Growing Wealth with Fixed-Mix Strategies

Igor V. Evstigneev          Klaus Reiner Schenk-Hoppé

First version: July 2009
Current version: November 2009

This research has been carried out within the NCCR FINRISK project on “Behavioural and Evolutionary Finance”
Growing wealth with fixed-mix strategies

Michael A.H. Dempster\textsuperscript{a}
Igor V. Evstigneev\textsuperscript{b}
Klaus Reiner Schenk-Hoppé\textsuperscript{c}

November 6, 2009

Abstract

This chapter surveys theoretical research on the long-term performance of fixed-mix investment strategies. These self-financing strategies re-balance the portfolio over time so as to keep constant the proportions of wealth invested in various assets. The main result is that wealth can be grown from volatility. Our findings demonstrate the benefits of active portfolio management and the potential of financial engineering.

\textit{JEL-Classification:} G11.

\textit{Key words:} Volatility; Constant proportions strategies; Fixed-mix strategies; Financial markets; Investment; Exponential growth; Transaction costs.

\textsuperscript{a}Centre for Financial Research, University of Cambridge, United Kingdom. E-mail address: mahd2@cam.ac.uk
\textsuperscript{b}Economics Department, School of Social Sciences, University of Manchester, United Kingdom. E-mail address: igor.evstigneev@manchester.ac.uk
\textsuperscript{c}Leeds University Business School and School of Mathematics, University of Leeds, United Kingdom. E-mail address: K.R.Schenk-Hoppe@leeds.ac.uk (Corresponding author)

1 Introduction

Investment advice is usually based on some optimality principle such as the maximization of expected utility or the growth rate. The present chapter takes a broader view by studying generic features of an investment style based on constant proportions, or fixed-mix, strategies. These self-financing strategies aim to maintain fixed proportions between the value of portfolio positions by trading in the market at specific points in time. It is an active portfolio management that rebalances positions by selling assets whose portfolio value exceed the given benchmark to finance the purchase of those assets with a too low weight in the portfolio.

The significance of constant proportions strategies for investment science was established in Kelly (1956)’s work on information theory and its application to betting markets. A detailed account is provided in the Section The early ideas and contributions in this Handbook (2009b). Kelly’s research was inspired by Claude Shannon’s lectures on investment problems in which the founder of the mathematical theory of information outlined his pioneering ideas in the field of investment science (though he never published in it); see Cover (1998) for the history of this approach.


In contrast to this literature on investment, we will ignore questions of optimality of trading strategies and will not make use of expected utility or related concepts. Our goal is to analyze the performance of arbitrary fixed-mix (constant proportions) strategies in markets with different characteristics. We are only interested in ‘generic’ properties of (not necessarily optimal) fixed-mix strategies and their performance relative to buy-and-hold strategies.¹

¹Of course it is well-known that optimally-chosen rebalancing strategies perform at
This survey is based on the authors’ research, Evstigneev and Schenk-Hoppé (2002) and Dempster, Evstigneev and Schenk-Hoppé (2003, 2007, 2008). In these papers, we study the wealth dynamics of investors employing fixed-mix strategies. Very general (and mostly counter-intuitive) results on fixed-mix strategies are obtained under the assumption that markets will exhibit some degree of stationarity either of (relative) prices or returns. This assumption of stationarity of asset returns is widely accepted in financial theory (and, thus, the basis for practical investment advice) allowing, as it does, expected exponential price growth and mean reversion, volatility clustering and very general intertemporal dependence, such as long memory effects, of returns. Stationarity of returns for instance is a salient feature of the Cox-Ross-Rubinstein binomial asset pricing model.

The remainder of this chapter is organized as follows. Section 1.1 invites the readers to test their intuition while Section 1.2 provides a very simple illustrative example. The general theory of constant proportions strategies in stationary markets is covered in Section 2. A generalization to fixed-mix strategies as well as an application to currency market models is given in Section 3. The most intriguing case of stock markets with stationary returns, covered in Section 4, delivers counter-intuitive results that even experienced researchers find puzzling. Section 5 discusses several prominent explanations for volatility-induced growth and provides an interpretation that agrees with our findings. Section 6 concludes.

1.1 A test of your intuition

There is barely any ‘better’ result than one that is (completely) counter-intuitive. But before one is tempted to make such a claim on one’s own findings, it seems appropriate to test the audience’s intuition. In the following we ask a few questions to which the reader has to promptly guess the answer. Many scientists have been asked these questions casually in private, at seminars and at conferences. The following insights were first reported in Dempster, Evstigneev and Schenk-Hoppé (2007). Non-technically minded readers can skip to the next section which provides the informal discussion of an intriguing example.

Stationary markets: puzzles and misconceptions. Consider a market with $K$ assets whose price process $p_t = (p_1^t, ..., p_K^t)$ is ergodic and stationary. (It is assumed that prices are log-integrable.) The assumption of stationarity of asset prices, perhaps after some detrending, seems plausible when modeling currency markets where ‘prices’ are determined by exchange rates of all the currencies with respect to some selected reference currency.

**Question 1.** Suppose vectors of asset prices $p_t = (p_1^t, ..., p_K^t)$ fluctuate ran-

least as well as any buy-and-hold portfolio, see e.g. the Section Classic papers and theories in this Handbook (2009a).
domly, forming a stationary stochastic process (assume even that the vectors $p_t$ are i.i.d. (independent identically distributed)). Consider a fixed-mix self-financing investment strategy prescribing rebalancing one’s portfolio at each of the dates $t = 1, 2, \ldots$ so as to keep equal investment proportions of wealth in all the assets. What is the tendency of the portfolio value in the long run, as $t \to \infty$? Will the value: (a) decrease; (b) increase; or (c) fluctuate randomly, converging (in some sense) to a stationary process?

The audience of our respondents was quite broad and professional, but practically nobody succeeded in guessing the correct answer, which is (b). Among those with a firm view, nearly all selected (c). There were also a couple of respondents who decided to bet on (a). Common intuition suggests that if the market is stationary, then the portfolio value for a constant proportions strategy must converge in one sense or another to a stationary process. The usual intuitive argument in support of this conjecture appeals to the self-financing property. The self-financing constraint seems to exclude possibilities of unbounded growth. This argument is also substantiated by the fact that in the deterministic case both the prices and the portfolio value are constant. This way of reasoning makes the answer (c) to the above question more plausible a priori than the others.

It might seem surprising that the wrong guess (c) has been put forward even by those who have known about examples of volatility pumping for a long time. The reason for this might lie in the non-traditional character of the setting where not only the asset returns but the prices themselves are stationary. Moreover, the phenomenon of volatility-induced growth is more paradoxical in the case of stationary prices, where growth emerges “from nothing.” In the conventional setting of stationary returns, volatility serves as the cause of an acceleration of growth, rather than its emergence from prices with zero growth rates.

A potentially promising attempt to understand the correct answer to Question 1 might be to refer to the concept of arbitrage. Getting something from nothing as a result of an arbitrage opportunity seems to be similar to the emergence of growth in a stationary setting where there are no obvious sources for growth. As long as we deal with an infinite time horizon, we would have to consider some kind of asymptotic arbitrage. All known concepts of this kind\(^2\) however are much weaker than what we would need in the present context. According to our results, growth is exponentially fast, unbounded wealth is achieved with probability one, and the effect of growth is demonstrated for specific (constant proportions) strategies. None of these properties can be directly deduced from asymptotic arbitrage.

Thus there are no convincing arguments showing that volatility-induced growth in stationary markets can be derived from, or explained by, asymp-

totico arbitrage over an infinite time horizon. But what can be said about relations between stationarity and arbitrage over finite time intervals? As is known, there are no arbitrage opportunities (over a finite time horizon) if and only if there exists an equivalent martingale measure. A stationary process can be viewed as an ‘antipodal concept’ to the notion of a martingale. This might lead to the conjecture that in a stationary market arbitrage is a typical situation. Is this true or not? Formally, the question can be stated as follows.

**Question 2.** Suppose vectors of asset prices \( p_t = (p_1^t, ..., p_K^t) \) form a stationary stochastic process (assume even that the vectors \( p_t \) are i.i.d.) Furthermore, suppose the first asset \( k = 1 \) is riskless with constant price \( p_1^t = 1 \). The market is frictionless and there are no portfolio constraints (in particular short selling is allowed). Does this market have arbitrage opportunities over a finite time horizon?

An arbitrage opportunity over a fixed time horizon is understood in the conventional sense: the existence of a trading strategy that does not require any investment at the initial time, is self-financing, does not incur a loss at the terminal time, and makes a gain with strictly positive probability. Again the answer to this question is practically never guessed immediately. The correct answer depends on whether the distribution of the price vector of the risky assets is continuous or discrete. For example, if \( (p_2^t, ..., p_K^t) \) takes on a finite number of values, then an arbitrage opportunity exists. But if its distribution is continuous, there are no arbitrage opportunities. For details see Evstigneev and Kapoor (2006).

If your intuition led you astray, the remainder of this chapter will help to offer insights into these counter-intuitive results. Even if your intuition led you to the correct answers, you might be interested in the precise formulation of the results and the ideas behind their proofs.

### 1.2 An illustrative example

The potential of constant proportions investment strategies for financial growth can be demonstrated by means of the following example which seems simple enough to be discussed at high-school level. A related example is presented in Dempster, Evstigneev and Schenk-Hoppé (2008) (see also Luenberger, 1998, Chapter 15). A more demanding illustration, though with a sound empirical background, is given in Ziemba (2008).

Two investment opportunities available at every point in time, 0,1,2,... The realized returns of the assets between two points in time are determined by flipping a fair coin (only one!). Think of the outcome (heads/tails) as a economy-wide event that effects the two investments in different ways. The net returns are defined in Table 1.
Investors who buy and hold the first asset will see their wealth decline (exponentially fast) over time. The growth rate is negative:

\[ g_1 := 0.5 \ln(1.5) + 0.5 \ln(0.6) \approx -0.05268 < 0. \]

Investment in only the second asset does not yield any better performance because the growth rate is negative as well and equal to \( g_1 \):

\[ g_2 := 0.5 \ln(0.45) + 0.5 \ln(2.0) = g_1 < 0. \]

Now consider an investor with a constant proportions strategy. Suppose the investor divides his wealth 50:50 between the two assets. After each change in the value of the two portfolio positions, the investor rebalances his portfolio by trading assets to restore the 50:50 split of wealth. The growth rate of his wealth is

\[ 0.5 \ln(1.5 \cdot 1.5 + 0.5 \cdot 0.45) + 0.5 \ln(0.5 \cdot 0.6 + 0.5 \cdot 2.0) \approx +0.1185 > 0. \]

Both passive investors (holding only one asset) will see their wealth halve in less than 14 periods on average. In contrast the investor following a constant proportions strategy with proportions 50:50 will, on average, double his wealth more often than every 6th period. A typical realization of the wealth dynamics is depicted in Figure 1.

The active investor’s choice of the proportions is arbitrary. Indeed any constant proportions investment strategy holding both assets will generate growth in excess of either buy-and-hold strategy. The 50:50 proportions create positive growth while strategies with proportions close to 100:0 (or 0:100) entail growth at negative rate but one that is higher than \( g_1 = g_2 \).

**Examples, examples, examples.** The above example is just one of many. Indeed most descriptions (and analyses) of the phenomenon of generating financial growth from volatility are restricted to examples involving specialized models. Since the deduction of general principles from examples is an unreliable route if a thorough mathematical analysis is feasible, it might come as no surprise that several terms have been coined to describe this phenomenon and many different explanations for its origins have been put forward. In the remainder of this chapter we will present mathematical results on the growth-volatility nexus in general (discrete-time) models and attempt to demystify its origins and explanations.
Figure 1: Wealth dynamics of the two passive investors (both decreasing) and the active investor with 50:50 constant proportions strategy (increasing). Initial wealth is 1.0. Logarithmic scale on y-axis.

1.3 Notation

All models discussed in this chapter share the basic setup. The definitions and concepts are listed and discussed here.

An investor observes prices and takes actions only at particular points in time. These time periods are denoted \( t = 0, 1, 2, \ldots \). The price dynamics are driven by random factors. These factors are described by a stochastic process \( s_t \) with \( t = 0, \pm 1, \pm 2, \ldots \). The process takes values in a measurable space \( S \). The realization of the random parameter \( s_t \) corresponds to the state of the world at time \( t \). Denote by \( P \) the probability measure induced by the stochastic process \( s_t \), \( t = 0, \pm 1, \pm 2, \ldots \), on the space of its paths. There are \( K \geq 2 \) assets traded in the market. Asset prices \( p_t = (p^1_t, \ldots, p^K_t) > 0 \), at the time periods \( t = 0, 1, 2, \ldots \) are described as a sequence of strictly positive random vectors with values in the \( K \)-dimensional linear space \( R^K \). We will assume that the price vector \( p_t \) depends on the history of the process \( s_t \) up to time \( t \):

\[
p_t = p_t(s^t) \in R^K, \quad s^t = (\ldots, s_{t-1}, s_t).
\]

It is supposed, without further mentioning, that all functions of \( s^t \) will be measurable.

A market is called \textit{stationary} if the process \( s_t \) is stationary and the price vectors \( p_t \) do not explicitly depend on \( t \), i.e., \( p_t = p(s^t) \). We will further assume that the process \( s_t \) is ergodic and that \( E[\ln p^K(s^t)] < \infty \) for all \( k = 1, \ldots, K \).

A stochastic process \( \xi_1, \xi_2, \ldots \) is \textit{stationary} if, for any \( m = 0, 1, 2, \ldots \) and any measurable function \( \phi(x_0, x_1, \ldots, x_m) \), the distribution of the random
variable $\phi_t := \phi(\xi_t, \xi_{t+1}, \ldots, \xi_{t+m})$ ($t = 0, 1, \ldots$) does not depend on $t$. According to this definition, all probabilistic characteristics of the process $\xi_t$ are time-invariant. If $\xi_t$ is stationary, then for any measurable function $\phi$ for which $E[\phi(\xi_t, \xi_{t+1}, \ldots, \xi_{t+m})] < \infty$, the averages

$$\frac{\phi_1 + \ldots + \phi_t}{t}$$

(1)

converge almost surely (a.s.) as $t \to \infty$ by Birkhoff’s ergodic theorem—see, e.g., Billingsley (1965). If the limit of all averages of the form (1) is non-random (equal to a constant a.s.), then the process $\xi_t$ is called ergodic. In this case, the above limit is equal a.s. to the expectation $E\phi_t$, which does not depend on $t$ by virtue of stationarity of $\xi_t$.

The exponential growth rate of a stochastic process $\xi_1, \xi_2, \ldots$ with $\xi_t > 0$ is defined by

$$\lim_{t \to \infty} \frac{1}{t} \ln \xi_t$$

(2)

provided the limit exists. Suppose the process $\xi$ is stationary and ergodic with $E \ln \xi_m < \infty$, then its growth rate zero because $t^{-1} \ln \xi_t \to 0$ a.s. by Proposition 4.1.3 in Arnold (1998). One also has

$$\frac{1}{t} \ln(\xi_1 \cdots \xi_t) = \frac{\ln \xi_1 + \ldots + \ln \xi_t}{t} \to E \ln \xi_m = \text{const. (a.s.)}.\quad (3)$$

This property is frequently used below.

At each time period $t$, the investor chooses a portfolio $h_t(s^t) = (h^1_t(s^t), \ldots, h^K_t(s^t))$ with a non-negative number of units of asset $k$, $h^K_t(s^t) \geq 0$. Short selling of assets is ruled out in our model by this assumption of non-negativity. A sequence $h_t(s^t)$, $t = 0, 1, 2, \ldots$, of portfolios is called a trading strategy. The value of a portfolio $h_t$ at time $t$ is $p_t h_t = \sum_k p^k_t h^k_t$.

A trading strategy $H = (h_0, h_1, \ldots)$ with initial wealth $w_0 > 0$ is self-financing if $p_0 h_0 = w_0$ and

$$p_t(s^t) h_t(s^t) \leq p_t(s^t) h_{t-1}(s^{t-1}), \quad t = 1, 2, \ldots \quad \text{(a.s.)}.\quad (4)$$

The inequalities in (4) are supposed to hold almost surely with respect to the probability measure $P$. The market value of the portfolio after trade does not exceed the vale of the yesterday’s portfolio at current prices. This budget constraint restricts the investor’s choice.

A balanced trading strategy $H$ is of the form

$$h_t(s^t) = \gamma(s^1) \cdot \ldots \cdot \gamma(s^t) \tilde{h}(s^t), \quad t = 1, 2, \ldots,\quad (5)$$

with a scalar-valued function $\gamma(\cdot) > 0$ and a vector function $\tilde{h}(\cdot) \geq 0$. If $t = 0$, we assume that $h_0(s^0) = \tilde{h}(s^0)$. It is assumed that $\ln \gamma(s^t)$ and $\ln |\tilde{h}(s^t)|$ are integrable with respect to the measure $P$, i.e. $E|\ln \gamma(s^t)|$ and $E|\ln |\tilde{h}(s^t)||$ are finite. Define $|h| = \sum_k |h^k|$ for a vector $h = (h^k)$.
These strategies are called balanced because they are of balanced growth: (5) implies that all proportions between the amounts of different assets in the portfolio
\[ \frac{h^t_i(s^t)}{h^t_k(s^t)} = \frac{\tilde{h}^t_i(s^t)}{\tilde{h}^t_k(s^t)}, \quad j \neq k, \] (6) form stationary stochastic processes. The random growth rate of the amount of each asset \( k = 1, ..., K \), in the portfolio
\[ \frac{h^t_k(s^t)}{h^{t-1}_k(s^{t-1})} = \gamma(s^t) \frac{\tilde{h}^k(s^t)}{\tilde{h}(s^{t-1})} \]
is a stationary process. A balanced strategy is self-financing (4) if and only if
\[ \gamma(s^t) p(s^t) \tilde{h}(s^t) \leq p(s^t) \tilde{h}(s^{t-1}) \quad \text{(a.s.)}. \] (7) Stationarity implies that if (7) holds for some \( t \), it holds for all \( t \).
Suppose the vector of asset prices is stationary. Then every non-negative vector function \( \tilde{h}(s^t) \) with \( E|\ln |\tilde{h}(s^t)|| < \infty \) defines a self-financing balanced strategy (5) by
\[ \gamma(s^t) := \frac{p(s^t) \tilde{h}(s^{t-1})}{p(s^t) \tilde{h}(s^t)} \] (8) since \( E|\ln p^k(s^t)| < \infty \), and relations (4) and (7) hold as equalities.
Balanced trading strategies have the property that the growth rate of wealth of any investor employing it is completely determined by the expected value of \( \gamma \). Proposition 1 in Evstigneev and Schenk-Hoppé (2002) states that for any balanced trading strategy (5)
\[ \lim_{t \to \infty} \frac{1}{t} \ln(p(s^t) h_i(s^t)) = \lim_{t \to \infty} \frac{1}{t} \ln |h_i(s^t)| = E \ln \gamma(s^0) \quad \text{(a.s.)}. \] (9)
This result shows strict positivity of \( E \ln \gamma(s^0) \equiv E \ln \gamma(s^t) \) implies exponential growth of wealth, i.e., \( p(s^t) h_i(s^t) \to \infty \) a.s. exponentially fast.

2 Asset markets with stationary prices

We first discuss the simplest model in which one can analyze generic properties of fixed-mix strategies: a market in which prices are stationary processes. It turns out that any constant proportions strategy produces growth, though in this stationary market the growth rate of each asset price is zero and, therefore, buy-and-hold strategies do not yield positive growth. This results might seem, at the first glance, counter-intuitive. This section is based on Evstigneev and Schenk-Hoppé (2002). All notations are introduced in Section 1.3.
2.1 The model

A constant proportions strategy in a market with $K$ assets is characterized by a vector $\lambda = (\lambda_1, ..., \lambda_K)$ in the set

$$\Delta = \left\{ (\lambda_1, ..., \lambda_K) \in \mathbb{R}^K : \lambda_k > 0, \sum_{k=1}^{K} \lambda_k = 1 \right\}. \quad (10)$$

To avoid pathological cases, we assume strict positivity of all components. Proportional investment rules of this kind are sometimes termed completely mixed.

The trading strategy $h_t$ is a constant proportions strategy if

$$p_t^k h_t^k = \lambda_k p_t h_{t-1}$$

for all $t = 1, 2, ..., K$. The investor rebalances the portfolio in every period in time by investing the constant share $\lambda_k$ of the wealth $p_t h_{t-1}$ into the $k$th asset. The investor’s wealth at the beginning of period $t$ is determined by evaluating the portfolio $h_{t-1}$ (bought in the previous period) at the current price system $p_t^k$. We assume that all the coordinates of $h_t$ are non-negative: short sales are ruled out.

Constant proportions strategies are self-financing because (11) implies

$$p_t h_t = p_t h_{t-1}.$$ 

Given a vector $\lambda$, the strategy is uniquely defined by its initial portfolio $h_0(s^0)$. Recursive application of (11) determines $h_t(s^t)$ for every $t = 1, 2, ...$.

In the study of the growth rate of wealth, the initial portfolio does not matter, see Evstigneev and Schenk-Hoppé (2002, page 568). Indeed if there are two constant proportions strategies $h_t$ and $\tilde{h}_t$ both generated by the same $\lambda \in \Delta$ but with different (non-zero) initial portfolios, then there exist $c, \overline{c} > 0$ such that $c h_t^k \leq \tilde{h}_t^k \leq \overline{c} h_t^k$ for all $k$ and all $t \geq 1$. Obviously, $\lim t^{-1} \ln |h_t| > 0$ a.s. if and only if $\lim t^{-1} \ln |\tilde{h}_t| > 0$ a.s. (and exponential growth is at the same speed).

2.2 The growth rate

Constant proportions strategies generate balanced portfolios. From the representation (5), the growth rate of the investors wealth is obtained as the expected value $E \ln \gamma(s^0)$. The notion of balanced portfolios goes back, though in a somewhat different form, to Radner (1971)’s study of stochastic generalizations of the von Neumann economic growth model. Arnold, Evstigneev and Gundlach (1999) study this concept of balanced paths in a general setting. The application to financial problems is recent.

The central result on the growth rate of constant proportions strategies in markets with stationary asset prices is obtained by showing that they can
be written as balanced strategies and that $E \ln \gamma(s^0) \geq 0$. To have a strictly positive growth rate, the price process $p(s^t)$ must be non-degenerate.

We impose the following assumption:

**(A)** With strictly positive probability, the random variable

$$p^k(s^t)/p^k(s^{t-1})$$

is not constant with respect to $k = 1, 2, ..., K$, i.e., there exist $m$ and $n$ (that might depend on $s^t$) for which

$$p^m(s^t) / p^m(s^{t-1}) \neq p^n(s^t) / p^n(s^{t-1}).$$

(12)

All results in this Handbook section will be proved under this assumption, though it will be convenient to repeat this condition in different disguises—each best suited for the particular application.

Under assumption (A), one has the following result, see Evstigneev and Schenk-Hoppé (2002, Theorem 1).

**Theorem 1** Fix any $\lambda = (\lambda_1, ..., \lambda_K) \in \Delta$, and let $w_0$ be a strictly positive number. Then there exists a vector function $h_0(s^0) \geq 0$ such that the constant proportions strategy $h_t$ generated by $\lambda$ and $h_0$ is a balanced strategy with initial wealth $w_0$, and we have

$$\lim_{t \to \infty} \frac{1}{t} \ln p_t h_t = \lim_{t \to \infty} \frac{1}{t} \ln |h_t| > 0 \quad (a.s.).$$

(13)

The construction of the vector function $h_0(s^0) \geq 0$ is as follows. Suppose the proportions $\lambda \in \Delta$ and initial wealth $w_0 > 0$ are given. To show that $h_t$ is a balanced trading strategy one needs to define a vector function $\tilde{h}(\cdot) \geq 0$ and a scalar-valued function $\gamma(\cdot) > 0$ such that $h_t$ coincides with the strategy defined in (5) (and that $h_t(s^0) = \tilde{h}(s^0)$). Define

$$\tilde{h}(s^t) = \left( \frac{\lambda_1 w_0}{p^1(s^t)}, ..., \frac{\lambda_K w_0}{p^K(s^t)} \right)$$

(14)

and

$$\gamma(s^t) = \frac{p(s^t) \tilde{h}(s^{t-1})}{w_0} = \left[ \sum_{k=1}^{K} \lambda_k \frac{p^k(s^t)}{p^k(s^{t-1})} \right].$$

(15)

This is a balanced trading strategy which coincides with $h_t$ recursively defined by (11).

Strictly positive growth ($E \ln \gamma(s^t) > 0$) holds because Jensen’s inequality (applied to the probability measure $\lambda_k$ on the set ${1, ..., K}$) implies that

$$\ln \sum_{k=1}^{K} \lambda_k \frac{p^k(s^t)}{p^k(s^{t-1})} \geq \sum_{k=1}^{K} \lambda_k \ln \frac{p^k(s^t)}{p^k(s^{t-1})}$$

(16)

12
with strict inequality on a set of positive probability by assumption (12), which ensures

$$E \ln \gamma(s^t) > \sum_{k=1}^{K} \lambda_k E \ln \frac{p_k^k(s^t)}{p_k^k(s^{t-1})} = 0.$$  

Strict positivity of $E \ln \gamma(s^t)$ means that wealth tends to infinity at an exponential rate. If the non-degeneracy condition (A) is not satisfied, the market is essentially deterministic and all prices are constant. In this case all strategies give zero growth. Indeed (12) is a very weak requirement that is satisfied in virtually every market.

The result presented in Theorem 1 holds under (sufficiently) small proportional transaction costs, see Evstigneev and Schenk-Hoppé (2002, Theorem 2).

2.3 Interpretation

Our analysis shows that constant proportions strategies provide growth in a market in which buy-and-hold strategies do not deliver any. The intuition behind this result is that constant proportions strategies ‘exploit’ the persistent fluctuation of prices. When keeping a fixed fraction of wealth invested in each asset, a change in prices leads the investor to sell those assets that are expensive relative to the other assets and to purchase relatively cheap assets. The stationarity of prices implies that this portfolio rule yields a strictly positive expected rate of growth, despite the fact that each asset price has growth rate zero. The notion of an asset being ‘cheap’ resp. ‘expensive’ makes sense when prices are stationary. If the current price is below the expected value, then the tomorrow’s price has a higher-than-average chance of being larger than today’s price. This asset is cheap today. In other words, if prices are stationary, there is reversion to the median (which is also true for price ratios). Rebalancing strategies, on average, buy low and sell high. In the model with stationary returns (rather than prices), accepting this interpretation would be falling victim to the gambler’s fallacy, see Section 4.

The result highlights the benefit of financial engineering. The implementation of a constant proportions strategy requires to invest actively in the available assets by rebalancing at discrete time periods—the growth is financially engineered by making use of Jensen’s inequality.

3 Fixed-mix strategies in stationary markets

The model with constant proportions strategies can be generalized to accommodate the transfer of fractions between different portfolio positions. An application to currency markets is provided. In this model the exchange rates fluctuate randomly in time as stationary stochastic processes. This
aspect of our analysis is inspired, in particular, by recent work of Kabanov (1999) and Kabanov and Stricker (2001). This section follows Dempster, Evstigneev and Schenk-Hoppé (2003). Again the main result holds under small proportional transaction costs.

3.1 The model

A fixed-mix strategy is determined by a (non-random) matrix \( \lambda_{kj}, k,j = 1, ..., K \), such that

\[
\lambda_{kj} > 0, \quad \sum_{k=1}^{K} \lambda_{kj} = 1. \quad (17)
\]

In each time period, this strategy prescribes the transfer of a fixed share \( \lambda_{kj} > 0 \) of the \( j \)th position of the portfolio to the \( k \)th position (\( k,j \in \{1, ..., K\} \)). The dynamics of the wealth invested in the portfolio positions are given by

\[
p_t^k h_t^k = \sum_{j=1}^{K} \lambda_{kj} p_t^j h_{t-1}^j. \quad (18)
\]

For any matrix \( \lambda_{kj} \) satisfying \((17))\), a strategy \( H \) is called a fixed-mix strategy associated with the matrix \( \lambda = (\lambda_{kj}) \), if \((18)\) holds for all \( k,t \) and \( s^t \).

Any fixed-mix strategy is self-financing: \( p_t h_t = p_t h_{t-1} \). As in the previous section, we are interested in the asymptotic behavior of the portfolio \( h_t \) (and its market value) of a trader employing a fixed-mix investment rule.

In the deterministic case the analysis of the long-term dynamics is simple. The price \( p_t = p = (p^1, ..., p^K) > 0 \) is a constant vector. The corresponding process \( h_t \) defined by \((18)\) is deterministic and will always converge to a steady state. This claim follows from results on positive matrices, Kemeny and Snell (1960). This result, combined with the assumption of stationarity, might seem to rule out unbounded growth and could lead to the (wrong) conjecture of the convergence of \( h_t \) to a stationary distribution in the stochastic case.

The case studied in Section 2 corresponds to the situation in which \( \lambda_{kj} \) does not depend on \( j \):

\[
\lambda_{kj} = \lambda_k \text{ with } \lambda_k > 0, \text{ and } \lambda_1 + ... + \lambda_K = 1 \quad (19)
\]

because \((18)\) reduces to \((11)\).

3.2 Currency markets

The foreign exchange market is arguably the real-world example closest to a market in which prices are stationary. We show to formulate the wealth dynamics of a currency trader with a fixed-mix strategy in the above model.
Currencies \( k = 1, 2, \ldots, K \) are traded in a frictionless market. The exchange rates \( \pi_{kj}^t = \pi^k(s^t) > 0 \) fluctuate randomly in time. The real number \( \pi_{kj}^t \) denotes the amount of currency \( k \) which can be purchased by selling one unit of currency \( j \) at time \( t \). We assume absence of arbitrage at each point in time \( t \), i.e. the exchange rates must satisfy

\[
\pi_{kj}^t = \pi_{km}^t \pi_{mj}^t \tag{20}
\]

for all \( k, m \) and \( j \).

Assume the trader follows any fixed-mix strategy (17) by dividing the holdings \( h_{jt}^{l-1} \geq 0 \) of currency \( j \) purchased at time \( t-1 \) according to the proportions \( \lambda_{kj} > 0 \), \( k = 1, \ldots, K \) at time \( t \). The amount \( \lambda_{kj} h_{jt}^{l-1} \) is exchanged into currency \( k \). After execution of all these transactions, the amount of currency \( k \) obtained at time \( t \) is equal to

\[
h_k^t = \sum_{j=1}^{K} \lambda_{kj} \pi_{kj}^t h_{jt}^{l-1}. \tag{21}
\]

The dynamics (21) can be written in the form (18). Take currency 1 as a numeraire and define

\[
p_k^t = \pi_{1k}^t.
\]

The relation (20) implies \( \pi_{kj}^t = 1/\pi_{jk}^t \) and \( \pi_{jj}^t = 1 \). Therefore, \( \pi_{kj}^t = p_j^t/p_k^t \).

Multiplying (21) by \( \pi_{1k}^t \) and using these relations, yields the formulation (18) of the wealth dynamics.

### 3.3 The growth rate

The analysis of the long-term growth of an investor following a fixed-mix strategy is conceptually identical to that of constant proportions strategies. First one shows that there is an initial portfolio such that the fixed-mix strategy leads to a balanced portfolio. Then one proves that the growth rate is independent of the initial portfolio. Finally one shows that this growth rate is strictly positive under a non-degeneracy condition. The mathematical tools required in the proof however are very different. The case of fixed-mix strategies requires considerably more advanced methods.

As in Section 2 we assume that the market is stationary and the state of the world \( s_t \) is ergodic. The price process \( p_t = p(s^t) \) satisfies \( E[\ln p^k(s^t)] < \infty \) for all \( k = 1, \ldots, K \). The growth rate of each asset price is zero because \( t^{-1} \ln p_t \to 0 \) a.s. and, therefore, buy-and-hold strategies do not yield positive growth.

We assume non-degeneracy of the price process \( p(s^t) \):

(B) The vector \( \bar{p}(s^t) = (\bar{p}^1(s^t), \ldots, \bar{p}^K(s^t)) \) of normalized prices

\[
\bar{p}^j(s^t) := \frac{p^j(s^t)}{\sum_m p^m(s^t)}, \quad j \in \{1, \ldots, K\},
\]
is not constant a.s. with respect to $s'$. This assumption says that there is no constant vector $c$ for which $p(s') = c$ almost surely. By virtue of stationarity of $(s_t)$, condition (B) holds for all $t$ if it is satisfied for some $t$. If $s_t$ is ergodic, the condition (B) is equivalent to the following requirement (Dempster, Evstigneev and Schenk-Hoppé, 2003, page 271): With positive probability, the ratios $p^k(s')/p^k(s'^{-1})$ are not constant with respect to $k$. This is condition (A).

The main result on the performance of an investor employing any fixed-mix strategy is as follows, Dempster, Evstigneev and Schenk-Hoppé (2003, Theorem 1).

**Theorem 2** Let $h_t(s')$, $t \geq 0$, be a fixed-mix strategy associated with the matrix $\lambda = (\lambda_{kj})$ satisfying (17). For each $k \in \{1, 2, \ldots, K\}$, the limit

$$\lim_{t \to \infty} \frac{1}{t} \ln h_t^k$$

exists and is strictly positive almost surely. Furthermore, this limit does not depend on $k$, and

$$\lim_{t \to \infty} \frac{1}{t} \ln h_t^k = \lim_{t \to \infty} \frac{1}{t} \ln p_t h_t > 0 \text{ (a.s.)} \quad (23)$$

The result ensures that the wealth of a fixed-mix investor tends to infinity at an exponential rate. All portfolio positions and the investor’s wealth grow at the same positive exponential rate. This finding generalizes the results in Section 2 to the general case of fixed-mix investment strategies.

The proof of Theorem 2 rests on the observation that one can define a balanced fixed-mix strategy for any given matrix $\lambda$ satisfying (17). Recall that a trading strategy $H$ is balanced if $h_t(s') = \gamma(s^1) \ldots \gamma(s^t) \tilde{h}(s')$ (a.s.) for $t \geq 1$. For $t = 0$, we assume $h_0(s^0) = \tilde{h}(s^0)$. The function $\gamma(\cdot) > 0$ is scalar-valued and $\tilde{h}(\cdot) > 0$ is a vector function such that $E(\ln \gamma(s')) < \infty$ and $|\tilde{h}(s')| = 1$. The norm $|h|$ of a vector $h = (h^k)$ is defined as $\sum_k |h^k|$. The assumption $|\tilde{h}(s')| = 1$, which was not imposed in (5), will be satisfied automatically in the present case.

For balanced strategies, all the proportions between the amounts of different assets in the portfolio and the random growth rate of the amount of each asset in the portfolio are stationary stochastic processes. To show existence of a balanced fixed-mix strategy associated with the matrix $\lambda = (\lambda_{kj})$, one needs to show there are appropriate functions $\gamma(\cdot)$ and $\tilde{h}(\cdot)$. Indeed Theorem 2 in Dempster, Evstigneev and Schenk-Hoppé (2003) ensures that for each matrix $\lambda$ satisfying (17) there is a unique balanced fixed-mix strategy with $|\tilde{h}(s')| = 1$.

The construction can be sketched as follows. Denote by $A_t = A(s') = (A_{kj}(s'))$ the positive random $K \times K$ matrix defined by

$$A_{kj}(s') = \lambda_{kj} p^j(s')/p^k(s').$$

(24)
We have $E|\ln A_{kj}(s')| < \infty$. With this definition, the fixed-mix strategy $H$ can be represented as

$$h_t(s') = A(s')A(s'^{-1})...A(s')h_0(s').$$

(25)

Stationarity of $(s_t)$ implies that functions $\gamma(\cdot)$ and $\tilde{x}(\cdot)$ (satisfying $E|\ln \gamma(s')| < \infty$ and $|\tilde{x}(s')| = 1$) generate a balanced $\lambda$-strategy if and only if

$$\gamma(s')\tilde{x}(s') = A(s')\tilde{x}(s'^{-1}) \text{ (a.s.)}. \quad (26)$$

The existence of a solution to this equation follows from a stochastic version of the Perron–Frobenius theorem, Theorem A.1 in Dempster, Evstigneev and Schenk-Hoppé (2003). The problem (26) cannot be solved by applying the conventional Perron–Frobenius theorem because the vector $\tilde{x}(s')$ on the left-hand side does not coincide with the vector $\tilde{x}(s'^{-1})$ on the right-hand side as the function $\tilde{x}(s')$ is obtained by ‘time-shifting’ $\tilde{x}(s'^{-1})$.

The independence of the growth rate from the initial portfolio is proved by couching the one-step forward portfolio between multiples of the portfolio $\tilde{x}(s')$, analogous to the outline in Section 2. All have the same growth rate.

Finally, the strict positivity of the fixed-mix strategies’ growth rate is asserted by proving that $E \ln \gamma(s') > 0$. This can be seen as follows. Denote by $(\gamma(\cdot), \tilde{x}(\cdot))$ the balanced strategy corresponding to the fixed mix strategy $\lambda$. The relation (26) implies

$$\gamma(s')\tilde{x}^k(s') = \sum_{j=1}^{K} \lambda_{kj} \frac{p_j(s')}{p_k(s')} \tilde{x}^j(s'^{-1}), \quad k \in \{1, ..., K\}. \quad (27)$$

The Perron–Frobenius theorem yields existence of a vector $r = (r_1, ..., r_K) > 0$ with

$$r_k = \sum_{j=1}^{K} \lambda_{kj} r_j, \quad k \in \{1, ..., K\}. \quad (28)$$

Put $\beta_{kj} = r_k^{-1}\lambda_{kj} r_j$. One finds that

$$\gamma(s') = \sum_{j=1}^{K} \beta_{kj} \frac{p_j(s')}{p_k(s'^{-1})} \frac{p_j(s'^{-1})\tilde{x}^j(s'^{-1})r_k}{p_k(s')\tilde{x}^k(s')r_j}, \quad k \in \{1, ..., K\}. \quad (28)$$

From this representation of $\gamma(s')$ one can conclude (using Jensen’s inequality and several other arguments, see Dempster, Evstigneev and Schenk-Hoppé, 2003, Section 3) that $E \ln \gamma(s') > 0$ under assumption (B).

### 3.4 Price processes with trend

The concept of a stationary market, where asset prices $p_t$ fluctuate as stationary stochastic processes, is an idealization. The assumption that only
the relative proportions $p_j^t/p_k^t$ are stationary seems more realistic. Following Dempster, Evstigneev and Schenk-Hoppé (2003), let us assume that the prices are of the form

$$p_t = \xi_t \hat{p}_t,$$

where $\hat{p}_t = \hat{p}(s^t)$ is a process satisfying the assumptions we previously imposed on $p_t$, and $\xi_t = \xi_t(s^t) > 0$ is any sequence of strictly positive random variables. The factors $\xi_t$ represent the dynamics of a price index. The normalized prices $\hat{p}_t$ are free of this trend.

The above analysis can directly be applied to this more general price process. The portfolio value $p_t h_t$ satisfies

$$\frac{1}{t} \ln p_t h_t = \frac{1}{t} \ln \xi_t + \frac{1}{t} \ln \hat{p}_t h_t.$$

The growth rate of $p_t h_t$ is determined by that of $\xi_t$ and $\hat{p}_t h_t$.

Under assumption (B), the process $\hat{p}_t h_t$ grows exponentially fast almost surely. Consequently, if the price index $\xi_t$ grows at an exogenous exponential rate $r$, then the investor’s wealth $p_t h_t$ will grow almost surely at a rate $r’$ strictly greater than $r$.

4 Stock markets with stationary returns

The observation that rebalancing a portfolio by following any constant proportions or, more generally, any fixed-mix strategy leads to a strictly higher growth rate of wealth than any buy-and-hold portfolio has been confirmed in markets with stationary prices. After seeing the results and following the intuition that a reversion to the mean is a powerful source of capital growth, one might wonder about the case in which returns (rather than prices) are stationary processes. Indeed, as explained in Section 1.1, there is no general agreement that the previous results hold for stationary returns. This is quite sensible because the any notion of cheap (or expensive) assets based on the argument that reversion to the long run trend renders an asset cheap after a series of low returns would mean to fall victim to the gambler’s fallacy. This lack of a simple intuition for the wealth dynamics of constant proportions strategies in markets where returns (not prices) are stationary demands a thorough analysis of this case. How can one rest without having gained an understanding of this problem? The presentation follows Dempster, Evstigneev and Schenk-Hoppé (2007) which also contains a proof of the main result under small (proportional) transaction costs.

4.1 The model

Denote the (gross) return on asset $k$ between time $t − 1$ and $t$ by

$$R_t^k := \frac{p_t^k}{p_t^{k-1}}, \quad k = 1, 2, ..., K, \quad t \geq 1.$$ (29)
Let $R_t := (R_1^1, ..., R_K^t)$. We impose the assumption:

(R) The vector stochastic process $R_t$, $t = 1, 2, ...,$, is stationary and ergodic. The expected values $E|\ln R_k^t|$, $k = 1, 2, ..., K$, are finite.

The typical example of a stationary ergodic process is a sequence of i.i.d. (independent identically distributed) random variables. To avoid misunderstandings, we emphasize that Brownian motion and a random walk are not stationary.

The asset price at time $t$ and the initial price are related as $p_k^t = p_0^k \cdot R_1^k \cdot ... \cdot R_t^k$, where the random sequence $R_k^t$ is stationary by (R). This assumption on the structure of the price process is a fundamental hypothesis in finance. Moreover, it is quite often assumed that the random variables $R_k^t$, $t = 1, 2, ...$, are independent, i.e., the price process $p_k^t$ forms a geometric random walk. This postulate, which is much stronger than the hypothesis of stationarity of $R_k^t$, lies at the heart of the classical theory of asset pricing (Black, Scholes, Merton), see e.g. Luenberger (1998).

Birkhoff’s ergodic theorem implies

$$\lim_{t \to \infty} \frac{1}{t} \ln p_k^t = \lim_{t \to \infty} \frac{1}{t} \sum_{n=1}^{t} \ln R_n^k = E \ln R_t^k \quad \text{(a.s.)} \quad (30)$$

for each $k = 1, 2, ..., K$. This means that the price of each asset $k$ has almost surely a well-defined and finite (asymptotic, exponential) growth rate, which turns out to be equal a.s. to the expectation $\rho_k := E \ln R_k^t$, the drift of this asset’s price. The drift can be positive, zero or negative. It does not depend on $t$ in view of the stationarity of $R_t$.

Consider an investor following a constant proportions strategy with proportions (10). Fix any vector $\lambda \in \Delta$. Given an initial portfolio $h_0 > 0$, the (self-financing) trading strategy $H$ is defined recursively by

$$h_k^t = \lambda_k p_t h_{t-1} / p_t^k \quad k = 1, 2, ..., K, \ t \geq 1. \quad (31)$$

This definition is equivalent to (11). Our aim is to study the asymptotic behavior of the portfolio value $V_t = p_t h_t$ as $t \to \infty$, i.e. the limit $\lim_{t \to \infty} t^{-1} \ln(V_t)$ describing the (exponential) growth rate of the strategy.

4.2 The growth rate

We impose the following non-degeneracy condition:

(C) With strictly positive probability,

$$\frac{p_k^t(s^t)}{p_t^m(s^t)} \neq \frac{p_{t-1}^k(s^{t-1})}{p_{t-1}^m(s^{t-1})} \text{ for some } 1 \leq k, m \leq K \text{ and } t \geq 1.$$
This condition is a very mild assumption on the existence of volatility of the price process. Condition (C) does not hold if and only if the relative prices of the assets are constant in time (a.s.), i.e. if, with probability one, the ratio \( \frac{p_k^t}{p^m_t} \) of the prices of any two assets \( k \) and \( m \) does not depend on \( t \). This condition is also equivalent to (B), see Dempster, Evstigneev and Schenk-Hoppé (2003, page 271).

The condition (C) on asset prices has an equivalent formulation for asset returns:

\[(D)\] For some \( t \geq 1 \) (and hence, by virtue of stationarity, for each \( t \geq 1 \)), the probability

\[ P\{R^k_t \neq R^m_t \text{ for some } 1 \leq k, m \leq K \} \]

is strictly positive.

Equivalence can be seen as follows. Note that \( \frac{p^k_t}{p^m_t} \neq \frac{p^k_{t-1}}{p^m_{t-1}} \) if and only if \( \frac{p^k_t}{p^k_{t-1}} \neq \frac{p^m_t}{p^m_{t-1}} \), i.e. \( R^k_t \neq R^m_t \). Denote by \( \delta_t \) the random variable that is equal to 1 if the event \( \{R^k_t \neq R^m_t \text{ for some } 1 \leq k, m \leq K \} \) occurs and 0 otherwise. Condition (C) means that \( P\{\max_{1 \leq t \leq 1} \delta_t = 1 \} > 0 \), while (D) states that, for some \( t \) (and hence for each \( t \)), \( P\{\delta_t = 1 \} > 0 \). The latter property is equivalent to the former because \( \{\max_{1 \leq t \leq 1} \delta_t = 1 \} = \bigcup_{t=1}^{\infty} \{\delta_t = 1 \} \).

We can now state the main result on the growth of wealth of investors following constant proportions strategies.

**Theorem 3** Fix any \( \lambda \in \Delta \).

(i) The growth rate of the constant proportions strategy is almost surely equal to a constant which is strictly greater than \( \sum \lambda_k \rho_k \), where \( \rho_k \) is the drift of the price of asset \( k \).

(ii) Suppose all the assets have the same drift (therefore, almost surely the same asymptotic growth rate), i.e. \( E \ln R^k_t = \rho \) for each \( k = 1, \ldots, K \) with some real number \( \rho \). Then the growth rate of the constant proportions strategy is almost surely strictly greater than the growth rate of each individual asset.

In Theorem 3, assertion (ii) immediately follows from (i). The result (ii) shows that any completely mixed constant proportions strategy grows at a rate strictly greater than \( \rho \), the growth rate of each particular asset. The growth of the investor’s wealth is only driven by the volatility of the price process. This result seems to contradict conventional finance theory which usually regards the volatility of asset prices as an impediment to financial growth. In the present context, volatility serves as an endogenous source of its acceleration. Theorem 3 (ii) asserts the validity of the conclusion drawn in the illustrative example discussed in Section 1.2. Indeed, the constant proportions strategy (.5, .5) (as well as any other completely mixed constant
proportions strategy) yields a higher growth rate than the price of each asset.

The first part of Theorem 3 places a floor under the constant proportions strategy’s growth rate. When asset prices grow at different rates, there is no general result that constant proportions strategy grow faster than any (completely diversified) buy-and-hold strategy. The latter grows at rate \( \max_k \rho_k \) which is not strictly dominated by \( \sum_k \lambda_k \rho_k \). The growth-optimal constant proportions strategy however will always grow at least as fast as any buy-and-hold in the model.

The proof of the above result is surprisingly simple and can be presented with success to an audience with knowledge of only elementary probability and calculus: Fix any vector \( \lambda \in \Delta \). The random wealth dynamics of the corresponding constant proportions strategy (31) are given by

\[
V_t = p_t h_t = \sum_{k=1}^{K} p_k^t h_{t-1}^k = \sum_{k=1}^{K} \frac{p}{p_t^{k-1}} p_t^k h_{t-1}^k =
\]

\[
\sum_{k=1}^{K} \frac{p}{p_t^{k-1}} \lambda_k p_t h_{t-1}^k = \left[ \sum_{k=1}^{K} R_t^k \lambda_k \right] V_{t-1} = (R_t \lambda) V_{t-1}
\]

for each \( t \geq 1 \). This implies,

\[
V_t = (R_t \lambda) \cdot \ldots \cdot (R_1 \lambda) V_0
\]

for all \( t \geq 1 \). The ergodic theorem ensures

\[
\lim_{t \to \infty} \frac{1}{t} \ln V_t = \lim_{t \to \infty} \frac{1}{t} \sum_{n=1}^{t} \ln(R_n \lambda) = E \ln(R_t \lambda) \quad \text{(a.s.)}
\]

It remains to show that \( E \ln(R_t \lambda) > \sum_{k=1}^{K} \lambda_k \rho_k \) under the condition (D). Indeed Jensen’s inequality and (D), ensure the relation

\[
\ln \sum_{k=1}^{K} R_t^k \lambda_k > \sum_{k=1}^{K} \lambda_k (\ln R_t^k)
\]

with strictly positive probability, while the non-strict inequality holds always. Consequently,

\[
E \ln(R_t \lambda) > \sum_{k=1}^{K} \lambda_k E(\ln R_t^k) = \sum_{k=1}^{K} \lambda_k \rho_k.
\]

This proves Theorem 3.
4.3 Interpretation

The rigorous proof of the presence of volatility-induced growth in markets where asset returns are stationary leaves open the question on the intuition behind this result.

It seems any explanation one can give is nothing but a repetition of the mathematical reasoning behind this result: If \( R_1^t, \ldots, R_K^t \) are the random returns of the \( K \) assets, then the asymptotic growth rates of these assets are \( E \ln R_k^t \), while the asymptotic growth rate of a constant proportions strategy is \( E \ln(\sum_k \lambda_k R_k^t) \), which is strictly greater than \( \sum_k \lambda_k E \ln(R_k^t) \) by Jensen’s inequality—because the logarithmic function is strictly concave. The proof in Section 2 confirms that Jensen’s inequality is the central tool used in the proof. Any explanation not based on this fact would be flawed.

It is well-known that a linear combination of assets can produce non-linearity in a portfolio’s characteristics. Indeed this feature drives mean-variance portfolio choice, cf. Luenberger (1998). In the present model, the rebalancing of the portfolio so as to maintain constant proportions causes a non-linear effect in the portfolio’s growth rate with the feature that \( E \ln(\sum_k \lambda_k R_k^t) - \sum_k \lambda_k E \ln(R_k^t) > 0 \) for any \( \lambda = (\lambda_1, \ldots, \lambda_K) \in \Delta \) provided assumption (C) holds.

Other attempts to appeal to intuition are discussed (and rejected) in the next section.

An interesting problem is the question under which (general) assumptions \( E \ln(\sum_k \lambda_k R_k^t) > 0 \) even if \( \max_k \rho_k \leq 0 \) (or, less restrictive, \( \min_k \rho_k \leq 0 \)). The example in Section 1.2 possesses this property. In that example, \( \rho_1 = \rho_2 < 0 \), but \( E \ln(\lambda_1 R_1^t + \lambda_2 R_2^t) > 0 \) for the constant proportions strategy \( \lambda = (.5, .5) \). But if \( \lambda \) is close to \( (1, 0) \) or \( (0, 1) \) the growth rate of wealth becomes negative as well. A partial answer to this ‘black swan’ question is provided in Section 5. We discuss a case in which the addition of a slower growing asset always enhances the growth of a constant proportions strategy and raises the growth rate over that of any buy-and-hold strategy.

5 Myths and misconceptions

The phenomenon of volatility-induced growth is simply counter-intuitive as confirmed by our test in Section 1.1. Providing a common-sense explanation for the mathematical results, can therefore not be a simple task. In this section we discuss some of the more prominent suggestions for intuition put forward in the literature.

5.1 Volatility pumping

The term ‘volatility pumping’ appears to have been first used by Luenberger (1998). His suggestion is that constant proportions strategies force the in-
vestor to ‘buy low and sell high’—the common sense dictum of stock market trading. Those assets whose prices have risen from the last rebalance date will be overweighted in the portfolio, and their holdings must be reduced to meet the required proportions and to be replaced in part by assets whose prices have fallen and whose holdings must therefore be increased. This behavior leads to growth if asset returns exhibit some stationarity properties.

In Section 2.3 we argued that this explanation is correct for stationary prices. In the present case, accepting this explanation would mean to fall victim to the gambler’s fallacy. This explanation is not valid because when the price follows a geometric random walk, the set of its values is generally unbounded and for every value there is a smaller as well as a larger value. Suppose returns are i.i.d., then ‘high’ and ‘low’ do not have any meaning. Reversion to the long run trend (as postulated by the ergodic theorem) is not an explanation either: arguing that the longer the run of black numbers, the higher the odds of red numbers at the next spin of the roulette wheel is the gambler’s fallacy. When returns are determined by the flip of a coin, an asset’s upside and downside potential does not change over time. Such an asset is not cheap or expensive at any point in time.

5.2 The importance of constancy

A more substantial lacuna in the above reasoning is that it does not reflect the assumption of constancy of investment proportions. This leads to the question: what will happen if the portfolio is rebalanced so as to sell all those assets that gain value and buy only those ones which lose it? (This should lead to an even higher growth rate—provided volatility could be ‘pumped.’)

Assume, for example, that there are two assets, the price $p_1^t$ of the first (riskless) is always 1, and the price $p_2^t$ of the second (risky) follows a geometric random walk, so that the gross return on it can be either 2 or $1/2$ with equal probabilities. Suppose the investor sells the second asset and invests all wealth in the first if the price $p_2^t$ goes up and performs the converse operation, betting all wealth on the risky asset, if $p_2^t$ goes down. Then the sequence $\lambda_t = (\lambda_{1,t}, \lambda_{2,t})$ of the vectors of investment proportions will be i.i.d. with values $(0,1)$ and $(1,0)$ taken on with equal probabilities. Furthermore, $\lambda_{t-1}$ will be independent of $R_t$. By virtue of (34), the growth rate of the portfolio value for this strategy is equal to

$$E \ln(R_t \lambda_{t-1}) = \frac{[\ln(0 \cdot 1 + 1 \cdot 2) + \ln(0 \cdot 1 + 1 \cdot \frac{1}{2}) + \ln(1 \cdot 1 + 0 \cdot 2) + \ln(1 \cdot 1 + 0 \cdot \frac{1}{2})]}{4} = 0,$$

which is the same as the growth rate of each of the two assets $k = 1, 2$. But this growth rate is strictly lower than that of any completely mixed constant proportions strategy.
5.3 Energy-interpretation of volatility

Our results highlight the importance of volatility because fixed-mix strategies do not produce growth beyond buy-and-hold portfolios in any of the above models when prices or returns are constant. It is tempting to see volatility as a source of energy that can be tapped to generate growth. Indeed Luenberger (1998) presents an intriguing continuous-time example which supports this view. (A closely related observation is made in Fernholz and Shay (1982).) There are $K$ assets whose prices follow independent (but identical) geometric Brownian motions. The constant proportions strategy with equal weights on all assets has a higher growth rate than any asset price. The remarkable feature however is that increasing the number of assets $K$ leads to a higher growth rate and –at the same time– to a reduction in the volatility of the return. It would seem as if fixed-mix strategies turn the classical return-volatility tradeoff on its head.

The framework for this example is the well-known continuous-time model developed by Merton and others, in which the price processes $S^k_t$, $t \geq 0$, of two assets $k = 1, 2$ are supposed to be solutions to the stochastic differential equations

$$dS^k_t/S^k_t = \mu dt + \sigma^k dW^k_t,$$

where the $W^k_t$ are independent (standard) Wiener processes and $S^k_0 = 1$. As is well-known, these equations admit explicit solutions

$$S^k_t = \exp[\mu t - (\sigma^k/2)t + \sigma^k W^k_t].$$

Given some $\theta \in (0, 1)$, the value $V_t$ of the constant proportions portfolio prescribing investing the proportions $\theta$ and $1 - \theta$ of wealth into assets $k = 1, 2$ is the solution to the equation

$$dV_t/V_t = [\theta \mu_1 + (1 - \theta)\mu_2]dt + \theta \sigma_1 dW^1_t + (1 - \theta)\sigma_2 dW^2_t.$$

Equivalently, $V_t$ can be represented as the solution to the equation $dV_t/V_t = \bar{\mu} dt + \bar{\sigma} dW_t$, where $\bar{\mu} := \theta \mu_1 + (1 - \theta)\mu_2$, $\bar{\sigma}^2 := (\theta \sigma_1)^2 + [(1 - \theta)\sigma_2]^2$ and $W_t$ is a standard Wiener process. Thus, $V_t = \exp[\bar{\mu} t - (\bar{\sigma}^2/2)t + \bar{\sigma} W_t]$, and so the growth rate and the volatility of the portfolio value process $V_t$ are given by $\mu - (\bar{\sigma}^2/2)$ and $\bar{\sigma}$. In particular, if $\mu_1 = \mu_2 = \mu$ and $\sigma_1 = \sigma_2 = \sigma$, then the growth rate and the volatility of $V_t$ are equal to

$$\mu - (\bar{\sigma}^2/2) \text{ and } \bar{\sigma} = \sigma \sqrt{\theta^2 + (1 - \theta)^2} < \sigma,$$

while for each individual asset the growth rate and the volatility are $\mu - (\sigma^2/2)$ and $\sigma$, respectively.

Thus, in this example, the use of a constant proportions strategy prescribing investing in a mixture of two assets leads (due to diversification) to an increase of the growth rate and to a simultaneous decrease of the volatility. When looking at the expressions in (36), the temptation arises even to say that the volatility reduction is the cause of volatility-induced growth. Indeed, the growth rate $\mu - (\bar{\sigma}^2/2)$ is greater than the growth rate $\mu - (\sigma^2/2)$ because $\bar{\sigma} < \sigma$. This suggests speculation along the following lines. Volatility is something like energy. When constructing a mixed
portfolio, it converts into growth and therefore decreases. The greater the volatility reduction, the higher the growth acceleration.

5.4 A counter-example

Do the observations made in the previous section have any grounds in the general case, or do they have a justification only in the above example? To formalize this question and answer it, let us return to the discrete-time framework. Suppose there are two assets with i.i.d. vectors of returns $R_t = (R^1_t, R^2_t)$. Let $\langle \xi, \eta \rangle := (R^1_1, R^2_1)$ and assume, to avoid technicalities, that the random vector $\langle \xi, \eta \rangle$ takes on a finite number of values and is strictly positive. The value $V_t$ of the portfolio generated by a fixed-mix strategy with proportions $x$ and $1-x$ ($0 < x < 1$) is computed according to the formula

$$V_t = V_1 \prod_{n=2}^{t} [x R^1_n + (1-x) R^2_n], \ t \geq 2,$$

see (33). The growth rate of this process and its volatility are given, respectively, by the expectation $E \ln \zeta_x$ and the standard deviation $\sqrt{\text{Var} \ln \zeta_x}$ of the random variable $\ln \zeta_x$, where $\zeta_x := x \xi + (1-x) \eta$. We know from the above analysis that the growth rate increases when mixing assets with the same growth rate. What can be said about volatility? Specifically, we consider the following question.

**Question 3.** (a) Suppose $\text{Var} \ln \xi = \text{Var} \ln \eta$. Is it true that $\text{Var} \ln \left[ x \xi + (1-x) \eta \right] \leq \text{Var} \ln \xi$ when $x \in (0,1)$? (b) More generally, is it true that $\text{Var} \ln \left[ x \xi + (1-x) \eta \right] \leq \max(\text{Var} \ln \xi, \text{Var} \ln \eta)$ for $x \in (0,1)$?

Query (b) asks whether the logarithmic variance is a quasi-convex functional. Questions (a) and (b) can also be stated for volatility defined as the square root of logarithmic variance. They will have the same answers because the square root is a strictly monotone function. Positive answers to these questions would substantiate the above conjecture of volatility reduction—negative, refute it.

In general (without additional assumptions on $\xi$ and $\eta$) the above questions 3(a) and 3(b) have negative answers. To show this consider two i.i.d. random variables $U$ and $V$ with values 1 and $a > 0$ realized with equal probabilities. Consider the function

$$f(y) := \text{Var} \ln[yU + (1-y)V], \ y \in [0,1].$$

(37)

By evaluating the first and the second derivatives of this function at $y = 1/2$, one can show the following. There exist some numbers $0 < a_- < 1$ and $a_+ > 1$ such that the function $f(y)$ attains its minimum at the point $y = 1/2$ when $a$ belongs to the closed interval $[a_-, a_+]$ and it has a local maximum.
at \( y = 1/2 \) when \( a \) does not belong to this interval. The numbers \( a_- \) and \( a_+ \) are given by

\[
a_{\pm} = 2e^4 - 1 \pm \sqrt{(2e^4 - 1)^2 - 1},
\]

where \( a_- \approx 0.0046 \) and \( a_+ \approx 216.388 \). If \( a \in [a_-, a_+] \), the function \( f(y) \) is convex, but if \( a \notin [a_-, a_+] \), its graph has the shape illustrated in Figure 2.

![Figure 2: Graph of the function \( f(y) \) in eq. (37) for \( a = 10^4 \).](image-url)

Fix any \( a \) for which the graph of \( f(y) \) looks like the one depicted in Figure 2. Consider any number \( y_0 < 1/2 \) which is greater than the smallest local minimum of \( f(y) \) and define \( \xi := y_0U + (1 - y_0)V \) and \( \eta := y_0V + (1 - y_0)U \). (\( U \) and \( V \) may be interpreted as “factors” on which the returns \( \xi \) and \( \eta \) on the two assets depend.) Then \( \text{Var} \ln[\xi + \eta] / 2 > \text{Var} \ln \xi = \text{Var} \ln \eta \), which yields a negative answer both to (a) and (b). In this example \( \xi \) and \( \eta \) are dependent. It would be of interest to investigate questions (a) and (b) for general independent random variables \( \xi \) and \( \eta \). It can be shown that the answer to (b) is positive if one of the variables \( \xi \) and \( \eta \) is constant. But even in this case the function \( \text{Var} \ln[x\xi + (1 - x)\eta] \) is not necessarily convex: it may have an inflection point in \((0, 1)\), which can be easily shown by examples involving two-valued random variables.

Thus it can happen that a fixed-mix portfolio may have a greater volatility than each of the assets from which it has been constructed. Consequently, the above conjecture and the ‘energy interpretation’ of volatility are generally not valid. It is interesting, however, to find additional conditions under which assertions regarding volatility reduction hold true. In this connection, we can assert the following fact, see Theorem 4 in Dempster, Evstigneev and Schenk-Hoppé (2007).

**Theorem 4** Let \( U \) and \( V \) be independent random variables bounded above and below by strictly positive constants. If \( U \) is not constant, then one has
that \( \text{Var} \ln[yU + (1 - y)V] < \text{Var} \ln U \) for all \( y \in (0, 1) \) sufficiently close to 1.

Volatility can be regarded as a quantitative measure of instability of the portfolio value. The above result shows that small independent noise can reduce volatility. This result is akin to a number of known facts about noise-induced stability, e.g., Abbott (2001) and Mielke (2000). An analysis of links between the topic of the present work and results about stability under random noise might constitute an interesting theme for further research.

### 5.5 Growth under transaction costs

In this section we consider an example (a binomial model) in which quantitative estimates for the size of the transaction costs needed for the validity of the result on volatility-induced growth can be provided. Suppose that there are two assets \( k = 1, 2 \): one riskless and one risky. The price of the former is constant and equal to 1. The price of the latter follows a geometric random walk. It can either jump up by \( u > 1 \) or down by \( u^{-1} \) with equal probabilities. Thus both security prices have growth rate zero.

Suppose the investor pursues the constant proportions strategy prescribing to keep 50% of wealth in each of the securities. There are no transaction costs for buying and selling the riskless asset, but there is a transaction cost rate for buying and selling the risky asset of \( \varepsilon \in [0, 1) \). Assume the investor’s portfolio at time \( t - 1 \) contains \( v \) units of cash; then the value of the risky position of the portfolio must be also equal to \( v \). At time \( t \), the riskless position of the portfolio will remain the same, and the value of the risky position will become either \( uv \) or \( u^{-1}v \) with equal probability. In the former case, the investor rebalances his/her portfolio by selling an amount of the risky asset worth \( A \) so that

\[
v + (1 - \varepsilon)A = vu - A. \tag{38}
\]

By selling an amount of the risky asset of value \( A \) in the current prices, the investor receives \((1 - \varepsilon)A\), and this sum of cash is added to the riskless position of the portfolio. After rebalancing, the values of both portfolio positions must be equal, which is expressed in (38). From (38) we obtain \( A = v(u - 1)(2 - \varepsilon)^{-1} \). The positions of the new (rebalanced) portfolio, measured in terms of their current values, are equal to \( v + (1 - \varepsilon)A = v[1 + (1 - \varepsilon)(2 - \varepsilon)^{-1}(u - 1)] \). In the latter case (when the value of the risky position becomes \( u^{-1}v \)), the investor buys some amount of the risky asset worth \( B \), for which the amount of cash \( (1 + \varepsilon)B \) is needed, so that

\[
v - (1 + \varepsilon)B = u^{-1}v + B.
\]

From this, we find \( -B = v(u^{-1} - 1)(2 + \varepsilon)^{-1} \), and so \( v - (1 + \varepsilon)B = v[1 + (1 + \varepsilon)(2 + \varepsilon)^{-1}(u^{-1} - 1)] \).
Thus, the portfolio value at each time $t$ is equal to its value at time $t-1$ multiplied by the random variable $\xi$ such that $P(\xi = g') = P(\xi = g'') = 1/2$, where $g' := 1 + (1 + \varepsilon)(2 + \varepsilon)^{-1}(u^{-1} - 1)$ and $g'' := 1 + (1 - \varepsilon)(2 - \varepsilon)^{-1}(u-1)$. Consequently, the asymptotic growth rate of the portfolio value, $E \ln \xi = (1/2)(\ln g' + \ln g'')$, is equal to $(1/2) \ln \phi(\varepsilon, u)$, where
\[
\phi(\varepsilon, u) := \left[1 + (1 + \varepsilon)\frac{u^{-1} - 1}{2 + \varepsilon}\right] \left[1 + (1 - \varepsilon)\frac{u - 1}{2 - \varepsilon}\right].
\]
We have $E \ln \xi > 0$, i.e., the phenomenon of volatility induced growth takes place, if $\phi(\varepsilon, u) > 1$. For $\varepsilon \in [0, 1)$, this inequality turns out to be equivalent to the following very simple relation: $0 \leq \varepsilon < (u - 1)(u + 1)^{-1}$. Thus, given $u > 1$, the asymptotic growth rate of the fixed-mix strategy under consideration is greater than zero if the transaction cost rate $\varepsilon$ is less than $\varepsilon^*(u) := (u - 1)(u + 1)^{-1}$. We call $\varepsilon^*(u)$ the threshold transaction cost rate. Volatility-induced growth takes place—in the present example, where the portfolio is rebalanced in every one period—when $0 \leq \varepsilon < \varepsilon^*(u)$.

The volatility $\sigma$ of the risky asset under consideration (the standard deviation of its logarithmic return) is equal to $\ln u$. In the above considerations, we assumed that $\sigma$—or equivalently, $u$—is fixed, and we examined $\phi(\varepsilon, u)$ as a function of $\varepsilon$. Let us now examine $\phi(\varepsilon, u)$ as a function of $u$ when the transaction cost rate $\varepsilon$ is fixed and strictly positive. For the derivative of $\phi(\varepsilon, u)$ with respect to $u$, we have
\[
\phi'_u(\varepsilon, u) = \frac{1 + \varepsilon}{4 - \varepsilon^2} \left[\frac{1 - \varepsilon}{1 + \varepsilon} - u^{-2}\right].
\]
If $u = 1$, then $\phi'_u(\varepsilon, 1) < 0$. Thus if the volatility of the risky asset is small, the performance of the constant proportions strategy at hand is worse than the performance of each individual asset. This fact refutes the possible conjecture that ‘the higher the volatility, the higher the induced growth rate.’ Further, the derivative $\phi'_u(\varepsilon, u)$ is negative when $u \in [0, u_*(\varepsilon))$, where $u_*(\varepsilon) := (1 - \varepsilon)^{-1/2}(1 + \varepsilon)^{1/2}$. For $u = u_*(\varepsilon)$ the asymptotic growth rate of the constant proportions strategy at hand attains its minimum value. For those values of $u$ which are greater than $u_*(\varepsilon)$, the growth rate increases, and it tends to infinity as $u \to \infty$. Thus, although the assertion ‘the greater the volatility, the greater the induced growth rate’ is not valid always, it is valid (in the present example) under the additional assumption that the volatility is large enough.

6 Conclusion

This chapter surveys the authors’ recent theoretical work on the phenomenon of volatility-induced growth. We study the performance of fixed-mix strategies in markets where asset prices or returns are stationary ergodic. We
have established the surprising result that these strategies generate portfolio growth rates in excess of the individual asset growth rates, provided some volatility is present. As a consequence, even if the growth rates of the individual securities all have mean zero, the value of a fixed-mix portfolio tends to infinity with probability one.

By contrast with the twenty five years in which the effects of ‘volatility pumping’ have been investigated in the literature by example, our results are quite general. They are obtained under assumptions which accommodate virtually all the empirical market return properties discussed in the literature. We have in this paper also dispelled the notion that the demonstrated acceleration of portfolio growth is simply a matter of ‘buying low and selling high’ or stems from some ‘volatility–energy’ link.

The example of Section 5.2 shows that our result depends critically on rebalancing to an arbitrary fixed mix of portfolio proportions. Any such mix defines the relative magnitudes of individual asset returns realized from volatility effects. This observation and our analysis of links between growth, arbitrage and noise-induced stability suggest that financial growth driven by volatility is a subtle and delicate phenomenon.

A major obstacle in practical applications of our results is the fact that investment is not over an infinite period. This point has been forcefully argued by Samuelson (1979), but its validity is not as clear cut as one may be led to believe, see MacLean, Zhao and Ziemba (2006), Ziemba (2008) and the discussion in Section Critics and good and bad properties in this Handbook (2009d). Other issues are the size of transaction costs in real markets. Though our findings do hold under sufficiently small proportional transaction costs, this issue can only be resolved empirically. Another intriguing question is how much stationarity is present in real markets. All of these questions are left for future research.

References


Kuhn, D. and Luenberger, D.G. (forth.) Analysis of the rebalancing frequency in log-optimal portfolio selection. Quantitative Finance. DOI: 10.1080/14697680802629400


32