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Equilibria in games with prospect theory preferences

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Abstract

We study how the framework of classical game theory changes when the preferences of the players are described by Prospect Theory (PT) and Cumulative Prospect Theory (CPT) instead of Expected Utility Theory. Specifically, we study the influence of framing effects and probability weighting on the existence and specific structure of equilibria in pure and mixed strategies for finite games. Although PT does not always select first order stochastic dominant lotteries, we prove that it preserves pure and mixed dominance of strategies, and that while CPT preserves pure dominance of strategies, this is not the case for mixed dominance. We give examples in which the set of best replies to some beliefs is not invariant with respect to the probability misestimation.

Keywords: Prospect theory, framing, probability weighting, Nash equilibria, stochastic dominance, dominance of strategies
JEL classification: C70, C73, D81.

1 Introduction

1.1 Games, Utilities and Prospects

The study of games as a cohesive theory has been brought into life by the work of von Neumann starting in 1928 and culminating in his seminal book with Morgenstern (v. Neumann 1928, v. Neumann & Morgenstern 1944). It has since then been widely applied as a prescriptive, but also as a descriptive model. Von Neumann and Morgenstern already observed that a study of games, strategies and outcomes, is only meaningful once we have understood the preferences that the players have on the possible outcomes

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of a game. They therefore started their considerations of games with a chapter on Expected Utility Theory, formalizing the ideas which Bernoulli had developed more than 200 years earlier (Bernoulli 1738), and thereby breaking the ground for yet another important foundation of modern economic theory.
The role of this decision model as a normative theory is uncontroversial. Recent years, however, have seen substantial progress on the understanding of differences between the actual, sometimes irrational decisions of individuals, and rational decisions according to the Expected Utility Theory. There are now several models to describe decisions under risk, e.g. Rank-Dependent Utility as introduced by Quiggin (1982) and Prospect Theory, as developed by Kahneman & Tversky (1979) and Tversky & Kahneman (1992), a model for which Daniel Kahneman was awarded with the Nobel Prize in economics in 2002. Since Prospect Theory and its variants are nowadays the most frequently used behavioral decision models, we concentrate on them.
Prospect Theory modifies classical Expected Utility Theory in several ways:

1. Unlike Expected Utility Theory, not the final wealth is evaluated, but the payoffs are framed as gains and losses with respect to a reference point; they are called “prospects”. Typically, subjects are risk-averse in gains and risk-seeking in losses.

2. Losses loom larger than gains, hence the marginal utility in losses is larger than in gains.

3. Small probabilities are overweighted and moderate to large probabilities are underweighted.

Mathematically, the first two features are reflected in a two-part S-shaped *value function* u (which replaces the usual utility function) – concave in gains and convex in losses and with larger slope for losses than for gains. The prototypical example has been given in Tversky & Kahneman (1992) for $\alpha, \beta \in (0,1)$ and $\lambda > 1$:

$$u(x) := \begin{cases} x^\alpha, & x \geq 0 \\ -\lambda(-x)^\beta, & x < 0. \end{cases} \tag{1}$$

The third feature is captured by weighting the probability distribution by an S-shaped function, the so-called *weighting function* w. The original example of Tversky & Kahneman (1992)[1] is given by

$$w(F) := \frac{F^\gamma}{(F^\gamma + (1-F)^\gamma)^{1/\gamma}}. \tag{2}$$

[1]This functional form later turned out to be problematic (Rieger & Wang 2006). Our subsequent analysis does, however, not depend on the precise form of w.

In the classical form of Prospect Theory (PT), the function w is applied directly to the probabilities of the different outcomes, resulting in an over-weighting of small probabilities, regardless of their associated outcome. The generalization of this form for outcomes x_i with probabilities p_i is given by

$$PT(p) = \sum_{i=1}^{n} w(p_i)u(x_i), \tag{3}$$

where $PT(p)$ is the *subjective utility* of a probability distribution p, has been implicitly suggested in Kahneman & Tversky (1979) and can explicitly be found in Schneider & Lopes (1986) and Wakker (1989).
The updated version of Prospect Theory, called *Cumulative Prospect Theory* (short: CPT), weights cumulative probabilities $F_i := \sum_{j=1}^{i} p_j$, where outcomes are ordered by their amount, and the weight factor for the i-th outcome is $w(F_i) - w(F_{i-1})$.[2] The result is that only low probability events *with extreme outcomes* are overweighted.
CPT helped in recent years to explain various effects in decision theory, economics and finance.

1.2 Why Prospect Theory Changes Game Theory

When we play a game, payoffs are often given in monetary (or similar) amounts. As von Neumann and Morgenstern noticed, game theory has therefore to take into account the players' preferences on sure and uncertain monetary outcomes of the game. This is why their famous work starts with a chapter on the notion of utility in which they present their formalization of Expected Utility Theory (v. Neumann & Morgenstern 1944). It is therefore not too surprising that this theory is somehow tailor-made for the application to game theory. In fact, the payoffs of a game are usually already *defined* to be given in utility units. Nevertheless it is an important and nontrivial feature that after a simple transformation of monetary outcomes into utility outcomes for each player, in the further analysis of the transformed game no additional considerations regarding the players' preferences have to be made.
This seamless interplay between Expected Utility Theory and game theory, however, obfuscates the fact that there is indeed something nontrivial in this connection, and that therefore changes might be necessary when using a different preference model in the study of games. This might explain why the problem of applying game theory in a PT setting has not been widely studied so far, although it seems very natural to substitute this successful behavioral model into the study of games. We are in fact only aware of two studies of specific games (Dekel, Safra & Segal 1991, Goeree, Holt & Palfrey 2003). On the other hand, there are several articles studying game theory in a general

[2] For a precise formula see Tversky & Kahneman (1992).

non-EUT context, in particular Dekel et al. (1991) and Chen & Neilson (1999) where sufficient conditions for the existence of Nash equilibria are studied and Fershtman, Safra & Vincent (1991) where inefficiency delays in reaching agreements by general deviations from EUT are considered. Even though these approaches are related, they will turn out to be non-applicable to our problem. For other lines of research on Behvaioral Game Theory that are not based on PT we refer the reader to (Camerer 2003).

Why is it non-trivial to bring game theory and PT together? The naive approach to incorporate Prospect Theory into an analysis of a game would be to transform the monetary outcomes via the value function and to transform actual probabilities for chance moves into experienced probabilities by applying the probability weighting functions.[3] This procedure mimics the method one successfully applied when dealing with Expected Utility Theory. Nevertheless in the case of Prospect Theory, there are two difficulties in this approach which pose interesting problems, namely:

1. The transformation of the monetary outcomes via the value function is not as harmless as it seems, since the reference point has to be chosen first.

2. The probabilities of chance moves are not the only probabilities that ought to be transformed by the probability weighting function, since the existence of mixed strategy Nash equilibria complicates considerations: the objective probability with which one player chooses a particular strategy will be transformed to a (different) subjective probability by another player who will choose his own strategy according to this subjective probability, rather than to the objective probability. This leads to an interesting interplay between the weighting functions of the players.[4]

We devote the following section to a discussion of these points. In particular we will study existence and *nonexistence* of Nash equilibria and derive how PT changes their precise values. In Section 3 we discuss further solution concepts, in particular stochastic dominance, best replies and risk dominance. Section 4 concludes.

[3]Here one has to be careful to transform the game into a form where the probabilities of the chance moves can be weighted separately for each player, since their probability weighting functions might differ.

[4]To be more precise, one should not speak about subjective or weighted probabilities, since PT is not about misestimation of probabilities, but states that decisions are made *as if* the underlying probabilities were misestimated or weighted. Mathematically, however, this results in the same formula, and so we keep for simplicity the slight abuse of language and talk about subjective or weighted probabilities as if they had a real meaning in PT and were not only auxiliary quantities.

2 Probability Weighting and Mixed Strategies

2.1 Fundamental concepts

One of the most interesting effects of Prospect Theory on the analysis of games is the interplay between probability weighting and mixed strategies. Let us consider a finite normal-form game with n players without chance moves. In this game, a player i can choose from the (finitely many) pure strategies S_i. We denote the set of all combinations of pure strategies $S := \times_{i=1}^n S_i$. The set of probability measures on S_i is denoted by M_i and describes the mixed strategies of player i. The combinations of mixed strategies are denoted by $M := \times_{i=1}^n M_i$. The payoff (in utility units) of the game for the i-th player is given by $u_i \colon S \to \mathbb{R}$. The game can then be written as $(S_i, u_i)_{i=1}^n$.
The total utility U that a player, say player 1, obtains for some mixed strategy play $m = (m_1, \dots, m_n) \in M$ depends on the underlying decision model. In the case of EUT, this utility becomes

$$U_1^{EUT}(m) = \sum_{s=(s_1,s_2,\dots)\in S} m_1(s_1) \cdot \dots \cdot m_n(s_n) u_1(s),$$

where $m_i(s_j)$ is the probability of player i to play strategy s_j. Using PT, we need to decide about the framing a player uses. Let us for the moment assume that his frame is fixed.[5] We consider probability weighting functions w_i. If the player weights the probabilities with which the other players choose their mixed strategies using this probability weighting function, we obtain for the utility of the first player

$$U_1^{PT}(m) = \sum_{s=(s_1,s_2,\dots)\in S} m_1(s_1) \cdot w_1(m_2(s_2)) \cdot \dots \cdot w_1(m_n(s_n)) u_1(s). \tag{4}$$

In the case of CPT, we need first to rank the possible outcomes, before we can compute the probability weighting. This turns out to be tricky, since in the case of more than two players the strategies of the opponents have per se no associated payoff, even if the first player considers his own strategies separately. In the case of two players we can, however, avoid this problem and define the CPT utility. Let us simplify our notation in this case and call the two players *Sally* and *Tom*, where Sally's set of pure strategies is $S = \{1, \dots, |S|\}$ with generic element $s \in S$ and Tom's set of pure strategies $T = \{1, \dots, |T|\}$ with generic element $t \in T$. Denote their mixed strategies by $\sigma = (\sigma(1), \dots, \sigma(|S|))$ and $\theta = (\theta(1), \dots, \theta(|T|))$ respectively. If Sally plays her strategy s, then her potential outcomes are denoted by $u_S(s,t)$, where $t \in T$ is the strategy played by Tom. In order to define cumulative

[5] We will discuss later, in Sec. 2.3, how different framing affects the following considerations.

probabilities, we need to sort these outcomes first, therefore for each of Sally's strategies s let us define permutations π^s on T such that

$$u_S(s, \pi^s(t)) \leq u_S(s, \pi^s(t+1)), \text{ for all } t = 1, \ldots, |T| - 1.$$

Now we can define the cumulative probabilities[6] of Tom's actions as seen from Sally by

$$\begin{aligned} F_S(s, t|\theta) &:= \sum_{l=1}^{t} \theta(\pi^s(l)), \\ F_S(s, 0) &:= 0, \end{aligned}$$

and the CPT-utility then becomes

$$U_S^{CPT}(\sigma, \theta) = \sum_{s=1}^{|S|} \left(\sigma(s) \sum_{t=1}^{|T|} \left(w_S(F_S(s, t|\theta)) - w_S(F_S(s, t-1|\theta)) \right) u_S(s, \pi^s(t)) \right). \tag{5}$$

Essentially, we follow here the formulation of Goeree et al. (2003)[7] and not the formulation which Dekel et al. (1991) proposed, since the latter cannot be extended to CPT which might not be surprising, given that CPT was only introduced a year later (Tversky & Kahneman 1992). We will come back to this point later.

An alternative model which can be defined for arbitrarily many players, is to assume that the players focus on the probability of certain outcomes, rather than on the probability of the opponents' strategies. In this case, the PT-utility would be

$$U_1^{alt}(m) = \sum_{s=(s_1, s_2, \ldots) \in S} m_1(s_1) \cdot w(m_2(s_2) \cdot \ldots \cdot m_n(s_n)) u_1(s), \tag{6}$$

and one could find an analogous but slightly complicated definition for the CPT-case, this time for arbitrarily many players. However, we prefer to assume that players think in strategies rather than only in outcomes and will consequently focus on the first model.

[6] We mention that there are slight differences in the precise definition of CPT in the literature. In the original formation (Tversky & Kahneman 1992), cumulative probabilities have been used in losses, but de-cumulative probabilities in gains. For our analysis, this difference would only make a quantitative difference (i.e. we would just need to work with a slightly different definition of w), but does not change the qualitative results.

[7] More precisely, Goeree et al. (2003) ranks the strategies of the second player uniformly according to the payoffs of the outcomes of the first strategy of the first players. This is not completely according to CPT, but a little bit simpler. We decided to work instead with the precise ranking according to outcomes.

We mention that all these models share the property that they are a sum of terms with the coefficients $m_1(s_1)$, therefore we can rewrite them by abbreviating $m' := (m_2, \dots)$, $s' := (s_2, \dots)$ as

$$U_1(m) = \sum_{s=(s_1,s_2,\dots)\in S} m_1(s_1) \cdot W(m', s') u_1(s), \tag{7}$$

for a certain continuous function W which depends on the precise model and the ranking of the outcomes. This form will be useful in the following considerations.

Let us now come back to the question why our approach is inherently different from the existence results studied by Dekel et al. (1991). On the one hand, our PT-decision model does not assume that a player uses the same decision rules to evaluate the probabilities associated to his and his opponents' strategies, but that he is perfectly rational when evaluating his own probabilities, but weights the (observed) probabilities of his opponent. This corresponds to the fact that in iterative game play, a player would *decide* rather than *observe* his own probability, so no weighting process will take place. In the alternative model (6) we assume, on the other hand, that the player integrates the probabilities correctly before weighting them. This corresponds to a player who observes the probabilities for all possible outcomes simultaneously.

Dekel et al. (1991) on the other hand assumes that also the probabilities of the player itself, i.e. his strategies are weighted: if, e.g., a player plays one of his strategies with a small probability, then he would behave as if he played it with a larger probability. It seems more natural to us to follow the ideas of Goeree et al. (2003) who distinguishes between the probabilities the player decides and the ones that he merely reacts to.

There is another, very practical reason why we cannot follow the approach that Dekel et al. (1991) applied to prospect theory: we want to extend our results to cumulative prospect theory. In order to do so, we need to rank the player's strategies according to outcomes. This is not possible if we simultaneously have to rank the outcomes of more than two players, thus the approach by Dekel et al. (1991) can unfortunately not be extended to CPT.

To let us illustrate this point, consider the following example:

Example 2.1. *Suppose player A faces two opponents B and C who choose among* (b_1, b_2) *and* (c_1, c_2)*, respectively. When choosing action* a_1*, player A obtains payoffs as indicated by the matrix below:*

	c_1	c_2
b_1	*1*	*0*
b_2	*0*	*1*

a_1

To write down player A's CPT-utility from choosing a_1, one needs to rank the outcomes associated with strategy b_1, namely $\binom{1}{0}$, with the outcomes associated with strategy b_2, namely $\binom{0}{1}$, but we cannot rank $\binom{1}{0}$ and $\binom{0}{1}$ independently on the strategy choice of player C.

If one wants to extend probability weighting to the player's own strategies (which has as mentioned certain conceptual problems), then the only alternative would be to give up both of the approaches by Dekel et al. (1991) and Goeree et al. (2003) and to extend the probability weighting in our alternative model (6) also to the player's own strategy. This is certainly possible, but we prefer the (at least to us) more natural approach of (4) and (5).

2.2 General Existence Results

Using the notation of the previous section, we can define our notion of Prospect Theory Nash equilibria:

Definition 2.2. *We call a strategy $\hat{m} \in M$ a* mixed PT-Nash equilibrium *if for all $i = 1, \ldots, n$ and all $m \in M$ with $m_k = \hat{m}_k$ for $k \neq i$ we have $U_i(\hat{m}) \geq U_i(m)$, where $U_i = U_i^{PT}$ is given by (4) or (6).*
Analogously, we say that $\hat{m} \in M$ is a mixed CPT-Nash equilibrium *if for $i = 1, 2$ and all $m \in M$ with $m_k = \hat{m}_k$ for $k \neq i$ we have $U_i(\hat{m}) \geq U_i(m)$, where $U_i = U_i^{CPT}$ is given by (5) or the CPT-variant of (6).*

The following result ensures the existence of such equilibria for fixed reference points. We will consider the case of general frames in Sec. 2.3.

Proposition 2.3. *Let the players' reference points be fixed, then every finite game admits a mixed PT- and a mixed CPT-Nash equilibirum.*

The proof will be given in the appendix.

2.3 Effects of Framing

One might wonder whether it is possible to extend the existence result of Prop. 2.3 to cases with general frames. This is certainly an interesting question, since in many applications the reference point is not fixed, e.g. it might change depending on previous payoffs. But even if we do not study consecutive, but only single payoffs, it is not always clear how the reference point should be chosen. There is e.g. evidence that suggests that in the same real-life decision different people might select different frames ("self-framing"), see Wang & Fischbeck (2004). In the case of games, the choice of the frame might depend on the situation, e.g. on the payoff of other players. But shouldn't all these considerations be irrelevant to obtain existence at least in simple normal form games? It is interesting to see that this is not the case: even in the simplest possible setting of a 2×2 game the framing effect can play a decisive role, as the following example demonstrates, which also shows that Prop. 2.3 can in fact *not* be generalized to arbitrary frames. – The assumption of a fixed frame is indeed necessary to state a general existence result!

Example 2.4. *Let* $[20, 0.5]$ *denote a lottery with 50% chance each to win a monetary amount of* 20 *or nothing. Let the game be given by the following payoff matrix for the two players A and B which have both PT-preferences:*

	t_1	t_2
s_1	*(1,20)*	*(0,20)*
s_2	*(0,10)*	*(1,[20,0.5])*

Let p denote the probability with which Player A plays the strategy s_1 and let q denote the probability with which Player B plays the strategy t_1, and let us assume that both players do not apply probability weighting. Let us imagine Player B knows the value of p. What is his best response? Well, this depends on his framing. Let us assume that $p > 0$, i.e. that there is a chance that Player A plays strategy s_1. Player B might argue now looking on the possible payoffs that a payoff of 20 is likely. It is even sure, when Player A plays s_1. Everything below 20 is therefore considered as a loss. Following this "framing logic", he can compute the subjective utility if he plays t_1 as

$$U_B^{t_1}(p) = u(20) + pu(0) + (1-p)u(-10),$$

and the subjective utility if he plays t_2 as

$$U_B^{t_2}(p) = u(20) + pu(0) + (1-p)\frac{1}{2}\left(u(-20) + u(0)\right).$$

Since Prospect Theory predicts risk-seeking behavior in losses, Player B would prefer the strategy t_2 which reduces his probability to lose the already "nearly" sure 20.
However, if $p = 0$, i.e. Player A only plays strategy s_2, the payoff matrix for Player B looks like

	t_1	t_2
s_2	(0,10)	(1,[20,0.5]).

There is now no reason for him to choose 20 as reference point. Instead, he might more likely choose zero as reference point.[8] The lottery $[20, 0.5]$, however, is then a lottery on gains, and according to Prospect Theory, he

[8]If he chooses 10 as reference point, the following result would still hold, thanks to loss aversion.

will be risk-averse in gains, thus he would prefer strategy t_1. Formally, this can be computed by comparing his new subjective utilities as follows:

$$\begin{aligned} U_B^{t_1}(0) &= u(10), \\ U_B^{t_2}(0) &= \frac{1}{2}\left(u(0) + u(20)\right). \end{aligned}$$

All together, we see that Player B will reply t_1 to $p = 0$ and t_2 to $p > 0$. What is now the best choice for Player A? Let us assume, Player A has a fixed frame and is always using zero as a reference point. Then his utility when choosing $p > 0$ will be

$$U_A(p) = pu(0) + (1 - p)u(1),$$

since Player B will play t_2 for sure. If he chooses $p = 0$, then Player B plays t_1, and U_A, the subjective utility of Player A becomes $U_A(0) = 0$. Therefore U_A does not admit a maximum, in other words: there exists no pure– or mixed strategy Nash equilibrium for this game, given the described framing of the two players! This is in stark contrast to the standard case where Nash equilibria always exist for finite games and underlines the necessity of assuming fixed frames in Prop. 2.3.

Considering this, we will concentrate in this article on situations where the choice of the reference point is fixed, and therefore existence of Nash equilibria is guaranteed.

3 Further Solution Concepts

3.1 Stochastic Dominance and Dominated Strategies in PT

Two our knowledge, Bawa (1975) is the first who uses first order stochastic dominance as a selection criterion for decisions under uncertainty. For two finite lotteries $A \sim \{a_i, \alpha_i\}_{i=1}^n$ and $B \sim \{b_j, \beta_j\}_{j=1}^m$ with outcomes a_i and b_j and probabilities α_i and β_j, lottery A (first order) stochastically dominates lottery B if $Prob(a \leq \bar{x}|A) \leq Prob(b \leq \bar{x}|B)$ for all $\bar{x} \in \mathbb{R}$ with strict inequality for at least one $\bar{x}$. Intuitively, the probability of observing an outcome of at least $\bar{x}$ is higher under lottery A than under lottery B for any value of $\bar{x}$. Bawa (1975) shows that any decision maker whose utility function is increasing and differentiable and who assesses the uncertainty correctly chooses the first order stochastic dominant lottery. While in our setting agents have increasing utility functions, they fail to have an unbiased perception of stochastic environments. Hence, an agent that uses PT does not necessarily prefer a stochastically dominant lottery A over lottery B, as the following classical example illustrates:

Example 3.1 (Stochastic Dominance and Prospect Theory)**.** *For lottery A let the valuation of outcomes be $a_i = 1 - i \cdot \epsilon$ and the probabilities be $\alpha_i = \frac{1}{n}$*

for $i = 1, \ldots, n$ and let B be the outcome b_1 evaluated with 1 and which occurs with probability 1. Then

$$PT(A) = \sum_{i=1}^{n}(1 - i \cdot \epsilon) \cdot w\left(\frac{1}{n}\right) = n \cdot w\left(\frac{1}{n}\right) \cdot \left(1 - \epsilon\frac{n+1}{2}\right)$$

and

$$PT(B) = 1 \ .$$

For any $\epsilon > 0$, lottery B stochastically dominates A. For ϵ small enough, $PT(A) > PT(B)$ as small probabilities are overweighted: $n \cdot w\left(\frac{1}{n}\right) > 1$.

This seemingly artificial result is implied by the inability of a PT-agent to categorize a class of similar outcomes and evaluate the probability of the whole category. In example (3.1), the agent does not categorize all outcomes under lottery A as one event, which would be plausible for ϵ close to zero. In general, there is a bias in favor of the lottery with a larger number of outcomes, because more outcomes imply lower probabilities which are overweighted by PT-agents.

In fact, this examples demonstrates that PT does not only violate stochastic dominance, but even the "in-betweenness axiom", that states that the certainty equivalent of a lottery should be between the smallest and largest possible outcomes. For further discussion of stochastic dominance in the original formulation of PT (that we discuss here) and in the variant by Karmarkar (1978) we refer to Rieger & Wang (2008).

In models of interaction, for each choice s_i the opponent's mixed strategy $m_{-i} \in \times_{j \neq i} \Delta S_j$ induces a lottery on the set of outcomes (s_i, s_{-i}). Note that given m_{-i} two different lotteries, one for strategy $\hat{s}_i$ and one for strategy $\tilde{s}_i$ say, have the same number of outcomes and have equal probability distributions. When choosing among two different lotteries, the agent faces the same distribution on the set of outcomes while he may attach different values to the outcomes.

Suppose now an agent who chooses strategy s_i is better off in the game $(\mathcal{N}, S, u)$ than choosing some strategy $\hat{s}_i$ no matter what the others do. Rapoport (1966) coined the term 'Dominating Strategy'. Formally, strategy $m_i \in M_i$ strictly dominates strategy $\hat{s}_i \in S_i$ if for all $m_{-i} \in M_{-i}$ we have $u_i(m_i, m_{-i}) > u_i(\hat{s}_i, m_{-i})$. The following result follows immediately:

Proposition 3.2. *If m_i strictly dominates $\hat{s}_i$, then*

$$PT_i(m_i, m_{-i}) > PT_i(\hat{s}_i, m_{-i}) \text{ for all } m_{-i} \in \times_{j \neq i} \Delta S_i \ .$$

Proof

$$\begin{aligned} PT_i(s_i, m_{-i}) - PT_i(\hat{s}_i, m_{-i}) \ &= \\ \textstyle\sum_{s_{-i} \in S_{-i}} u_i(m_i, s_{-i}) \cdot w(m_{-i}(s_{-i})) \ &- \ \textstyle\sum_{s_{-i} \in S_{-i}} u_i(\hat{s}_i, s_{-i}) \cdot w(m_{-i}(s_{-i})) \\ &= \ \textstyle\sum_{s_{-i} \in S_{-i}} (u_i(m_i, s_{-i}) - u_i(\hat{s}_i, s_{-i})) \cdot w(m_{-i}(s_{-i})) \\ &> 0 \text{ for all } m_{-i} \in \times_{j \neq i} S_j \ \square \end{aligned}$$

How does strict dominance of strategies relate to stochastic dominance of a lottery? As strict dominance is a dominance relation restricted to lotteries that are induced by the strategy choice in a normal form game it implies 'state dominance' which is a stronger concept than stochastic dominance. Hence strict dominance implies stochastic dominance but not vice versa.

When it comes to Cumulative Prospect Theory, the argument changes slightly while the result remains the same, if the dominating strategy is pure. For each alternative strategy s, Sally ranks the outcomes $\{u_S(s,t)\}_{t\in T}$ such that each successor is larger than its predecessor. This order is captured by the permutation π^s on T. Obviously, the order does not need to be the same for two different strategies $\tilde{s}$ and $\hat{s}$. Therefore, the lottery $\mathcal{L}^{\tilde{s}} = \big((\tilde{s},\pi^{\tilde{s}}), w(\theta(\pi^{\tilde{s}}))\big)$ neither has the same vector of outcomes nor the same vector of probabilities as the lottery $\mathcal{L}^{\hat{s}} = \big((\hat{s},\pi^{\hat{s}}), w(\theta(\pi^{\hat{s}}))\big)$. To prove stochastic dominance we construct two compound lotteries with the same set of outcomes by adding probability zero events. Define the compound order $c\pi^{\tilde{s},\hat{s}} : \{1,\dots,2|T|\} \to \{\tilde{s},\hat{s}\}\times T$ such that $u_S(c\pi^{\tilde{s},\hat{s}}(l)) \le u_S(c\pi^{\tilde{s},\hat{s}}(l+1))$ for all $l = 1,\dots,2|T|-1$. If $c\pi^{\tilde{s},\hat{s}}(l) = (\hat{s},t)$ for some $t\in T$, denote by $k(l)$ the unique index such that $c\pi^{\tilde{s},\hat{s}}(k(l)) = (\tilde{s},t)$. If $c\pi^{\tilde{s},\hat{s}}(l) \notin \{\hat{s}\}\times T$, let $k(l)=\emptyset$. For compound lottery $c\mathcal{L}^s = \big(c\pi^{\tilde{s},\hat{s}}(l), p^s(l)\big)_l$, $s\in\{\tilde{s},\hat{s}\}$ define

$$p^s(l) = \begin{cases} 0 & \text{if } c\pi^{\tilde{s},\hat{s}}(l) \notin \{s\}\times T \\ w(\theta(t)) & \text{if } c\pi^{\tilde{s},\hat{s}}(l) = (s,t)\ . \end{cases}$$

Proposition 3.3. *If $\tilde{s}$ strictly dominates $\hat{s}$, then lottery $\mathcal{L}^{\tilde{s}}$ stochastically dominates lottery $\mathcal{L}^{\hat{s}}$.*

Proof

As $\tilde{s}$ strictly dominates $\hat{s}$ it must hold that $k(l) > l$ for any $l : k(l) \neq \emptyset$. For any l such that $p^{\hat{s}}(l) > 0$ it holds that $p^{\hat{s}}(l) = p^{\tilde{s}}(k(l))$. Hence for any $L = 1,\dots,2|T|$:

$$\sum_{l=1}^{L} p^{\hat{s}}(l) = \sum_{\substack{l=1:\\ k(l)\le L}}^{L} p^{\hat{s}}(l) + \sum_{\substack{l=1:\\ k(l)>L}}^{L} p^{\hat{s}}(l) = \sum_{l=1}^{L} p^{\tilde{s}}(l) + \sum_{\substack{l=1:\\ k(l)>L}}^{L} p^{\hat{s}}(l) \ge \sum_{l=1}^{L} p^{\tilde{s}}(l).$$

For any L such that $p^{\hat{s}}(L) > 0$, the inequality is strict. We conclude that compound lottery $c\mathcal{L}^{\tilde{s}}$ stochastically dominates compound lottery $c\mathcal{L}^{\hat{s}}$. As lotteries $\mathcal{L}^s$ and $c\mathcal{L}^s$ differ only with respect to events with probability zero, $\mathcal{L}^{\tilde{s}}$ stochastically dominates $\mathcal{L}^{\hat{s}}$. □

Corollary 3.4 (strict dominance of pure strategies)**.** *If $\tilde{s}$ strictly dominates $\hat{s}$, then $U_S^{CPT}(\tilde{s},\theta) > U_S^{CPT}(\hat{s},\theta)$ for all $\theta \in \Delta(T)$.*

The next simple example shows that this result does not hold for mixed domination.

Example 3.5. *Let the payoffs of the row player be given*

	t_1	t_2
s_1	$2+\epsilon$	0
s_2	0	$2+\epsilon$
s_3	1	1

by the matrix above with $\epsilon > 0$. Given any mixed strategy (θ_1, θ_2), an expected utility maximizer gains $\frac{2+\epsilon}{2}$ from choosing the mixed strategy $\sigma := (1/2, 1/2, 0)$ (i.e. playing s_1 and s_2 each with probability $1/2$) and payoff 1 from choosing s_3. For any $\epsilon > 0$, σ strictly dominates s_3. For CPT-agents, payoffs are $U_S^{CPT}(s_1, \theta) = (1 - w(\theta_2)) \cdot (2+\epsilon)$ and $U_S^{CPT}(s_2, \theta) = (1 - w(\theta_1)) \cdot (2+\epsilon)$; the payoff from palying the mixed strategy σ is therefore

$$U_S^{CPT}(\sigma, \theta) = (2+\epsilon) \cdot \left(1 - \frac{1}{2}(w(\theta_1) + w(\theta_2))\right).$$

On the other hand,

$$U_S^{CPT}(s_3, \theta) = w(\theta_1) \cdot 1 + (1 - w(\theta_1)) \cdot 1 = 1.$$

Consider now a probability wheighting function with $w\left(\frac{1}{7}\right) > \frac{12}{35} (> \frac{1}{7})$ and $w\left(\frac{6}{7}\right) > \frac{23}{35} (< \frac{6}{7})$. Clearly, $w\left(\frac{1}{7}\right) + w\left(\frac{6}{7}\right) > 1$. For ϵ small enough, $U_S^{CPT}\left(s_3, \left(\frac{1}{7}, \frac{6}{7}\right)\right) > U_S^{CPT}\left(\sigma, \left(\frac{1}{7}, \frac{6}{7}\right)\right)$, contradicting dominance.

Knowing that PT-agents abstain from choosing purely dominated strategies, we can push this logic further to iterated domination. If an agent believes his opponent to be at least PT-rational, he can consider the game without any strategy that is dominated for his opponent. In other words dominated strategies can be eliminated from the game. Now in the new game some strategies can be dominated that have not been dominated before. This process of iterated elimination of dominated strategies will end after finitely many steps as the strategy sets are finite. If only a single strategy profile is left, the game is called dominance solvable.

Corollary 3.6 (Pure iterated domination)**.** *PT-agents will not choose strategies that are iteratively strictly dominated. In particular, if the game is dominance solvable, PT-agents choose the unique strategy that is iteratively strictly dominant.*

3.2 Best Replies and Nash equilibria in PT

In a game $(\mathcal{N}, S, u)$, a best reply to the belief $m_{-i} \in \Delta(S_{-i})$ is the strategy that maximizes the expected payoffs:

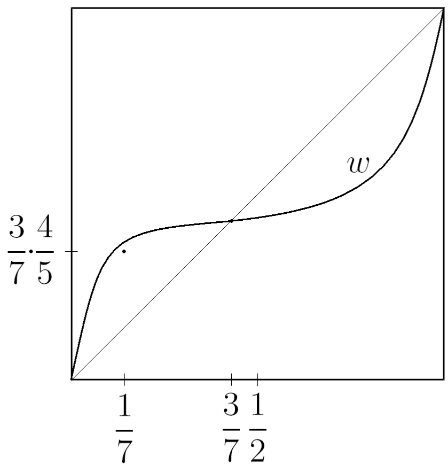
$$BR_i(m_{-i}) = \arg\max_{s_i \in S_i} \sum_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}) m_{-i}(s_{-i})$$

For agents that do not maximize expected utility, the set of best replies can differ. The next example illustrates the role of the probability wheighting function in changing the set of best replies.

Example 3.7. Non-best reply caused by probability wheighting *Consider a symmetric coordination game with three strategies and payoffs*

	t_1	t_2	t_3
s_1	(10,10)	(1,0)	(1,0)
s_2	(0,1)	(10,10)	(0,0)
s_3	(0,1)	(0,0)	(10,10)

For a weighting function w *with* $w(\frac{3}{7}) = \frac{3}{7}$ *and* $w(\frac{1}{7}) > \frac{3}{7} \cdot \frac{4}{5}$ *as depicted below, consider a mixed strategy* $\theta(t_1) = \frac{1}{7}$ *and* $\theta(t_2) = \theta(t_3) = \frac{3}{7}$.



Then $BR(\theta) = \{s_2, s_3\}$ *but* $U_S^{PT}(s_1, \theta) = 10 \cdot w(\frac{1}{7}) + w(\frac{3}{7}) + w(\frac{3}{7}) > 10 \cdot \frac{3}{7} \cdot \frac{4}{5} + \frac{6}{7} = 10\frac{3}{7} = U_S^{PT}(s_2, \theta) = U_S^{PT}(s_3, \theta)$.

(Counter-) example 3.7 demonstrates that the best reply correspondences do not coincide for expected utility maximizers and PT-agents.

As the best response correspondences do not need to coincide for EUT- and PT-agents, one is inclined to be sceptical with respect to the equivalence of "never best responses". A strategy $\tilde{s}$ is a *never best response* if for any belief m there is another strategy s (possibly mixed and different for each belief) such that the agent does not prefer $\tilde{s}$ over s. Indeed the example in section 3 in which strategy s_3 is preferred over mixed (and EUT-dominant) strategy m illustrates that the concept never best response points at different sets of strategies for EUT and PT.

Despite this non-invariance we can state the following:

Lemma 3.8. *The following three statements are equivalent:*

(i) $s_i \in S_i$ *is a Best Reply against pure strategies* $s_{-i} \in S_{-i}$.

(ii) $U_i^{PT}(s_i, s_{-i}) \geq U_i^{PT}(\tilde{s}_i, s_{-i})$ for all $\tilde{s}_i$.

(iii) $U_i^{CPT}(s_i, s_{-i}) \geq U_i^{CPT}(s_i, s_{-i})$ for all $\tilde{s}_i$.

Proof
Let $u_i(s_i, s_{-i}) \geq u_i(\tilde{s}_i, s_{-i})$. This is equivalent to

$$U_i^{PT}(s_i, s_{-i}) = U_i^{CPT}(s_i, s_{-i}) = u_i(s_i, s_{-i}) \geq$$

$$u_i(\tilde{s}_i, s_{-i}) = U_i^{PT}(\tilde{s}_i, s_{-i}) = U_i^{CPT}(\tilde{s}_i, s_{-i}) \text{ for all } \tilde{s}_i \in S_i \ . \ \square$$

This trivial statement allows us to conclude that the Nash equilibrium solution restricted to pure strategy spaces defines the same outcomes for expected utility maximizers, PT- and CPT-agents.

Corollary 3.9. *A pure strategy* $s \in S$ *is a Nash equilibrium in the game* $(\mathcal{N}, S, u)$ *if and only if it is a Nash equilibrium for PT-agents and CPT-agents.*

4 Conclusions

We analyze decisions of agents who use Prospect Theory or Cumulative Prospect Theory (Kahneman & Tversky 1979, Tversky & Kahneman 1992) when they face strategic interaction. These agents differ from traditional expected utility maximizers with respect to two dimensions of irrationality. Firstly, they are risk averse in gains and risk loving in losses. Secondly, those agents overestimate small probabilities and underestimate large probabilities.
(Cumulative) Prospect Theory describes behavior under uncertainty which usually is modeled as a choice among various exogenous lotteries. In our strategic setting – normal form games – a lottery is imposed by the potentially mixed expectation on the choice of the strategic opponents. If not the choice itself, its expectation certainly fails to be independent of the own decision. Hence, when analyzing solution concepts of game theory we have to admit endogenous lotteries. In this setting, we analyze irrational choices in the sense of Expected Utility Theory and identify the effects caused by probability misestimation.

An immediate finding is that pure best replies are equivalent for EUT- and (C)PT-agents. This implies that purely dominant strategies or pure Nash

Equilibria are invariant with respect to any monotone value function or probability weighting function.
When it comes to mixed strategies, less properties carry over from Expected Utility Theory to (Cumulative) Prospect Theory. We give examples in which the set of best replies to some beliefs is not invariant with respect to the probability misestimation. While a dominated strategy is dominated for agents who maximize according to Prospect Theory in any case, this does not need to be the case for agents who apply Cumulative Prospect Theory, if the dominating strategy is mixed.

A Proofs

Proof of Proposition 2.3:
The proof follows very closely the one given in Jehle & Reny (2001) in the context of classical game theory. For the convenience of the reader we present nevertheless the main steps.
For ease of notation we denote $(\hat{m}_1, \dots, \hat{m}_{i-1}, m_i, \hat{m}_{i+1}, \dots)$ by $(m_i, \hat{m}_{-i})$. We define $m_{ij} := m_i(s_j)$,

$$f_{ij}(m) := \frac{m_{ij} + \max\left(0, U_i(m_j, m_{-i}) - U_i(m)\right)}{1 + \sum_{j'=1}^{n} \max\big(0, U_i(m_{j'}, m_{-i}) - U_i(m)\big)}$$

and

$$f(m) := \begin{pmatrix} f_{11}(m) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ f_{1n}(m) & \cdots & f_{nn}(m) \end{pmatrix}.$$

The function $f\colon M \to M$ is continuous. Since M is compact, Brouwer's Fixed Point Theorem implies that there exists some $\hat{m} \in M$ with $f(\hat{m}) = \hat{m}$. This is our candidate for a Nash equilibirum. We reformulate the fixed point equation and arrive at

$$\hat{m}_{ij} \sum_{j'=1}^{n} \max\big(0, U_i(\hat{m}_{j'}, \hat{m}_{-i}) - U_i(\hat{m})\big) = \max\left(0, U_i(\hat{m}_j, \hat{m}_{-i}) - U_i(\hat{m})\right).$$

Summing over all strategies j, we conclude that

$$\begin{aligned} & \sum_{j=1}^{n} \hat{m}_{ij}(U_i(\hat{m}_j, \hat{m}_{-i}) - U_i(\hat{m})) \sum_{j'=1}^{n} \max\big(0, U_i(\hat{m}_{j'}, \hat{m}_{-i}) - U_i(\hat{m})\big) \\ = & \sum_{j=1}^{n} \hat{m}_{ij}(U_i(\hat{m}_j, \hat{m}_{-i}) - U_i(\hat{m})) \max\left(0, U_i(\hat{m}_j, \hat{m}_{-i}) - U_i(\hat{m})\right). \end{aligned} \tag{8}$$

We compute

$$\sum_{j=1}^{n} \hat{m}_{ij}(U_i(\hat{m}_j, \hat{m}_{-i}) - U_i(\hat{m})) = \left(\sum_{j=1}^{n} \hat{m}_{ij} U_i(\hat{m}_j, \hat{m}_{-i}) \right) - U_i(\hat{m})$$
$$= U_i(\hat{m}) - U_i(\hat{m}) = 0,$$

which follows from the fact that PT- and CPT-utility can be written such that they are linear combinations with coefficients $\hat{m}_{ij}$, compare (7).
Plugging this identity into (8), we arrive at

$$\sum_{j=1}^{n} \hat{m}_{ij}(U_i(\hat{m}_j, \hat{m}_{-i}) - U_i(\hat{m})) \max(0, U_i(\hat{m}_j, \hat{m}_{-i}) - U_i(\hat{m})) = 0.$$

From this we derive $U_i(\hat{m}_j, \hat{m}_{-i}) - U_i(\hat{m}) \leq 0$ for all j, in other words, no pure strategy is better than the mixed strategy $\hat{m}$. We use again the representation (7) and deduce that $\hat{m}$ is a PT- respectively CPT-Nash equilibrium. □

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References

Bawa, V. (1975), 'Optimal rules for ordering uncertain prospects', *Journal of Financial Economics* **2**(1), 95–121.

Bernoulli, D. (1738), *Specimen Theoriae de Mensura Sortis*, Commentarii Academiae Scientiarum Imperialis Petropolitanae (Proceedings of the royal academy of science, St. Petersburg).

Camerer, C. F. (2003), *Behavioral Game Theory: Experiments in Strategic Interaction*, Princeton University Press.

Chen, H.-C. & Neilson, W. S. (1999), 'Pure-strategy equilibria with non-expected utility players', *Theory and Decision* **46**, 199–209.

Dekel, E., Safra, Z. & Segal, U. (1991), 'Existence and dynamic consistency of Nash equilibirum with non-expected utility preferences', *Journal of Economic Theory* **55**, 229–246.

Fershtman, C., Safra, Z. & Vincent, D. (1991), 'Delayed agreements and nonexpected utility', *Games and economic behavior* **3**, 423–437.

Goeree, J. K., Holt, C. A. & Palfrey, T. R. (2003), 'Risk averse behavior in generalized matching pennies games', *Games and Economic Behavior* **45**(1), 97–113.

Jehle, G. A. & Reny, P. J. (2001), *Advanced Microeconomic Theory*, Addison Wesley.

Kahneman, D. & Tversky, A. (1979), 'Prospect Theory: An analysis of decision under risk', *Econometrica* **47**, 263–291.

Karmarkar, U. (1978), 'Subjectively weighted utility: A descriptive extension of the expected utility model', *Organizational Behavior and Human Performance* **21**(1), 69–72.

Quiggin, J. (1982), 'A theory of anticipated utility', *Journal of Economic Behavior & Organization* **3**(4), 323–343.

Rapoport, A. (1966), *Two-person game theory: The essential ideas*, University of Michigan Press.

Rieger, M. O. & Wang, M. (2006), 'Cumulative Prospect Theory and the St. Petersburg paradox', *Economic Theory* **28**, 665–679.

Rieger, M. O. & Wang, M. (2008), 'Prospect Theory for continuous distributions', *Journal of Risk and Uncertainty* **36**, 83–102.

Schneider, S. L. & Lopes, L. L. (1986), 'Reflection in preferences under risk: who and when may suggest why', *Journal of Experimental Psychology: Human Perception and Performance* **12**, 535–548.

Tversky, A. & Kahneman, D. (1992), 'Advances in Prospect Theory: Cumulative representation of uncertainty', *Journal of Risk and Uncertainty* **5**, 297–323.

v. Neumann, J. (1928), 'Zur Theorie der Gesellschaftsspiele', *Mathematische Annalen* **100**(1), 295–320.

v. Neumann, J. & Morgenstern, O. (1944), 'Theory of games and economic behavior', *Princeton University Press* .

Wakker, P. P. (1989), Transforming probabilities without violating stochastic dominance, *in* 'Mathematical Psychology in Progress', Springer, Berlin, pp. 29–47.

Wang, M. & Fischbeck, P. S. (2004), 'Incorporating framing into prospect theory modeling: A mixture-model approach', *Journal of Risk and Uncertainty* **29**(2), 181–197.