Swiss Finance Institute and from NCCR FINRISK of the Swiss National Science Foundation is acknowledged.
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after correction for risk, leading to important violations of the Capital Asset Pricing Model. The combination of randomness in expected rates of return and proportional transactions costs is a serious blow to existing frictionless pricing models.

**Introduction**

Investors, who have to pay transactions costs, optimally rebalance their portfolio at points in times that are random and are not easily observable. The financial econometrician, instead, measures rates of return on financial assets over regular, fixed intervals in time. Investors compare the rates of return on assets over the forthcoming holding periods while the econometrician testing the validity of an asset pricing models, arbitrarily attempts to compare them over successive weeks, months or years.

We would like to know whether it is possible meaningfully to compare the rates of return on two otherwise similar assets when the rates are measured at regular intervals, while investors trade at random times. The question cannot be addressed without a model of the way in which investors choose to rebalance or not their portfolios. We first consider the case of two riskless assets in a portfolio. Then we extend the analysis to risky, long-lived assets such as equities.

If two interest rates on deposits were to remain unequal forever, it would pay to arbitrage out their difference immediately, even if transactions costs had to be incurred in so doing. In the absence of discounting, and in the absence of any costs for rolling over the deposits, the interest differential earned by the arbitrage would eventually outweigh any finite transactions costs incurred at the outset of the arbitrage operation.

If, however, the spread between the two rates fluctuates randomly, it may no longer pay to start an arbitrage. The interest differential may not last long enough to cover profitably the transactions costs. This basic idea was put forth originally in Baldwin (1990) who argues that very small transactions costs help in accounting for the failure of foreign exchange market efficiency tests and shows that the problem mathematically resembled Dixit’s (1989) problem of stochastic entry and exit. More recently, Bacchetta and van
Wincoop (2009) contribute to this literature by examining the impact of infrequent portfolio decisions on the forward discount puzzle. They show that asset management costs discourage investors from active trade, accounting for large deviations from the uncovered interest parity.

The purpose of the present paper is to re-formulate this idea of no-arbitrage spread between the rates of return on two riskless assets and exploit it in the context of an optimal portfolio choice problem with transactions costs.

We examine the portfolio choice of an investor with given relative risk aversion who has access to two riskless investments with instantaneous returns (infinitesimal maturity). One of these brings a rate of interest that is constant over time while the other yields a rate that varies according to a stochastic process. The process incorporates a reversion force, which in the long run pulls the second rate towards the first one. We approach this problem of portfolio choice in the manner of Dumas and Luciano (1991), postponing final consumption to a point infinitely far into the future. For a given portfolio imbalance, the investors allow some gap between the two rates to survive; this gap is called “the hysteresis band”. We are interested in the size of this gap. We intend to show that the gap is much larger (i.e. of a smaller order of magnitude) than the transactions costs.

Because deposits are not forcibly refunded and can be rolled over costlessly, the period over which a given investor holds the deposit - the “holding period” - is a decision variable.\(^1\) As smaller and smaller transactions costs are considered, the allowable spread measured over the holding period is gradually compressed but the anticipated optimal holding period shrinks because smaller transactions make it less costly to switch from one asset to the other. Depending on the rates at which these two variables approach zero, the allowable annualized quoted spread may become small slowly or quickly. We

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\(^1\)The analysis is not limited to bank deposits. In fact, it applies to all assets. Shares of stock that pay no dividend are automatically “rolled over” until the investor explicitly sells them. Section 3 will be devoted to the analysis of rates of returns on equities. The analysis could, but will not, be generalized to shares that pay a dividend. Bonds would require a separate study because they are 100% refunded at the maturity date. That is one “transaction” that is forced on the bondholder.
show that it becomes small at a cubic-root rate.

Later on, we consider an arbitrage between a riskless asset with a constant rate and a risky asset with a stochastic mean-reverting conditionally expected rate of return. We conclude that the CAPM must be badly violated because of the existence of transactions costs. This conclusion contrasts with the final observations of Constantinides (1986) who holds the view that small transactions costs only produce small deviations from the CAPM. The difference in the results is traceable to the difference in the assumed behavior of the conditionally expected return on the risky asset. Constantinides considers an expected return that is constant; we consider a stochastic, mean reverting one. Other evidence that small transaction costs have a strong impact on asset pricing is provided by Jang, Koo, Liu and Loewenstein (2007). They propose a regime-switching model of portfolio choice and show that jumps in regime, by entailing time-varying investment opportunity set, generate first-order effects on liquidity premia.

Mean reversion in expected returns on stocks has been studied empirically by Fama and French (1988), Poterba and Summers (1988), and Bekaert and Hodrick (1992) among others. We contribute to the asset-allocation literature solving a portfolio-choice problem with transaction costs and mean-reverting expected returns. In this sense, we extend the works of Davis and Norman (1990), Dumas and Luciano (1991), Akian, Menaldi, and Sulem (1996), Eastham and Hastings (1988), Liu (2004) who determine the optimal portfolio policy in case of proportional transaction costs and constant investment set, and Kim and Omberg (1996), Campbell and Viceira (1999), and Wachter (2002) who instead consider mean reverting expected returns but no transaction costs.

The paper is organized as follows. In Section 1 we solve the basic portfolio problem considered by Baldwin (1990) in which investors are constrained to investing their entire wealth in one riskless asset or the other; we measure the resulting gap in interest rates. In Section 2, we allow continuous adjustment of the portfolio while still considering only two riskless assets. In Section 3,

\footnote{Fama and French have shown that long-holding period returns display mean reversion. The behavior of long-period returns is the combined result of short-period mean behavior and volatility behavior. In our model, short-period volatility is assumed constant.}
we optimize a portfolio made up of one riskless asset with a constant rate and one risky asset with a mean reverting expected return; we evaluate the deviation from the costless CAPM. Section 4 shows the optimal portfolio policy when the investor faces some idiosyncratic shocks. Finally, Section 5 presents a calibrated numerical illustration.

1 The case of two riskless assets and all-or-nothing portfolio holdings

1.1 Problem Formulation

Consider two assets. One of them has a constant riskless rate of return, which, without loss of generality in our context, we can set equal to zero. The other brings, over a small, fixed period of time, a rate of return, $\alpha$, which is also riskless but which follows a mean-reverting stochastic process:

$$d\alpha = -\lambda \alpha dt + \sigma dz.$$  

At any given time $t$, the dollar value of an investor’s holding of the first asset is denoted $x$ and the dollar value of his holding of the second asset is denoted $y$. Proportional transactions costs at the rate $1 - s$ are incurred when exchanging one asset into the other; these costs are proportional to the dollar value of the trade.

We seek an optimal portfolio policy in which the objective is to maximize the utility of terminal consumption at some later date $T$. The utility of terminal consumption is logarithmic so that the objective is stated as:

$$L(x, y; t; T) \equiv \max \mathbb{E}_t [\ln(c_T)],$$  

where $c_T = x_T$.

In an attempt to discover a stationary optimal policy, we take $T$ to infinity. Furthermore, we assume that the function $L$ asymptotically exhibits linear growth, at some rate, $\beta$, to determined:

$$L(x, y, \alpha; t; T) - \beta(T - t) \rightarrow J(x, y, \alpha).$$  

$$T \rightarrow \infty$$
In this section we restrict the investor to holding all his wealth in the form of one asset or the other. Hence, the portfolio, apart from its size, can only be in one of two states. The only decision to be made at any given time is whether or not to switch the entire portfolio from one asset to the other. The investor will make that switch when $\alpha$ and the fixed rate are sufficiently far apart from each other. We seek the optimal choice of the trigger values $\alpha$ and $\bar{\alpha}$ on each side of the constant value, 0, of the fixed rate of interest.

Exploiting the obvious homogeneity of the problem, define:

$$J(x, y, \alpha) = \ln(x + y) + I(\theta, \alpha), \quad \text{where } \theta \equiv \frac{y}{x + y}. \quad (4)$$

In light of the restrictions imposed on the portfolio, $\theta$ is a binary variable which takes the value 0 or the value 1. For the remainder of this section we denote: $I_0(\alpha) \equiv I(0, \alpha)$ and $I_1(\alpha) \equiv I(1, \alpha)$. $I_1$ is the discounted utility function for a unit wealth that obtains when the investor is invested in the variable-rate asset; $I_0$ is the discounted utility for a unit of wealth that obtains when he is invested in the fixed interest-rate asset.

### 1.2 Probabilistic approach: backward induction

The relationship between these two functions $I_1$ and $I_0$ is given by Equations (5) and (6) below. In Equation (5) a backward probabilistic reasoning gives the current value, $I_1(\alpha)$ of $I_1$. It is equal to:

- the value, $I_0(\alpha)$, of utility when the next switch out of the variable-rate asset occurs,

- plus the logarithm of the per-unit loss in wealth produced by the transactions costs,

- plus the expected extra log-earnings, $\mathbb{E} \left[ \int_0^\tau \alpha_t dt \mid \alpha \right]$, produced by the variable-rate asset during the time until the switch,

- minus the effect of discounting over the expected time till the switch:

$$I_1(\alpha) = \ln(s) + I_0(\alpha) + \mathbb{E} \left[ \int_0^\tau \alpha_t dt \mid \alpha \right] - \beta \mathbb{E} [\tau \mid \alpha] \quad ; \quad \alpha > \bar{\alpha}. \quad (5)$$

Here, $\tau$ is the first-passage time of $\alpha$ to $\bar{\alpha}$. A similar backward reasoning, in (6), gives the current value, $I_0(\alpha)$, of the utility function $I_0$ when not
invested:

\[ I_0 (\alpha) = \ln(s) + I_1 (\overline{\alpha}) - \beta \mathbb{E} [\tau | \alpha]; \quad \alpha < \overline{\alpha}. \tag{6} \]

In (6), \( \tau \) is the first-passage time of \( \alpha \) to \( \overline{\alpha} \).

### 1.3 Equivalent analytical approach

Parenthetically, Equations (5) and (6) can equivalently be obtained by imposing the condition that the value of the function \( L \), defined in (2), executes a martingale process and subsequently introducing the changes of unknown function (3) and (4).

Hence, \( L \) has zero growth; \( J \) grows linearly at the unknown rate \( \beta \); \( I_1 \) grows at the rate \( \beta - \alpha \) because the log of earnings grows at the rate \( \alpha \) when the portfolio is entirely made up of the variable-rate asset; similarly \( I_0 \) grows at the rate \( \beta \).

These restrictions are written successively as follows:

\[
L_t - \lambda \alpha L_{\alpha} + \frac{1}{2} \sigma^2 L_{\alpha\alpha} = 0, \tag{7}
\]

\[
-\beta - \lambda \alpha J_{\alpha} + \frac{1}{2} \sigma^2 J_{\alpha\alpha} = 0, \tag{8}
\]

\[
\begin{aligned}
-\beta + \alpha - \lambda \alpha I_1 + \frac{1}{2} \sigma^2 I_1'' &= 0, \tag{9} \\
-\beta - \lambda \alpha I_0'' + \frac{1}{2} \sigma^2 I_0'' &= 0. \tag{10}
\end{aligned}
\]

Equations (9) and (10), plus Value-Matching boundary conditions, are equivalent to (5) and (6) by virtue of the Feynman-Kac formula, but they are more easily generalizable to the cases of Sections 2 and 3 below than the probabilistic approach would be.

### 1.4 Solution

Returning to the backward, probabilistic approach, we first calculate the expected-earnings integral, \( \mathbb{E} [\int_0^T \alpha dt | \alpha] \), which appears in Equation (5).
An analogous calculation is performed in Karlin and Taylor (1981). The answer in our case is:

\[ E \left[ \int_0^\tau \alpha \, dt \mid \alpha \right] = \frac{\alpha - \underline{\alpha}}{\lambda}; \quad \alpha > \underline{\alpha}. \]  

(11)

For the purpose of interpretation, recall that the value of this integral is the expected cumulative earnings on the variable-rate asset until the next switch to the fixed-rate asset, which will occur at time \( \tau \), the first time that \( \alpha \) reaches \( \underline{\alpha} \) from above.\(^3\)

These expected earnings are always non-negative, which may be surprising. In order to understand this result, it is important to keep in mind that the event \( \alpha = \underline{\alpha} \) stops the sample paths over which the integral is calculated. Hence, earnings that are below \( \underline{\alpha} \) are censored out, whereas excursions of large positive earnings are included in the sum. It may also be surprising to the reader that these expected earnings increase as \( \underline{\alpha} \) is set to a lower, presumably negative value. The answer to this puzzle is again that setting \( \alpha \) lower takes the earnings into a somewhat lower negative zone but also allows some additional, possibly long excursions into positive values that would otherwise be censored out.\(^4\)

The calculation of the expected first-passage time of an Ornstein-Uhlenbeck process, \( E [\tau \mid \alpha] \), is performed in Ricciardi and Sato (1988). In contrast to a standard Brownian motion, an Ornstein-Uhlenbeck process always has a finite expected hitting time. Ricciardi and Sato define a function \( \phi_1(\alpha) \) as follows:

\[ \phi_1(\alpha) = \frac{1}{2\lambda} \sum_{n=1}^\infty \left[ \frac{2\sqrt{\lambda}}{\sigma} \alpha \right]^n \frac{\Gamma(n/2)}{n!}, \]  

(12)

which is easily programmed on a computer. Depending on the situation, \( \phi_1(\alpha) \) or \( -\phi_1(-\alpha) \) serves to compute expected hitting time.

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\(^3\)On pages 196-197.

\(^4\)We expect that \( \underline{\alpha} < 0 \).

\(^5\)The reader might also wonder why the investor would ever want to switch to a zero-rate of return asset when the value of his earnings on the variable-rate asset till the next switch is currently expected to be negative. He will do this (see optimization below) when \( \alpha \) is negative enough because that will enhance his expected earnings. Earnings of the near future are negative; by switching he avoids those. Later, the switch back to the variable-rate asset will occur only when \( \alpha \) is positive and large enough again (\( \alpha = \bar{\alpha} \)).
In Equation (5), the expected earnings and the expected hitting time are inserted as follows:

\[
I_1 (\alpha) = \ln(s) + I_0 (\alpha) + \frac{\alpha - \alpha}{\lambda} - \beta [\phi_1 (-\alpha) + \phi_1 (-\alpha)], \tag{13}
\]

while, in Equation (6), the correct expression is:

\[
I_0 (\alpha) = \ln(s) + I_1 (\pi) - \beta [\phi_1 (\pi) - \phi_1 (\alpha)]. \tag{14}
\]

The functions \( I_0 \) and \( I_1 \) given by (13) and (14) are solutions of the differential equations (9) and (10).

The values of \( I_1 (\pi) \) and \( I_0 (\alpha) \) are easily eliminated between equations (13) and (14) to get a single equation:

\[
0 = 2 \ln(s) + \frac{\alpha - \alpha}{\lambda} - \beta [\phi_1 (\pi) - \phi_1 (-\pi) - \phi_1 (\alpha) + \phi_1 (-\alpha)] \tag{15}
\]

This equation lends itself to a satisfactory interpretation. The sum of the first two terms of the right-hand side, \( 2 \ln(s) + \frac{\alpha - \alpha}{\lambda} \), equals the expected net log-earnings (per unit of wealth) from a round-trip between the two assets: \( 2 \ln(s) \) is the per unit log-transactions costs and \( \frac{\pi - \beta}{\lambda} \) is the expected log-earnings during the part of the round trip where the variable-rate asset is held. \( \beta \) is, of course, the expected rate of growth of wealth (or the expected increment of log utility per unit of time). \( \beta \) is multiplied by the term between square brackets, which is simply equal to the expected duration of a round trip.

Hence, Equation (15) serves to calculate the expected rate of growth of wealth produced by a given \((\alpha, \pi)\) switching policy; it is equal to the expected net earnings during a round trip divided by the expected time that the round trip takes:

\[
\beta = \frac{2 \ln(s) + \frac{\pi - \alpha}{\lambda}}{\phi_1 (-\pi) - \phi_1 (-\pi) + \phi_1 (\pi) - \phi_1 (\pi)}. \tag{16}
\]

### 1.5 Optimization

We need to write that the choice of \( \alpha \) and \( \pi \) is optimal. Two Smooth-pasting conditions will accomplish that task. They are:
\[ I'_1(\alpha) = I'_0(\alpha), \]

and

\[ I'_1(\overline{\alpha}) = I'_0(\overline{\alpha}), \]

otherwise written (based on (13) and (14)) as:

\[ \frac{1}{\lambda} - \beta \phi'_1(-\overline{\alpha}) = \beta \phi'_1(\overline{\alpha}) \quad (17a) \]

and

\[ \frac{1}{\lambda} - \beta \phi'_1(-\overline{\alpha}) = \beta \phi'_1(\overline{\alpha}). \quad (17b) \]

It is easy to check that Equations (17) are the straightforward first-order conditions of the maximization of the rate of growth, \( \beta \), calculated as in (16), with respect to the choice of \( \alpha \) and \( \overline{\alpha} \).

Because we have been able to express the functions \( I_1 \) and \( I_0 \) explicitly in (13) and (14), the difficult variational problem that we were facing has been reduced to the solution of a system of three algebraic equations (15-17) in three real numbers. Furthermore, in that system the unknown number \( \beta \) appears linearly so that it can be easily eliminated leaving two equations in two unknowns. A further simplification is reached since we can easily show symmetry: \( \alpha = -\overline{\alpha} \). Hence, we are left with just one equation in one unknown number. That number must be found numerically.

### 1.6 The hysteresis band

We have solved the system (15, 17) repeatedly for various values of \( s \), from the value 1 downward, corresponding to increasing rates of transactions costs. Figure 1 shows the values of \( \alpha \) and \( \overline{\alpha} \) against the value of \( s \) (outer curve).\(^6\) The interesting result is that, as \( s \to 1 \), the slope of these curves

\(^6\)We discuss the choice of parameter values in Section 4 below. For the time being, we focus on the qualitative features of the solution.
approaches infinity. As the rate of transactions costs goes to zero, the spread that the investor lets survive between the two riskless rates, goes to zero at a slower pace.

In the absence of transactions costs, arbitrage would force $\alpha$ to be pegged at the value 0. Transactions costs allow wide deviations from the arbitrage result. We can quantify the rate at which the range of deviations approaches zero:

**Statement 1:** As $\ln(s)$ approaches zero, the range of fluctuations of $\alpha$, over which no transaction takes place, approaches zero like $\ln(s)^{1/3}$.

**Proof:** Call $z = \alpha = -\alpha$ the common unknown value of the interest rate bounds. Eliminate $\beta$ between (15) and (16) or (17), to get:

$$- \ln(s) = \frac{z}{\lambda} - \frac{1}{\lambda} \frac{\phi_1(z) - \phi_1(-z)}{\phi_1'(z) + \phi_1'(-z)}. \quad (18)$$

The expansion of $\phi_1(z)$ was provided in (12). The expansion of $\phi_1'(z)$ is:

$$\phi_1'(z) = \frac{1}{2\lambda} \sum_{n=1}^{\infty} \left[ \frac{2\sqrt{\lambda}}{\sigma} \right]^n z^{n-1} \frac{\Gamma(n/2)}{(n-1)!}. \quad (19)$$

From these we can get the expansion of the right-hand side of (18). The result is:\footnote{$\Gamma(1/2)/\Gamma(3/2) = 2.$}

$$- \ln(s) = \frac{1}{6\lambda} \left[ \frac{2\sqrt{\lambda}}{\sigma} \right]^2 z^3, \quad (20)$$

or:

$$z = \left( -\frac{3}{2} \sigma^2 \ln(s) \right)^{1/3}. \quad (21)$$

Q.E.D.

Cubic rates of convergence for similar limit problems have been found in different contexts by Dixit (1990), Fleming et al (1990) and Svensson (1991).

Equation (21) shows that, for small transaction costs, two parameters only play a role in the determination of the hysteresis band, viz. $\sigma$ and $s$. 
Mean reversion parameter, $\lambda$, is not present. For finite transactions costs, the band remains very insensitive to the value of $\lambda$. Figure 1, displays the approximate values of $\sigma$ and $\bar{\sigma}$ as given by (21) (inner curves); they are virtually identical to the exact values over the range of transactions costs shown.

1.7 The expected rate of growth and the expected frequency of transactions

Equation (12) makes it plain that the expected time between two transactions is of the same order of magnitude, i.e. $1/3$, as the barrier position itself. From the identity (15) and the leading term in the expansion (12) of the function $\phi_1$, one can deduce that the limit of the expected rate of growth as transaction costs are taken to zero is equal to the following number $\beta^*$:

$$\beta^* = \left[\frac{2\sqrt{\lambda}}{\sigma} \Gamma(1/2)\right]^{-1}.$$ (22)

Substituting (21) into the first terms of (16), the expected duration of a round trip is approximately equal to:

$$\frac{(-6 \ln(s)\sigma^2)^{1/3}}{\lambda \beta^*} + \frac{1}{\lambda} \left[\frac{2\sqrt{\lambda}}{\sigma}\right]^3 \frac{\Gamma(3/2)}{6} \left(-3 \ln(s)\sigma^2\right).$$ (23)

Finally, the value of the expected growth rate in a neighborhood of $s = 1$ is given by:

$$\beta = \beta^* - 2\beta^* \left[\frac{1}{\lambda} \left(3\sigma^2\right)^{1/3}\right]^{-1} \left(-\ln(s)\right)^{2/3}.$$ (24)

As in the case of the boundary positions, these approximate expressions are extremely accurate over a range of transactions costs from zero to several percentage points. The assumption of “small” transactions costs allows the derivation of accurate analytical expressions.
2 The case of two riskless assets and continuous portfolio holdings

When the two asset holdings $x$ and $y$ are allowed to vary continuously, the state transition equations are:

$$dx = sdl - dw;$$  \hspace{1cm} (25) \hspace{1cm}

$$dy = \alpha ydt - dl + sdu;$$ \hspace{1cm} (26)

$$d\alpha = -\lambda \alpha dt + \sigma dz.$$ \hspace{1cm} (27)

Here $u$, and $l$, are two nondecreasing stochastic processes which increase only when (respectively) some amount of fixed-rate, or variable-rate asset is sold. We call $\theta (\theta)$ and $\underline{\theta} (\theta)$ the upper and lower trigger values of which depend on the current composition, $\theta \equiv \frac{y}{x+y}$, of the portfolio.

Between transactions, $dx = 0$ and $dy = \alpha ydt$ so that the portfolio composition, $\theta$, satisfies the following time-differential equation:

$$d\theta = \alpha \theta (1 - \theta) dt.$$ \hspace{1cm} (28)

Over the domain of no transactions, therefore, the value function, $I (\alpha, \theta)$, satisfies the following partial differential equation:\footnote{This P.D.E. is analogous to the pair of Equations (9) and (10) above.}

$$-\beta + \alpha \theta - \lambda \alpha I_\alpha + \frac{1}{2} \sigma^2 I_{\alpha\alpha} + \alpha \theta (1 - \theta) I_\theta = 0.$$ \hspace{1cm} (29)

We solve this partial differential equation by first discretizing it over the values of $\theta$. We pick $\theta \in \{\theta_i; i = 0, ..., n\}$. Then we need to find $n+1$ functions $I (\alpha, \theta_i)$, analogous to the functions $I_0 (\alpha)$ and $I_1 (\alpha)$ in the previous section. At any time $t$, and for any portfolio composition $\theta_i$, the agent drops his holdings to $\theta_{i-1}$ whenever $\alpha$ reaches $\underline{\alpha}_{i-1} \equiv \underline{\alpha} (\theta_i)$, whereas he increases the portfolio proportion to $\theta_{i+1}$ when $\alpha$ reaches $\overline{\alpha}_i \equiv \overline{\alpha} (\theta_i)$.$^{9}$

\footnote{This also means that two agents characterized by the same log-utility function and the same investment opportunity set, but endowed with a different initial-portfolio composition $\theta_0$, will necessarily have the same portfolio policy.}
Given the existence of proportional transactions costs, the utility impact of switching may be computed as follows. First, on the way down from $\theta_i$ to $\theta_{i-1}$:

\[ \theta_i = \frac{y}{x+y}; \quad \theta_{i-1} = \frac{y - \Delta y}{x + s\Delta y + y - \Delta y}, \]

which implies:

\[ \Delta y = (x + y) \frac{\theta_i - \theta_{i-1}}{\theta_{i-1}(s - 1) + 1}. \]  

(31)

Matching the values of the indirect utility before and after the change in portfolio composition, we have:

\[ \ln (x + y) + I(\alpha_i, \theta_i) = \ln (x + s\Delta y + y - \Delta y) + I(\alpha, \theta_{i-1}). \]

(32)

From (31):

\[ \frac{x + s\Delta y + y - \Delta y}{x + y} = 1 + (s - 1)\frac{\theta_i - \theta_{i-1}}{\theta_{i-1}(s - 1) + 1}. \]

(33)

Call $\pi_{i-1}$ the right-hand side of (33). Since (32) may be rewritten as:\footnote{Assume $\theta_{i-1} < \frac{1}{1-s}$, so that $\pi_{i-1} > 1$.}

\[ I(\alpha_{i-1}, \theta_i) = \ln (\pi_{i-1}) + I(\alpha_{i-1}, \theta_{i-1}), \]

(34)

we conclude that the transaction-cost related utility loss on the way down is $\ln (\pi_{i-1})$. Equation (34) is a Value-Matching condition.

The transition on the way up from $\theta_i$ to $\theta_{i+1}$ is handled in a similar way. Let:\footnote{Assume $\theta_i > \frac{s}{1-s}$, so that $\pi_i > 1$.}

\[ \pi_i = 1 + (s - 1)\frac{\theta_i - \theta_{i+1}}{\theta_{i+1}(s - 1) - s}, \quad i = 0, \ldots, n - 1. \]

(35)

The transaction-cost related utility loss on the way up is $\ln (\pi_i)$, resulting in a second set of Value-Matching conditions.

Finally, we need to write that the choice of $\alpha_{i-1}$ and $\pi_i$ is optimal for each $i$. Smooth-pasting necessary conditions accomplish that task:

\[ \text{Assume } \theta_{i-1} < \frac{1}{1-s}, \text{ so that } \pi_{i-1} > 1. \]

\[ \text{Assume } \theta_i > \frac{s}{1-s}, \text{ so that } \pi_i > 1. \]
\[ I_a(\alpha_{i-1}, \theta_i) = I_a(\alpha_{i-1}, \theta_{i-1}), \ i = 1, \ldots, n; \]  

\[ I_a(\alpha_i, \theta_i) = I_a(\alpha_i, \theta_{i+1}), \ i = 0, \ldots, n - 1. \]  

(36)  

(37)

It is worthwhile to underline that, at \( \theta = 0 \) and \( \theta = 1 \), the P.D.E. (29) is locally an ordinary differential equation. Hence, the functions \( I(\alpha, 0) \) and \( I(\alpha, 1) \), already obtained in Section 1, namely:

\[ I(\alpha, 1) = I(\alpha(1), 1) + \frac{\alpha - \alpha(1)}{\lambda} - \beta [\phi_1 (-\alpha) + \phi_1 (-\alpha(1))], \]  

\[ I(\alpha, 0) = I(\alpha(0), 0) - \beta [\phi_1 (\alpha(0)) - \phi_1 (\alpha)], \]  

(38)  

(39)

serve as boundary conditions at these values of \( \theta \).

The solution to the system (29, 34 - 39), i.e. the functions \( I(\alpha, \theta) \), \( \bar{\alpha}(\theta) \) and \( \alpha(\theta) \), has been obtained numerically using a finite-difference method and considering two different scenarios for the portfolio composition: one (Scenario 1) in which we have assumed no short-selling, i.e. \( \theta_0 = 0 \) and \( \theta_n = 1 \), and the other (Scenario 2) in which we have allowed \( \theta \) to take values outside the \([0, 1]\) range, i.e. \( \theta_0 < 0 \) and \( \theta_n > 1 \).

We proceeded in three steps. First, for every \( \theta_i \), we have discretized the values of \( \alpha \) within the range \([\alpha(\theta_i), \bar{\alpha}(\theta_i)]\), therefore obtaining a grid of points in the variables \( \theta \) and \( \alpha \). Then, for each arbitrary position of the barriers, we have computed \( I(\alpha, \theta) \) by solving simultaneously the system of linear equations given by (29, 34 - 39). More precisely, in Scenario 1 we have used Condition 29 at all points strictly inside the grid, the Value-Matching conditions corresponding to the upper and the lower barriers, and Equations 38 and 39 at \( \theta = 0 \) and \( \theta = 1 \). In Scenario 2, for \( \theta \in [0, 1] \) we have used the same conditions as Scenario 1, while, outside this range, only Equation 29 and Value-Matching conditions have been employed. Furthermore, at \( \theta = \theta_0 \) and at \( \theta = \theta_n \), we have computed the partial derivative \( I_\theta \) using exclusively the information inside the grid, thus implicitly assuming as aside condition that the second partial derivative is zero. Finally, in the last step, we have determined the functions \( \bar{\alpha}(\theta) \) and \( \alpha(\theta) \) using an iterative procedure,
updating the position of the barriers on the basis of the violations of the smooth-pasting conditions.

The resulting portfolio-adjustment boundary is shown in Figure 2.

FIGURE 2 GOES HERE

Figure 2 shows the optimal position of the barrier corresponding to both scenarios. It is worthwhile to mention that, for every values of $\theta_0$ and $\theta_n$, when short-selling is allowed, the no-arbitrage spread between the interest rates is simply the continuation of the portfolio-adjustment boundary obtained when $\theta$ is restricted to be in the $[0, 1]$ range. This result is not a surprise and is consistent with the myopic behavior of agents exhibiting a logarithmic utility function. Myopic investors do not look ahead to time at which they will be constrained to not sell short.

In both cases, numerical experiments indicate that the barrier has the following property:

**Statement 2:** The optimal barrier is a flat straight line whose middle point is located at the optimal switching point of the binary policy.

The location of the boundary implies that the “cubic” property (Statement 1) applies equally to the width of the hysteresis band in this case and confirms the symmetric behavior (around the line $\alpha = 0$) of the thresholds functions $\bar{a}(\theta)$ and $\underline{a}(\theta)$, consistent with the results of Section 1.

FIGURE 3 GOES HERE

To better understand the intuition behind this last point we plot the optimal position of the barrier as function of the transaction cost parameter $s$, in the case in which $\theta$ is restricted to be between 0 and 1 (Scenario 1). Figure 3 confirms Statement 2 and shows that, for the same percentage reduction in the transaction costs $1 - s$, the size of the hysteresis bands shrinks at a decreasing pace. In fact, this reduction is 0.0056 when the transaction costs pass from 2% to 1%, while it is only 0.0044 in case of further halving, i.e. from 1% to 0.5%. 

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3 The case of one riskless and one risky, mean-reverting asset

3.1 Problem formulation and solution

When not only the expected rate of return on one of the two assets follows a stochastic process but its rate of return is also risky, the state transition equations are:

\[ dx = sdl - du; \]  
\[ dy = ydt + \sigma_1ydz_1 - dl + sdu; \]  
\[ d\mu = \lambda (\gamma - \mu) dt + \sigma dz. \]

Here, \( \mu \) is the conditionally expected rate of return on the risky asset, \( \sigma_1 \) is the conditional standard deviation of the rate of return on that asset. The expected rate of return, \( \mu \), is assumed to be mean reverting. We call \( \gamma \) the center of reversion. The white-noise shock, \( dz_1 \), affecting the rate of return on the asset is assumed independent on the white noise shock, \( dz \), affecting the expected rate of return.

Introduce a change of state variable:\(^{12}\)

\[ \alpha_t = \mu - \gamma, \]  
and observe that the investor’s frictionless demand for the risky asset at any given time would be given by:

\[ \theta_t = \frac{\mu_t}{\sigma_1^2} = \frac{\alpha_t + \gamma}{\sigma_1^2}, \]

or:

\[ \theta_t - \frac{\gamma}{\sigma_1^2} = \frac{\alpha_t}{\sigma_1^2}, \]

\(^{12}\)Notice that the case of two riskless assets studied in Section 2 can be seen as a special case of the model described here. In fact, under the assumption that \( \gamma = 0 \) and \( \sigma_1 = 0 \), Equations 40-42 are equivalent to 25-27.
which means that the frictionless demand schedule is symmetric around the point \((\alpha = 0, \theta = \frac{\pi}{\sigma_1})\). The portfolio demand with transactions costs will inherit the same symmetry property.

Between transactions, the stochastic differential equation governing the evolution of the portfolio composition, \(\theta\), is:

\[
d\theta = \theta (1 - \theta) \left( \alpha + \gamma - \theta \sigma_1^2 \right) dt + \sigma_1 \theta (1 - \theta) d\xi_1. \tag{45}
\]

Over the domain of no transactions, the value function, \(I(\alpha, \theta)\), satisfies the following partial differential equation:

\[
0 = -\beta + (\alpha + \gamma) \theta - \frac{1}{2} \theta^2 \sigma_1^2 - \lambda \alpha I_\alpha + \frac{1}{2} \sigma_1^2 I_{\alpha\alpha} + \theta (1 - \theta) (\alpha + \gamma - \theta \sigma_1^2) I_\theta + \frac{1}{2} \sigma_1^2 \theta^2 (1 - \theta)^2 I_{\theta\theta}. \tag{46}
\]

The Value-matching and Smooth-pasting boundary conditions remain as in (34 - 37). The boundary conditions at \(\theta = 1\) and \(\theta = 0\) become now:

\[
I(\alpha, 1) = I(\alpha(1), 1) + \frac{\alpha - \alpha(1)}{\lambda} + (-\beta + \gamma - \sigma_1^2/2) \left[ -\phi_1 (-\alpha) + \phi_1 (-\alpha(1)) \right], \tag{47}
\]

\[
I(\alpha, 0) = I(\pi(0), 0) - \beta \left[ \phi_1 (\pi(0)) - \phi_1 (\alpha) \right]. \tag{48}
\]

**FIGURE 4 GOES HERE**

**FIGURE 5 GOES HERE**

The optimal policy that solves this system has been obtained numerically by the method that has been outlined in the previous section. The same scenarios seen in Section 2 are analyzed here. In Figure 4 we plot the position of the barrier assuming that short-selling is not allowed, while the hysteresis bands resulting from extending the portfolio holdings outside the \(0 - 1\) range are plotted in Figure 5.

The following statement summarizes the property of the portfolio-adjustment boundary.\(^\text{13}\)

\(^{13}\)The parameter values underlying Figures 4-6 are discussed in Section 4.
Statement 3: The optimal barrier is a steep straight line positioned slightly outside the hysteresis band of the riskless case constructed around the frictionless demand.

From Statement 3, it follows that the cubic property (Statement 1) is valid again.

FIGURE 6 GOES HERE

As before, to provide evidence of the cubic property we plot the optimal position of the barrier as function of the transaction cost parameter $s$. Figure 6 confirms Statement 3 showing that as $\ln(s)$ approaches zero, the size of the hysteresis bands converges to zero at a slower pace.

3.2 Equilibrium and deviations from the C.A.P.M

We now briefly discuss the equilibrium of an economy with two production techniques available in infinitely elastic supply. One is riskless and brings a zero return; the other is risky and brings a mean-reverting expected return. The economy is populated with identical logarithmic investors, each one of them choosing a portfolio of investments in the manner that we have just described. In this economy, provided its value\textsuperscript{14} is between 0 and 1, let the variable $\theta$ describe the composition of the aggregate, “market” portfolio. This variable changes over time as the expected return, $\mu$, on the risky technology fluctuates. The classic Capital Asset Pricing Model would require that:\textsuperscript{15}

$$\mu_t = \sigma_t^2 \theta_t,$$

which is simply the inverse of the frictionless demand (43), and which is shown as the straight line on Figure 4.

In an economy with transactions costs, the expected return, $\mu_t = \alpha_t + \gamma$, is allowed to fluctuate within a wide interval given horizontally by the hysteresis band of Figure 4, without any adjustment in the aggregate portfolio. Any

\textsuperscript{14}When $\theta = 0$ or 1, a corner occurs: only one asset is available to all investors and no portfolio decision has to be made by any of them.

\textsuperscript{15}The portfolio composition, $\theta$, and the risk measurement, beta, traditionally used in writing the CAPM, would be equal to each other in our case example.
fluctuation within that band is interpretable as a deviation from the CAPM. This shows that deviations from the CAPM can be large, even with small transactions costs, provided expected returns fluctuate randomly.

4 The hysteresis bands in presence of idiosyncratic risk

In this section we compute the optimal portfolio policy when the investor faces some idiosyncratic shocks to the amount invested in the constant riskless asset. When the investment opportunity set is characterized by one riskless asset and one risky, mean-reverting asset, the state transition equations are:

\[
\begin{align*}
    dx &= \sigma_x x dz_x + sdl - du; \\
    dy &= \mu y dt + \sigma_y dz_1 - dl + sdu; \\
    d\mu &= \lambda (\gamma - \mu) dt + \sigma d\zeta.
\end{align*}
\]

(50) (51) (52)

Here, \(dz_x\) is a standard Brownian-motion independent on any other sources of risk in the economy, \(\sigma_x\) is the sensitivity of the investor money bank account to these shocks.

As in Section 3, we introduce the state variable \(\alpha_t = \mu_t - \gamma\). Between transactions, the portfolio composition \(\theta\) evolves according to the following stochastic differential equation:

\[
d\theta = \theta (1 - \theta) \left( \alpha + \gamma - \theta \sigma_1^2 + (1 - \theta) \sigma_x^2 \right) dt + \theta (1 - \theta) (\sigma_1 dz_1 - \sigma_x dz_x).
\]

(53)

Over the domain of no transactions, the value function, \(I(\alpha, \theta)\), satisfies the following partial differential equation:

\[
0 = -\beta + (\alpha + \gamma) \theta - \frac{1}{2} \theta^2 \sigma_1^2 - \frac{1}{2} (1 - \theta)^2 \sigma_x^2 - \lambda \alpha I_\alpha + \frac{1}{2} \sigma^2 I_{\alpha\alpha}
\]

\[
+ \theta (1 - \theta) \left( \alpha + \gamma - \theta \sigma_1^2 + (1 - \theta) \sigma_x^2 \right) I_\theta + \frac{1}{2} \theta^2 (1 - \theta)^2 (\sigma_1^2 + \sigma_x^2) I_{\theta\theta}.
\]

(54)

The Value-matching and Smooth-pasting boundary conditions remain as in (34 - 37), while the boundary conditions at \(\theta = 1\) and \(\theta = 0\) become now:
\[ I(\alpha, 1) = I(\alpha(1), 1) + \frac{\alpha - \alpha(1)}{\lambda} + (\beta + \gamma - \sigma_1^2/2) \left[ -\phi_1(-\alpha) + \phi_1(-\alpha(1)) \right], \]  
\[ I(\alpha, 0) = I(\alpha(0), 0) - (\beta + \sigma_0^2/2) \left[ \phi_1(\alpha(0)) - \phi_1(\alpha) \right]. \]  

Figure 7 shows the resulting portfolio-adjustment boundaries when short selling is not allowed, for both the investment opportunity sets considered above. In the case of two riskless assets, the presence of idiosyncratic risk alters exclusively the position of the optimal barriers but has only a minor effect on the size of the hysteresis bands. In fact, the upper and the lower boundaries are now steep straight lines and not flat as shown in Section 2, while the gap between the two interest rates is 2.04%/year, i.e. slightly higher the 2.01% obtained before. On the contrary, when the economy consists of one riskless and one risky, mean-reverting asset, the idiosyncratic shocks have a stronger impact on the optimal portfolio policy, increasing significantly not only the slopes of these barriers but also the size of the hysteresis bands, which passes from 2.27%/year to 3.3%/year.

5 Calibration

We now wish to quantify the gap in expected returns that can survive in an economy with realistic parameter values. A calibration exercise was conducted both for the problem with two riskless assets (as in Section 1 and 2) and for the problem with one risky asset and one riskless asset (as in Section 3). Parameter values were obtained from the empirical literature on mean reversion in interest rates and stock returns. Our principal sources were Jegadeesh (1991) and Chan et al. (1992). The crucial parameters are the degree of mean reversion, \( \lambda \), and the volatility, \( \sigma \), in expected returns.

For the case of two riskless assets, we have chosen the values: \( \lambda = 1.125%/year \) and \( \sigma = 2.6%/year \). The value of \( \lambda \) implies that it takes eighty years on average for the interest rate to revert to its long-run value. This is
a very small value of the reversion parameter, which is equal to half of the value \( \lambda \) estimated by Chan et al..

Figures 1-3 were drawn for these parameter values. Consistent with the evidence provided by Feldhutter (2009), who measured the average transaction costs paid in corporate bond markets,\(^{16}\) we consider the situation where transaction costs are collected at the rate of 0.1%, \( s = 0.999 \). The figures indicate that the combined effect of such small transactions costs and fluctuating expected returns is enough to produce a dramatic hysteresis effect in the rebalancing of the portfolio. Specifically, a gap of 1.005%/year in interest rates must exist before a decision is made to switch from one asset to the other.

The case of one risky and one riskless asset is calibrated in a like manner. The evidence concerning mean reversion in stock returns is not as conclusive as that concerning interest rates; nonetheless, we used the same value of the mean reversion parameter, which was in any case a low one. The parameter values chosen are: \( \lambda = 1.125\%/\text{year} \), \( \sigma_1 = 15\%/\text{year} \), \( \sigma = 2.6\%/\text{year} \), \( \gamma = \left(\frac{0.15}{2}\right)^2 = 15\%/\text{year} \). Again, we use transactions costs of 0.1%. With these values, at \( \theta = 1/2 \), some wealth is transferred from the risky asset to the riskless as soon as the expected return on the risky asset falls below -1.25%/year. It must be 1.02% before the investor wishes to transfer some wealth from riskless to risky asset. Finally, the sensitivity \( \sigma_x \) of the constant riskless asset to the idiosyncratic shocks is set to 10%/year.

6 Conclusion

Hysteresis bands have not been discovered in this paper. Neither has been the fact that hysteresis bands tend to remain large even when the costs that created them become small. The new result of this paper is that these ideas apply to pricing models so that classic CAPMs are subject to wide hysteresis-band violations when conditionally expected returns follow a stochastic, mean reverting process. Our results also imply that arbitrage models

\(^{16}\)Feldhutter (2009) shows that the round-trip transaction costs depend on the amount traded, varying from 0.4% for extremely small trade sizes (<$5K) to 0.05% for very high sizes (>$$1000K)$.
must be drastically revised to take into account the combined effect of stochastic expected returns and transactions costs.

The qualitative point made by this paper regarding violations of frictionless pricing models does not depend on our assumption that investors have unit risk aversion. Regardless of his degree of risk aversion, a risk averse investor would always choose hysteretic rebalancing decisions. The unit risk aversion assumption has only simplified the calculations and allowed us in the simplest cases to obtain closed-form solutions.
References


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Figure 1: The effect of transaction costs

Figure 1 shows the effect of the transaction cost $s$ on the size of the hysteresis bands. The mean reversion $\lambda$ is set to 1.125%/year, while the conditional volatility is 2.6%/year.
Figure 2: The optimal position of the barrier. Case of two riskless assets and continuous portfolio holdings

Figure 2 shows the optimal position of the thresholds functions $\bar{\tau}$ and $\alpha$ corresponding to Scenario 1 and 2. The mean reversion $\lambda$ is set to 1.125%/year, while the conditional volatility is 2.6%/year. The transaction costs are collected at the rate of 0.1%, implying $s = 0.999$. 
Figure 3: The effect of transaction costs on the portfolio-adjustment boundary. Case of two riskless assets and continuous portfolio holdings.

Figure 3 shows the optimal position of the barrier for different values of the transaction cost parameter $s$. To simplify the analysis we restricted $\theta$ between zero and one. The mean reversion $\lambda$ is set to $1.125\%/year$, while the conditional volatility is $2.6\%/year$. The transaction costs are collected at the rate of $0.1\%$, implying $s = 0.999$. 
Figure 4: The optimal position of the barrier under short-selling restrictions. Case of one riskless and one risky, mean-reverting asset.

Figure 4 shows the optimal position of the thresholds functions $\bar{\sigma}$ and $\sigma$ when short-selling is not allowed. The mean reversion $\lambda$ is set to 1.125%/year, while the conditional volatility is 2.6%/year. The transaction costs are collected at the rate of 0.1%, implying $s = 0.999$. Finally, both the conditional volatility of the risky asset $\sigma_1$ and the risk premium $\gamma$ are set equal to 15%/year.
Figure 5: The optimal position of the barrier when short-selling is allowed. Case of one riskless and one risky, mean-reverting asset

Figure 5 shows the optimal position of the thresholds functions $\bar{\alpha}$ and $\alpha$ when short-selling is allowed. The mean reversion $\lambda$ is set to 1.125%/year, while the conditional volatility is 2.6%/year. The transaction costs are collected at the rate of 0.1%, implying $s = 0.999$. Finally, both the conditional volatility of the risky asset $\sigma_1$ and the risk premium $\gamma$ are set equal to 15%/year.
Figure 6: The effect of transaction costs on the portfolio-adjustment boundary. Case of one riskless and one risky, mean-reverting asset.

Figure 6 shows the optimal position of the barrier for different values of the transaction cost parameter $s$. To simplify the analysis we restricted $\theta$ between zero and one. The mean reversion $\lambda$ is set to 1.125%/year, while the conditional volatility is 2.6%/year. The transaction costs are collected at the rate of 0.1%, implying $s = 0.999$. Finally, both the conditional volatility of the risky asset $\sigma_1$ and the risk premium $\gamma$ are set equal to 15%/year.
Figure 7: The optimal position of the barriers in presence of idiosyncratic risk

Figure 7 shows the optimal position of the barriers when the investor faces some idiosyncratic shocks to the amount invested in the constant riskless asset. To simplify the analysis we restricted $\theta$ between zero and one. The mean reversion $\lambda$ is set to 1.125%/year, while the conditional volatility is 2.6%/year. The transaction costs are collected at the rate of 0.1%, implying $s = 0.999$. The conditional volatility of the risky asset $\sigma_1$ and the risk premium $\gamma$ are set equal to 15%/year. Finally, the sensitivity $\sigma_x$ of the constant riskless asset is set to 10%/year.