Market Belief, Trading Volume, Price Volatility, and Liquidity

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First version: June 2010
Current version: June 2010

This research has been carried out within the NCCR FINRISK project on “Credit Risk and Non-Standard Sources of Risk in Finance”
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June 30, 2010

Abstract

This paper studies the effects of investors’ heterogeneous beliefs on the trading volume, price volatility, and liquidity of stocks. Following Kurz and Motolese (2008), we propose a simple theoretical model to show that the equilibrium stock price is linearly and positively correlated with market belief (investors’ average belief) about stock future payoffs. Further, it is shown in this paper that when market belief is more volatile, trading volume and liquidity decline while price volatility increases. Using the analyst forecast data on quarterly earnings per share provided by the Institutional Brokers’ Estimate System, we obtain empirical results which support the theoretically predicted relations between the volatility of market belief about stock future payoffs and the trading volume, price volatility, and liquidity of underlying stocks, and these empirical results are robust to various methods of estimating market belief and its volatility and to alternative illiquidity measures.

JEL codes: G12, G17, G32.

Keywords: Heterogeneous Beliefs, Market Belief, Analysts’ Forecasts, Trading Volume, Price Volatility, Liquidity

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1 Introduction

Economic agents differ in their endowments, preferences and beliefs. Despite these differences and despite strong and persuasive arguments put forward for including heterogeneous agents in finance and macroeconomics, the representative agent paradigm with identical agents is still the leading structural approach to asset pricing, and standard asset pricing models such as the capital asset pricing model (CAPM) and the consumption-based CAPM (CCAPM) developed in the representative agent paradigm, given their empirical tractability, have generated extensive empirical tests and subsequent theoretical extensions. But, as mentioned by Williams (1977), “difficulties remain, significant among which is the restrictive assumption of homogeneous agents”.

In this paper, we focus on the heterogeneity of beliefs. The heterogeneity of beliefs capture how individual agents interpret commonly observed information differently. In reality, agents often receive common information, but the ways in which they interpret the information are different, and each agent only believes in the validity of his or her interpretation. For example, financial analysts and macroeconomists make divergent forecasts about future movements of earnings per share, interest rates, exchange rates, and gross national products despite the fact that all of them have access to the same information set, Kandel and Pearson (1995) find that identical interpretation of information seems to be not consistent with the empirical data. Models with agents who have heterogeneous beliefs have been previously studied by Harrison and Kreps (1978), Varian (1985), De Long et al. (1990), Harris and Raviv (1993), Detemple and Murthy (1994), Kurz (1994), Kandel and Pearson (1995), Zapatero (1998), Basak (2000), Scheinkman and Xiong (2003), Berrada (2006), Li (2007), Kurz and Motolesi (2008). It is by now well recognized that heterogenous beliefs across agents play an important role in the formation of asset prices and their dynamics and in the generation of trades among agents. Empirical works also strongly support this fact. For example, Buraschi and Jiltsov (2002) find that differences of opinion can help explain the dynamics of option trading volume, while Pavlova and Rigobon (2003) provide empirical support for a model of international stock prices and exchange rates with heterogeneity in beliefs.

The primary objective of this paper is to address the issue of how investors’ heterogeneous beliefs affect the trading volume, price volatility, and liquidity of (common) stocks in a pure exchange economy. The existing literature related to stock trading volume and price volatility is mostly casted within the framework of rational expectation models with asymmetric information. These models generate disagreement through private information, and involve trading among privately informed traders, uninformed traders, and noise traders (see, e.g., Grossman and Stiglitz (1976), Pfleiderer (1984), Kyle (1985), Admati and Pfleiderer (1988), Wang (1992), Foster and Viswanathan (1993)). The trading by noise traders or the stochastic supply of stock causes stock prices to be more volatile than the dividends (see, e.g., Wang
(1993), Campbell and Kyle (1993)). In an overlapping generation, symmetric information
model, Spiegel (98) shows that there is an equilibrium in which stock prices exhibit excessive
volatility. However, these models place severe restrictions on the circumstances under which
speculative trade can occur, and trade is generally not generated by public information sig-
nals, which is not consistent with the empirical finding by, for example, Kandel and Pearson

Previous works on heterogeneous beliefs show that models with agents who have hetero-
geneous beliefs are able to generate empirical patterns in stock trading volume and excessive
volatility. Harris and Raviv (1993) develop a model of trading in speculative markets based
on announcements of public information by assuming that there are two types of risk neutral,
speculative traders with a common prior belief about the returns to a particular asset. When
information about the asset becomes available, each type of trader updates his belief about
returns using his own model (or likelihood function) of the relationship between the news and
the asset’s returns. These authors show that the approach they employ can help to explain
some of the empirical regularities concerning the relationship between volume and price and
the time-series properties of volume and price. Kandel and Pearson (1995) consider a similar
model except that traders in their model have both different prior beliefs and different like-
lihood functions and are risk averse. They show that volume and absolute price change are
positively correlated and volume can be positive even when price does not change. Zapatero
(1998) studies the effects of financial innovations on the volatility of interest rates in a pure
exchange economy with two log-arithmetic utility maximizer agents who observe aggregate
consumption but disagree about its expected rate of change. He shows that financial innova-
tions lead to an increase in the volatility of interest rates, but the volatility of stock is equal
to that of its dividend. Li (2007) focuses on “investors who have heterogeneous beliefs about
the structure of a dividend process”. He demonstrates that stock price volatility depends on
the difference of beliefs and the wealth distribution, and the dependence enables the model
to produce several well-known stylized facts such as excessive volatility and positive relations
between price and trading volume and between volatility and volume.

Although the focus of this paper is the heterogeneity of beliefs, the approach that it uses
to model heterogeneous beliefs and their effects on stock trading volume and price volatility is
generally different from those employed by other authors. First, we take heterogeneous beliefs
as exogenously given and will not study where the subjective beliefs of investors come from.
Jouini and Napp (2007) explain that “the models with learning as in Detemple and Murthy
(1994) and Basak (2000) are not more endogenous given that the investor updating rule and
the corresponding probabilities can be determined separately from his or her optimization
problem”. Second, this paper focuses on market belief (i.e., investors’ average belief) as well
as its volatility and how they can affect stock trading volume and price volatility, while most
of other works on heterogeneous beliefs study belief dispersion across investors, this is one of main contribution of this paper.

Liquidity is an important research topic in the finance literature, and market microstructure theory suggests that there are two potential affecting factors on stock liquidity, namely, inventory risk and/or information asymmetry (see, e.g., Garman (1976), Ho and Stoll (1981), Glosten and Milgrom (1985), Kyle (1985), etc). This paper shows that even when there are no problems of inventory risk and information asymmetry, liquidity problem can still occur in financial markets as a consequence of heterogeneous beliefs across investors, this issue has never been studied before.

This paper is primarily built upon Kurz and Motoles (2008). Consider an economy where the true probability measure of sequence of stock payoffs is neither stationary nor observable. The nonstationarity of economic systems can cause the sequence of stock payoffs to follow a nonstationary process which is also impossible to be identified by rational investors due to limited data. Historical data can be used to infer a unique stationary empirical measure, but most investors would not believe that such an empirical measure is adequate to forecast stock future payoffs. Instead, investors will hold their own beliefs when they make investment decisions. For each investor, the transition function of his expectation about next period economic variables is uniquely pinned down by his own belief. The equilibrium can be solved by treating investors’ individual beliefs and its average as state variables.

In the equilibrium, the stock price is linearly and positively correlated with market belief. When investors’ average belief about stock future payoffs is higher than the econometrician’s belief\(^1\), they will value the stock more aggressively and the equilibrium stock price will appear “expensive” to the econometrician. This result is similar to those obtained by Detemple and Murthy (1994), Basak (2000), and Xiong and Yan (2009). For example, Xiong and Yan (2009) find that bond prices aggregate investors’ heterogeneous beliefs and particularly reflect their wealth-weighted average belief about future short rates. A trivial but important result following the market belief related stock price is that stock price volatility is positively correlated with the innovations to market belief, and thus can be higher than that of its payoffs.

The optimal stock demand of each investor is a linear increasing function of the difference between his own belief and market belief, that is, the more optimistic than the market he is, the higher stock demand he has. The amount of stock shares he would trade is determined by changes in the belief difference, and its expectation decreases with the volatility of market belief. In this model, trading volume is independent of belief dispersion, this result is different from previous findings that trading volume increases with belief dispersion (e.g., Chordia et al. (2007)).

We further show that stock liquidity, measured by the absolute value of stock transaction

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\(^1\)The econometrician’s belief is derived with empirical data using statistical models. Refer to Section 3.2 for the details.
price minus its fundamental value, declines with the volatility of market belief. Intuitively, more volatile market belief will lead to more volatile stock price since stock price is a linear function of market belief, and thus to a larger deviation of stock transaction price from its fundamental value.

Although the theoretical model in this paper is similar to that one developed by Kurz and Motolese (2008), the objectives of these two papers are different: Kurz and Motolese (2008) intend to use market belief premia to explain the “equity premium puzzle” documented first by Mehra and Prescott (1985) while we are interested in how investors’ heterogeneous beliefs affect the trading volume, price volatility, and liquidity of stocks.

Empirically, we use the analyst forecast data on quarterly earnings per share provided by the Institutional Brokers’ Estimate System to examine in the time-series context whether the theoretically predicted relations between the volatility of market belief about stock future payoffs and the trading volume, price volatility, and liquidity of underlying stocks hold in the real financial markets. Analyst’s individual belief is estimated as the difference between his earnings forecast and the forecast made with historical earnings data, and market belief is defined as the average of individual beliefs across analysts. The volatility of market belief can be estimated by using both a rolling regression method and a GARCH model. The empirical findings are convincing: for a majority of the stocks in the sample consisting of 114 S&P500 Index stocks, when market belief is more volatile, trading volume and liquidity decline while price volatility increases. Precisely, price volatility is positively correlated with the volatility of market belief for 75%-80% of the stocks, and the correlation is significant for more than half of the stocks. The negative correlation between the volatility of market belief and stock liquidity measured by the Roll’s (1984) effective spread are pronounced for about two thirds of the stocks and is significant for about 35% of the stocks. In both the price volatility and liquidity cases, the explanatory power of the volatility of market belief is impressive. The trading volumes of 60%-70% of the stocks tend to decline with the volatility of market beliefs, it is worth noting that this negative correlation is significant only for few of sample stocks, suggesting that there should be a large component of noise and/or other influences in stock trading volume. All these empirical results are robust to alternative methods of estimating market belief and its volatility and to alternative illiquidity measures.

The remaining of this paper is organized as follows: Section 2 develops a simple theoretical model to study how investors’ heterogeneous beliefs affect the trading volume, price volatility, and liquidity of stocks. Empirical results are presented in Section 3. Section 4 concludes this paper with further discussions and comments.
2 The Model

We adopt the model of heterogeneous beliefs developed by Kurz and Motolesi (2008). In the model, the true probability measure governing the generation of the exogenous risky sequence of stock payoffs is assumed to be nonstationary and unobservable. Although historical data can be used to infer a unique stationary probability measure, most investors will not believe that such an empirical measure is adequate to forecast stock future payoffs, and will hold their own beliefs about the true stock payoffs generating probability measure when making investment decisions. Each investor optimizes consumption and investment decisions based on his own belief. Market clearing conditions determine the equilibrium stock price.

2.1 The Setting

Consider an infinite horizon exchange economy where a single stock is traded with a market price $p_t$ and delivers an exogenous risky sequence of payoffs \{\(D_t, \ t = 1, 2, \cdots\)\} under a true probability measure \(\hat{\Pi}\). For simplicity, the total supply of stock shares is normalized to one. In this economy, a key assumption is that investors do not know the true \(\hat{\Pi}\) of the process \{\(D_t, \ t = 1, 2, \cdots\)\}, and they hold heterogeneous beliefs about \(D_t\). Instead, the historical data on a set of observable variables including \(D_t\) is known to all investors and plays the role of the common knowledge basis of investors with heterogeneous beliefs. With a long history of observations of these variables, all investors compute the same empirical moments and the same finite dimensional distributions of the observed variables. Using the standard extension of measures they deduce from the data a unique empirical probability measure denoted by \(\hat{\Pi}\) on \{\(D_t, \ t = 1, 2, \cdots\)\}. Kurz (1994) has shown that the probability measure \(\hat{\Pi}\) is stationary.

This is the empirical knowledge shared by all investors. Furthermore, we also assume that the data reveals that, under the probability measure \(\hat{\Pi}\), \{\(D_t, t = 1, 2, \cdots\)\} constitutes a first order Markov process where \(D_{t+1}\) is conditionally normally distributed with mean \(\mu + \lambda_d (D_t - \mu)\) and variance \(\sigma_d^2\). To simplify, let’s define \(d_t = D_t - \mu\), the sequence \{\(d_t, t = 1, 2, \cdots\)\} is hence zero mean with an unknown true probability measure \(\Pi\) and a unique empirical probability measure \(\hat{\mu}\). The assumption that \{\(D_t, t = 1, 2, \cdots\)\} constitutes a first order Markov process implies that, under the empirical probability measure \(\hat{\mu}\), the dynamics of \{\(d_t, t = 1, 2, \cdots\)\} is also characterized by a first order Markov process with transition function

\[
d_{t+1} = \lambda_d d_t + \epsilon^d_{t+1}, \tag{1}
\]

where \(\epsilon^d_{t+1} \sim N(0, \sigma_d^2)\). It is clear that \(E_{t}^{\hat{\mu}} (d_{t+1}) = \lambda_d d_t\).

It is well known that our society has been undergoing changes in production technology and firm organization structure, which are rapid with important economic effects, making the sequence \{\(d_t, t = 1, 2, \cdots\)\} to be a nonstationary process. This implies that the distribution of
$d_t$ is time dependent, such variability makes it almost impossible to learn the true probability measure, this is why investors do not know the $\Pi$.

We assume that there are $N$ investors in the economy. At date $t$, investor $i$ buys $\theta^i_t$ shares of stock which will pay him $d_{t+1} + \mu$ at date $t+1$ for each unit of stock. The risk free interest rate $R = 1 + r_t$ is assumed to be constant over time such that there exists a riskless asset by which an investor can invest the amount $B_t$ at date $t$ and receive with certainty the amount $B_tR$ at date $t + 1$. The consumption of investor $i$ at date $t$ is set equal to the income earned from the portfolio $(\theta^i_{t-1}, B^i_{t-1})$ held from date $t - 1$ to $t$ minus the cost spent for the portfolio $(\theta^i_t, B^i_t)$ built at date $t$, and it takes the following form

$$c^i_t = \theta^i_{t-1}(p_t + d_t + \mu) + B^i_{t-1}R - \theta^i_tp_t - B^i_t$$

and his wealth at date $t$ is defined as

$$W^i_t = c^i_t + \theta^i_t p_t + B^i_t$$

and hence the transition formula of wealth takes a form as follows

$$W^i_{t+1} = (W^i_t - c^i_{t-1})R + \theta^i_{t-1}Q_t,$$  \hspace{1cm} (2)

where $Q_t = p_t + (d_t + \mu) - Rp_{t-1}$. Given some initial values $(\theta^i_0, W^i_0)$ and with an exponential utility function, investor $i$ will choose $(\theta^i, c^i)$ to maximize the expected utility

$$U = E^i_t \left[ \sum_{s=0}^{\infty} -\beta^{t+s-1}e^{-\frac{1}{2}c^i_{t+s}}|\mathcal{F}_t} \right]$$  \hspace{1cm} (3)

subject to a vector of state variables $\psi^i_t$ and their transition functions which will be specified below. $\beta$ is a time discount factor and $\mathcal{F}_t$ is a $\sigma$-field generated by the information available up to time $t$.

### 2.2 The Equilibrium without Heterogeneous Beliefs

Before solving the model with heterogeneous beliefs in the next subsections, let’s first consider the case in which investors’ beliefs are homogeneous, and all investors only use the common information embedded in the empirical data of $d_t$ to make their investment decisions.

Assume that $R > 1$ and $0 < \lambda_d < 1$ hold, the method developed by Blanchard and Kahn (1980) can be adopted to solve the equilibrium\(^4\). The optimal stock demand of each investor

\(^2\)Like Wang (1994), we call $Q_{t+1}$ the excess share return since it is the excess return on one share of stock instead of the excess return on one dollar invested in the stock.

\(^3\)Following the idea in the section 1.5.2 of Kurz and Motoles (2008), we can show that many results in this paper are general and do not depend on the choice of specific utility function.

\(^4\)To ensure the existence of the equilibrium of a linear difference model, it is required that the exogenous variable, here $d_t$, does not exponentially grow, $d_t$ satisfies this condition when $0 < \lambda_d < 1$. 

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and the equilibrium stock price are stated in the following proposition:

**PROPOSITION I:** Given the empirical transition equation (1) for $d_t$ and the constrained conditions $R > 1$ and $0 < \lambda_d < 1$, the optimal stock demand is identical across investors

$$\theta_i^* = \frac{R\tau}{r\sigma^2_Q} [E^m_t (Q_{t+1}) + u_0^* + u_1^* d_t], \tag{4}$$

where $[u_0^*, u_1^*] = -\bar{b}\tilde{\Omega}\tilde{\Lambda}_0$, and $\tilde{\sigma}^2_Q = \bar{b}^2\tilde{\Omega}$ is the adjusted conditional variance of the excess share return $Q_{t+1}$. $\bar{b}$, $\tilde{\Omega}$ and $\tilde{\Lambda}_0$ are defined as follows

$$\bar{b} = a_d^* + 1, \quad \tilde{\Omega} = \sigma_d^2/\left[1 + \sigma_d^2\tilde{v}_{11}\right], \quad \tilde{\Lambda}_0 = [\tilde{v}_{01}, \lambda_d\tilde{v}_{11}], \quad \tilde{V} = [\tilde{v}_{00}, \tilde{v}_{01}; \tilde{v}_{01}, \tilde{v}_{11}],$$

where $\tilde{V}$ is determined in the following equation

$$\frac{\tilde{M}}{R} - \tilde{V} + 2 \left[\gamma\tilde{C} + \ln \left(\frac{R}{\tilde{R}}\right)\right] \tilde{\iota}_{11} = 0,$$

$$\tilde{M} = \frac{1}{b\tilde{\Omega}} \left[\tilde{a}^T - \bar{b}\tilde{\Omega}\tilde{\Lambda}_0\right]^T \left[\tilde{a}^T - \bar{b}\tilde{\Omega}\tilde{\Lambda}_0\right] + \left[\tilde{\Lambda}_0^T \tilde{V} \tilde{\Lambda}_0 - \tilde{\Lambda}_0^T \tilde{\Omega} \tilde{\Lambda}_0\right],$$

where $\tilde{C} = \frac{\ln(\gamma\bar{b}\tilde{G})}{R\tilde{R}}$, $\tilde{a} = [P_0^* (1 - R) + \mu, (a_d^* + 1) \lambda_d - a_d^* R]^T$, $\tilde{\Lambda}_0 = [1; 0; 0; \lambda_d]$, $\tilde{G} = (1 + \sigma_d^2\tilde{v}_{11})^{-\frac{1}{2}}$ and $\tilde{\iota}_{11}$ is a $2 \times 2$ matrix with the element $(1, 1)$ being one and other elements being zero. The equilibrium stock price is unique and takes a form as follows

$$p_t^* = P_0^* + a_d^* d_t, \tag{5}$$

where the constant term $P_0^*$ and the coefficients $a_d^*$ are defined as follows

$$P_0^* = \frac{\mu + u_0^*}{R - 1} - \frac{\tilde{\sigma}^2_Q}{RN\tau}, \quad a_d^* = \frac{\lambda_d + u_1^*}{R - \lambda_d}$$

**Proof:** This proposition can be proved as a special case of the appendix A.1. ||

It is obvious that when investors’ beliefs are homogeneous, the optimal demand of stock is identical across investors. The demand can be divided into two parts: one is the investment demand driven by the expected excess share return $E^m_t (Q_{t+1})$ and another is driven through hedging against the fluctuations of $d_t$, the coefficients $u_0^*$ and $u_1^*$ determine the magnitude of hedging demands. It is shown in Figure 1 that the values of $u_0^*$ and $u_1^*$ for the relevant values of the model parameters are close to zero and therefore negligible. To insert the equilibrium stock price $p_t^*$ (5) into the demand function (4) and to simplify lead to $\theta_i^* = \frac{1}{N}$, that is, when investors’ beliefs are homogeneous, each investor will constantly demand $\frac{1}{N}$ of the total stock supply over time, and no trading will occur among investors except in the first period when the initial stock endowments of investors may be different from $\frac{1}{N}$.  

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The equilibrium stock price $p_t^*$ derived in this case consists of two components: the first component is $P_0^*$ that can be further divided into two parts: the first part $\frac{u^0_u}{R - 1}$ approximates the present value of the expected total future cash flows discounted at the risk free interest rate since $u^*_0$ is close to zero, and the second part $-\frac{\hat{\alpha}_Q}{RN\tau}$ represents the discount on the stock price to compensate for the risk in future payoffs, and the discount decreases with investor’s risk tolerance $\tau$ and the number of investors $N$ and increases with the adjusted conditional variance $\hat{\sigma}_Q^2$ of the excess share return $Q_{t+1}$, these results are intuitive; the second component of $p_t^*$ depends on $d_t$. The effect of $d_t$ on $p_t^*$ is determined by the coefficients $a_k^*_0$. As we can see above, the coefficient $u^*_1$ is almost equal to zero, therefore the coefficient $a_k^*_0$ is always strictly positive as shown in Figure 1, as a result, $p_t^*$ increases with $d_t$.

2.3 Modeling Heterogeneous Beliefs

Most investors will not believe that the empirical stationary model (1) is adequate to forecast the future since $\{d_t, t = 1, 2, \cdots\}$ is nonstationary with an unknown true probability measure $\Pi$. Each investor does hold his own belief about $d_{t+1}$ when he makes investment decisions, and investors’ beliefs are heterogeneous. Heterogeneity in investors’ beliefs is due to the fact that different investors react differently to the common information of economic and political factors which may influence firm’s production. For example, the earnings forecasts made by financial analysts diverge although all analysts have access to the same data. The belief of an investor is defined to be rational if his subjective model is consistent with historical data and if simulated, its simulated data can reproduce the stationary probability measure $\alpha^*$ deduced from the historical data. All investors are assumed to be rational in the economy. Note that unlike in private information models, investors are willing to disclose their forecasts in this model, but other investors do not view such forecasts as a source of new information and then will not update their beliefs. Since investors are willing to reveal their own forecasts, samples of individual forecasts are thus taken, and their distributions become publicly available. We can then make a realistic assumption that forecast distributions are public observations over time.

The key analytical step in this model is to treat individual beliefs as state variables. An individual belief about an economic state variable could be described as a personal state of belief which will uniquely pin down the transition function of the investor’s belief about next period’s economic state variables. Assume that investor $i$’s state of belief, denoted by $g^*_i$, is

\[The theory of Rational Beliefs (RB in short) adopted in this paper is due to Kurz (1994, 1997) and Kurz and Motoleso (2008), and Kurz and Motoleso (2008) present mathematical conditions with which investors are rational. These conditions are not presented in this paper for simplicity.

\[For an investor, the forecasts of other investors are based on the same information set as he has and are expressions of subjective opinions and don’t contain any new information that he does not know.
characterized by the following dynamics

\[ g_{t+1}^i = \lambda_Z g_t^i + \epsilon_{t+1}^g, \]  

(6)

where \( \epsilon_{t+1}^g \sim N(0, \sigma_g^2) \) are correlated across \( i \) reflecting the correlation of individual beliefs. The unconditional expectation of \( g_t^i \) is zero. Investor \( i \)'s state of belief \( g_t^i \) pins down his own perception of date \( t + 1 \) payoff \( d_{t+1}^i \) in the following way

\[ d_{t+1}^i = \lambda_d d_t + \lambda_d^g g_t^i + \epsilon_{t+1}^{id} \]  

(7)

where \( \epsilon_{t+1}^{id} \sim N(0, \hat{\sigma}_d^2) \) and \( \hat{\sigma}_d^2 \) is the same for all investors. Obviously, we have that

\[ E^i [d_{t+1}^i | \mathfrak{S}_t, g_t^i] - E^m [d_{t+1} | \mathfrak{S}_t] = \lambda_d^g g_t^i \]  

(8)

where the first and second terms in the left side are respectively the investor \( i \)'s forecast of \( d_{t+1} \), and the forecast made under the empirical probability measure \( m \). Given that forecast distributions are public observations over time, in addition to \( g_t^i \), investor \( i \) also knows the market distribution of \( g_t^i \) across \( l \) for all time \( \tau \leq t \).

Average (6) over investors and denote by \( Z_t \) the mean of the cross-sectional distribution of \( g_t^i \) and refer to it as the market state of belief (or market belief in brief). The dynamics of \( Z_t \) is given by

\[ Z_{t+1} = \lambda_Z Z_t + \epsilon_{t+1}^Z \]  

(9)

with \( \epsilon_{t+1}^Z \overset{m}{\sim} N(0, \sigma_Z^2) \). Assume that the correlation of individual beliefs is nonstationary and the true distribution of \( \epsilon_{t+1}^Z \) is unknown, then the process \( \{Z_t, t = 1, 2, \cdots \} \) is nonstationary as well. Since both \( d_t \) and \( Z_t \) are observable, so does the joint empirical distribution of these two state variables. Assume that this joint distribution is described by the following system of equations

\[ d_{t+1} = \lambda_d d_t + \epsilon_{t+1}^d \]  

(10)

\[ Z_{t+1} = \lambda_Z Z_t + \epsilon_{t+1}^Z \]  

(11)

where the error terms \( \epsilon_{t+1}^d \) and \( \epsilon_{t+1}^Z \) are jointly normally distributed as follows

\[ \begin{bmatrix} \epsilon_{t+1}^d \\ \epsilon_{t+1}^Z \end{bmatrix} \overset{m}{\sim} N \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_d^2 & 0 \\ 0 & \sigma_Z^2 \end{bmatrix} \]  

(12)

Again, most investors do not believe that the stationary models (10) and (11) are the truth, and they hold their own beliefs. Investor \( i \)'s state of belief \( g_t^i \) pins down his perception of the

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\(^7\)Kurz (2008) deduces (6) as a limit posterior of a Bayesian inference, meaning that the assumption of (6) is consistent with the Bayesian learning method used in the finance literature of heterogeneous beliefs.
two state variables \((d_{t+1}^i, Z_{t+1}^i)\) as follows

\[
d_{t+1}^i = \lambda_d d_t + \lambda_d^g g_t^i + \epsilon_{t+1}^d
\]

\[
Z_{t+1}^i = \lambda_Z Z_t + \lambda_Z^g g_t^i + \epsilon_{t+1}^Z
\]

\[
g_{t+1}^i = \lambda_Z g_t^i + \epsilon_{t+1}^g
\]

where the error terms \(\epsilon_{t+1}^d, \epsilon_{t+1}^Z\) and \(\epsilon_{t+1}^g\) are jointly normally distributed with

\[
\begin{bmatrix}
\epsilon_{t+1}^d \\
\epsilon_{t+1}^Z \\
\epsilon_{t+1}^g
\end{bmatrix}
\sim N
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
,
\begin{bmatrix}
\hat{\sigma}_d^2 & \hat{\sigma}_{dZ} & 0 \\
\hat{\sigma}_{dZ} & \hat{\sigma}_Z^2 & 0 \\
0 & 0 & \sigma_g^2
\end{bmatrix}
\]

The stochastic transition functions \((13)-(15)\) together with the wealth transition process \((2)\) constitute the constrained conditions of investor \(i\)'s optimization problem \((3)^8\).

### 2.4 The Equilibrium with Heterogeneous Beliefs

In this section, we will solve the equilibrium with heterogeneous beliefs. When investors hold heterogeneous beliefs, each investor maximizes his optimal consumption-investment choice based on his own belief about \((d_t, Z_t)\) and then the stock demand of investors can be different from each other since their beliefs may be different.

We use the perception models \((13)-(15)\) about the state variables \((d_t, Z_t, g_t)\), average them over investors and use the definition of \(\hat{\mathcal{E}}_t\) to deduce the following relations

\[
\mathcal{E}_t(d_{t+1}) = \lambda_d d_t + \lambda_d^g Z_t
\]

\[
\mathcal{E}_t(Z_{t+1}) = (\lambda_Z + \lambda_Z^g) Z_t
\]

where \(\mathcal{E}_t(\bullet)\) is an average market expectation operator. Besides the constrained conditions \(R > 1\) and \(0 < \lambda_d < 1\) as in the model without heterogeneous beliefs, it is also assumed that \(0 < \lambda_Z + \lambda_Z^g < 1\) to make sure that \(Z_t\) is on average stationary.

**PROPOSITION II:** Given the perception equations \((13)-(15)\), the relations \((17)-(18)\) and the constrained conditions \(R>1, 0 < \lambda_d < 1\) and \(0 < \lambda_Z + \lambda_Z^g < 1\), the optimal stock demand of investor \(i\) is:

\[
\theta_i^t = \frac{R \tau}{r \hat{\sigma}_Q^2}
\left[
\mathcal{E}_t^i(Q_{t+1}) + u_0 + u_1 d_t + u_2 Z_t + u_3 g_t^i
\right]
\]

where \(\hat{\sigma}_Q^2\) is the adjusted conditional variance of the excess share return \(Q_{t+1}\). The equilibrium stock price is unique and takes a form as follows

\[
p_t = P_0 + a_d d_t + a_Z Z_t,
\]
where the constant term \( P_0 \), the coefficients \( a_d \) and \( a_Z \) are defined as

\[
P_0 = \frac{\mu + u_0}{R - 1} - \frac{\hat{\sigma}_Q^2}{RN\tau}, \quad a_d = \frac{\lambda_d + u_1}{R - \lambda_d}, \quad a_Z = \frac{(a_d + 1) \lambda_d^g + u_2 + u_3}{R - (\lambda_d + \lambda_Z^g)}.
\]

**Proof:** See Appendix A.1. ||

Note that the results in this proposition are similar to the previous results derived without heterogeneous beliefs except that in the heterogeneous beliefs model, besides the investment demand associated with \( E_t^i (Q_{t+1}) \) and the hedging demand against the fluctuations of \( d_t \), the optimal stock demand of investor \( i \) is also related to market belief \( Z_t \) and his own belief \( g^i_t \) which are both subjective, investors with different beliefs will have different demands. The equilibrium stock price \( p_t \) is also dependent of \( Z_t \), and how \( Z_t \) affects \( p_t \) is determined by the coefficient \( a_Z \). There are no closed-form solutions for the parameters \( u = [u_0, u_1, u_2, u_3] \) and then \( a_d \) and \( a_Z \), we use the Monte-Carlo method to numerically compute these coefficients for relevant values of the model parameters, and the results are displayed in Figure 2. This figure shows that \( a_Z \) is strictly positive for relevant values of the model parameters, it can be thus concluded that \( p_t \) increases when the market is more optimistic.

In this model, individual beliefs about \( d_t \) equally affect stock price, this is slightly different from the findings of Detemple and Murthy (1994), Basak (2000), Xiong and Yan (2009), etc that the equilibrium prices have a wealth-weighted average structure. Jouini and Napp (2007) show that under the hypothesis of a positive correlation between wealth and optimism, as e.g. in Miller (1977), asset prices reflect the more optimistic view, meaning that assets with a higher belief dispersion should yield lower returns, this result is supported by the empirical findings of Diether et al. (2002) that “stocks with higher dispersions in analysts earnings forecasts earn lower future returns than otherwise similar stocks”. However, in the economy we define, as Kurz and Motolesse (2008) show, asset returns are determined by the average of individual beliefs over investors rather than belief dispersion.

### 2.5 Trading Volume

To insert the equilibrium stock price (20) into the demand function of investor \( i \) (19) and to simplify, we obtain

\[
\theta^i_t = \frac{1}{N} + \frac{R\tau}{r\hat{\sigma}_Q^2} \left[ (a_d + 1) \lambda_d^g + a_Z \lambda_Z^g + u_3 \right] (g^i_t - Z_t)
\]

(21)

The optimal stock demand of investor \( i \) in the model with heterogeneous beliefs is not equal to \( \frac{1}{N} \) any more, and it depends on the difference between his own belief \( g^i_t \) and market belief \( Z_t \). As shown in Figure 2, the numerically computed value of \( (a_d + 1) \lambda_d^g + a_Z \lambda_Z^g + u_3 \) is strictly positive for relevant values of the model parameters, this means that when investor \( i \) is more
optimistic than the market, his demand will then exceed \( \frac{1}{N} \), and vice versa. If beliefs are just homogeneous across investors, then, similar as in the model without heterogeneous beliefs, the demand is identical across investors and is equal to \( \frac{1}{N} \). Trading will take place among investors only when the differences between individual beliefs and market belief change over time. Note that the concept of ‘trading’ is different from the demand, for an investor, high demand doesn’t mean actual buying in, and vice versa. Since the total supply of stock shares is normalized to one, what refers to as trading volume is actually turnover rate. For investor \( i \), the stock volume he will trade at time \( t \), denoted by \( V^i_t \), is

\[
V^i_t = |\theta^i_t - \theta^i_{t-1}| = \overline{C}_V \left| (g^i_t - g^i_{t-1}) - (Z_t - Z_{t-1}) \right|
\]  \hspace{1cm} (22)

where \( \overline{C}_V = \frac{R}{\sigma^2_Z} | (a_d + 1) \lambda^g_d + a_Z \lambda^g_Z + u_3 | \), and the aggregate trading volume, denoted by \( V_t \), is equal to

\[
V_t = \frac{1}{2} \sum_{i=1}^{N} V^i_t
\]  \hspace{1cm} (23)

The sum of \( V^i_t \) over investors is divided by two since buying and selling can be only accounted once. Note that, in this model, trading volume is not directly dependent of belief dispersion across investors\(^9\). This result is different from previous findings by, for example, Chordia et al. (2007) who show that trading volume increases with belief dispersion. For investor \( i \), we have

\[
\overline{V}^i = E^i (V^i_t) = \overline{C}_V \sqrt{\frac{2}{\pi}} \sigma_V
\]  \hspace{1cm} (24)

\[
\text{var}^i (V^i_t) = \left( \frac{\pi}{2} - 1 \right) (\overline{V}^i)^2
\]  \hspace{1cm} (25)

where \( \sigma^2_V = \frac{2}{(1+\lambda_Z)(1-\lambda_Z)} \left[ (1 + \lambda_Z + \lambda^g_Z)(1 - \lambda_Z) - \lambda_Z (\lambda^g_Z)^2 \right] \sigma^2_Z + \frac{2}{1+\lambda_Z} \sigma^2_Z \) and it is identical across investors. The variance of trading volume increases quadratically with its mean. The aggregate expected trading volume denoted by \( \overline{V} \) is given by:

\[
\overline{V} = \frac{1}{2} \sum_{i=1}^{N} \overline{V}^i = \frac{1}{2} N \overline{C}_V \sqrt{\frac{2}{\pi}} \sigma_V
\]  \hspace{1cm} (26)

Figure 3 plots the numerically computed aggregate expected trading volume \( \overline{V} \) against the variance of market belief \( \sigma^2_Z \) for relevant values of the model parameters. Increasing \( \sigma^2_Z \) has two opposite effects on \( \overline{V} \): \( \sigma_V \) increases while \( \overline{C}_V \) decreases. That \( \overline{C}_V \) decreases with \( \sigma^2_Z \) is due to the fact that the adjusted variance of excess share return \( \hat{\sigma}^2_Q \) increases with \( \sigma^2_Z \) while \( (a_d + 1) \lambda^g_d + a_Z \lambda^g_Z + u_3 \) decreases with \( \sigma^2_Z \) as shown in Figure 2B. The net effect is that \( \overline{V} \) decreases with \( \sigma^2_Z \), in other words, the expected trading volume declines when market belief is more volatile.

\(^9\)To explain this, assume that in the economy, there are only two investors \( i \) and \( j \) (i.e., \( N=2 \)), then, from the definition of \( Z_t \), we obtain that \( g^i_t - Z_t = \frac{1}{2} (g^j_t - g^i_t) \) where \( g^i_t - g^j_t \) exactly measures belief dispersion. Denote \( \Delta g_t = g^j_t - g^i_t \), we have that \( V_t = V^{i(j)} = \frac{1}{2} |\Delta g_t - \Delta g_{t-1}| \), this means that trading volume depends on the change of belief dispersion, but not belief dispersion itself.
2.6 Price Volatility

First, we consider the price volatility for the case in which there are no heterogeneous beliefs. Given the price process (5), price volatility, denoted by $\sigma_p^2$, is given by

$$\sigma_p^2 = Var(p^*_t) = \frac{2a_d^2}{1 + \lambda_d} \sigma_d^2$$

Figure 4A plots the numerically computed price volatility $\sigma_p^2$ for relevant values of the model parameters. Clearly, stock price tends to become more volatile when the variance of $d_t$ under the empirical probability measure $m$ increases. The relation between $\sigma_p^2$ and $\sigma_d^2$ looks like linear, this is because the effect of $\sigma_d^2$ on $a_d^*$ is infinitesimal, and for any relevant values of the model parameters, $a_d^*$ almost does not change with $\sigma_d^2$. In the model without heterogeneous beliefs, price volatility is uniquely determined by the innovations to $d_t$.

However, in the real financial markets, the stock price is more volatile. To see this point, we calculate the volatility of market price $p_t$ (denoted by $\sigma_p^2$) under the probability measure $m$ as follows:

$$\sigma_p^2 = Var(p_t) = \frac{2a_d^2}{1 + \lambda_d} \sigma_d^2 + \frac{2a_Z^2}{1 + \lambda_Z} \sigma_Z^2$$

Therefore, the volatility of market price is attributed to not only the innovations to $d_t$, but also to the innovations to $Z_t$. The numerically computed $\sigma_p^2$ is plotted in Figure 4B. The first obvious result to be observed from this figure is that the volatility of market price increases with both $\sigma_d^2$ and $\sigma_Z^2$. Second, $\sigma_Z^2$ has a larger effect on $\sigma_p^2$ than $\sigma_d^2$ does while the effects of $\sigma_d^2$ and $\sigma_Z^2$ are nonlinear due to the nonlinear effects of $\sigma_d^2$ and $\sigma_Z^2$ on $a_Z$ as shown in Figure 2B. The third and also most important result is that the market price $p_t$ could be much more volatile than the payoff $d_t$ and the hypothetical price (5) when $Z_t$ is sufficiently volatile, this can be found out by comparing the results in Figure 4A and Figure 4B.

2.7 Liquidity

Kyle (1985) argues that liquidity is a slippery and elusive concept, in part because it encompasses a number of transactional properties of financial markets. These properties include tightness (the cost of turning around a position over a short time period), depth (the size of an order flow innovation required to change prices a given amount), and resiliency (the speed with which prices recover from a random, uninformative shock).

The illiquidity measure used in this paper is closely related to its first property—tightness. Define $\Lambda_t$ as follows:

$$\Lambda_t = p_t - p^*_t$$

which is the signed deviation of the market price (20) from the hypothetical price (5), and our measure of stock illiquidity is defined as the absolute value of $|\Lambda_t|$. In the financial markets,

10There is no measure which is able to capture all the three types of liquidity properties.
at any time, the real value of a stock is unobservable, \( p_t^* \) can be however regarded as an over time average of the real values of the stock provided that the empirical probability measure \( m \) is an average over an infinite sequence of true probability measure \( \hat{m} \) regimes, reflecting long term frequencies. The illiquidity measure in this paper is similar to that one adopted by Brunnermeier and Pedersen (2009). Essentially, \( |\Lambda_t| \) measures price pressure or transitory price effect which is an important aspect of liquidity properties according to Grossman and Miller (1988), and Hendershott and Menkveld (2009).

Given the price formulas (5) and (20), we have

\[
|\Lambda_t| = |\Delta P_0 + \Delta a_d d_t + a_Z Z_t| \tag{30}
\]

where \( \Delta P_0 = P_0 - P_0^* \) and \( \Delta a_d = a_d - a_d^* \). Thus, the illiquidity measure \( |\Lambda_t| \) is a function of market belief. Setting \( |\Lambda_t| = 0 \), that is \( p_t = p_t^* \), leads to

\[
Z_t^E = - \frac{\Delta P_0}{a_Z} - \frac{\Delta a_d}{a_Z} d_t \tag{31}
\]

Illiquidity problem will arise when \( Z_t \) deviates from \( Z_t^E \). Given the definition (30) and the joint normality assumption about \( d_t \) and \( Z_t \), we have

\[
\mu_{|\Lambda|} = E^m (|\Lambda_t|) = \sigma_\Lambda \sqrt{\frac{2}{\pi}} \exp \left( - \frac{\Delta P_0^2}{2 \sigma_\Lambda^2} \right) + \mu_\Lambda \left[ 1 - 2 \Phi \left( \frac{\mu_\Lambda}{\sigma_\Lambda} \right) \right], \tag{32}
\]

and the unconditional variance of \( |\Lambda_t| \) is

\[
\sigma_{|\Lambda|}^2 = var^m (|\Lambda_t|) = \mu_\Lambda^2 + \sigma_\Lambda^2 - (\mu_{|\Lambda|})^2, \tag{33}
\]

where \( \mu_\Lambda = \Delta P_0 \) and \( \sigma_\Lambda^2 = (\Delta a_d)^2 \frac{\sigma_d^2}{1 - \lambda_d^2} + a_Z^2 \frac{\sigma_Z^2}{1 - \lambda_Z^2} \). The proof is in Appendix A.2. As plotted in Figure 5, the expected value of \( |\Lambda_t| \) increases with \( \sigma_Z^2 \), this means that the expected stock liquidity declines when market belief is more volatile. Higher volatility of market belief leads to higher market price volatility and then higher price pressure since the market price \( p_t \) is linearly correlated with market belief. This result, together with the results in Section 2.6, allows us to conclude that larger \( \sigma_Z^2 \) is accompanied by both higher price volatility and less liquidity. The phenomenon of the coexistence of higher price volatility and less liquidity has been documented in the financial markets, this model offers a potential explanation for this phenomenon.

Different from the inventory risk and asymmetric information theories, it is shown in this model that even when there is neither inventory risk nor information asymmetry, liquidity problem can still occur in the financial markets as a consequence of heterogeneous beliefs across investors. To our knowledge, this paper is the first one to show such a relation between investors’ beliefs and stock liquidity.
3 Empirical Analysis

In the last section, the model shows that the volatility of market belief about stock future payoffs has significant effects on the trading volume, price volatility, and liquidity of underlying stocks. In this section, we are going to empirically examine these theoretically predicted relations in the time-series context\(^{11}\). Precisely, the objective of this section is to test whether the following hypothesis holds for stocks:

- **Hypothesis**: when market belief is more volatile, trading volume and liquidity decline while price volatility increases;

In the following, we will first construct market belief about stock future payoffs and then estimate its time-varying volatility using various methods like rolling regression method and GARCH model, and with the estimated volatility of market belief, the above hypothesis will be examined. Additionally, we have also done robustness tests.

3.1 Data

We will use (financial) analysts’ forecasts to approximate investors’ beliefs in that the latter data is difficult to collect. Previous studies find that analysts are able to effectively record a sentiment diffused in the financial markets, and their forecasts are good proxies for investors’ beliefs\(^{12}\). A potential issue with analysts’ forecasts is that analysts are biased in their forecasts\(^{13}\): analysts are generally optimistic about annual and longer-term forecasts, but there is also evidence that as the forecast period declines, analysts will become slightly pessimistic in their forecasts. Moreover, analysts tend to overreact to positive news and underreact to negative news. There are many reasons for this bias, including agency issues and behavioral biases. Whatever the verdict about biases, we focus on the volatility of market belief. While the mean of forecasts among analysts is affected by biases, its volatility is less likely to suffer from the bias problem.

The analyst forecast data are collected from the Institutional Brokers’ Estimate System (I/B/E/S) U.S. Summary History database that contains the summary statistics of analysts’ earnings forecasts, including forecast mean, median, standard deviation, and the number of financial analysts. This database also contains the revision date on which the forecast was last confirmed to be accurate. These data are usually disclosed on the third Tuesday of each month\(^{14}\).

\(^{11}\)Due to the limitation of sample size and the fact that our sample is not diversified, to test these relations in the cross-sectional context would not be that meaningful.

\(^{12}\)See, for example, Goetzmann and Massa (2005), and Anderson et al. (2005).

\(^{13}\)The biases of analysts’ forecasts have been documented by, for example, De Bondt and Thaler (1985, 1987, 1990), LaPort (1996), Easterwood and Nutt (1999), Brown (2001), Matsumoto (2002), etc.

\(^{14}\)Diether et al. (2002) have a detailed description of the I/B/E/S database.
In the I/B/E/S database, there are two main types of analysts’ forecast data about stock future payoffs: one is the dividend per share (DPS) forecast and another is the earnings per share (EPS) forecast, both of them being often used in the literature. It is known that DPS is affected very much by a firm’s dividend policy whose effect is difficult to be controlled in the empirical study, and more importantly, the DPS forecast data only has a short history and the coverage of analysts for DPS forecast is low. Because of these reasons, only the EPS forecast data will be used in our empirical analysis.15

As can be seen later, to construct market belief about stock future earnings, we also need the actual EPS data which are again collected from the I/B/E/S database. The actual EPS data provided by the I/B/E/S is called the ‘Street’ EPS because it is tracked by analysts and priced by investors. The COMPUSTAT provides another type of actual EPS data known as the GAAP EPS that is reported in firms’ financial statements. Bradshaw et al. (2000) show that there exists a large and growing gap between the ‘Street’ EPS data and the GAAP EPS data because the former excludes cost items such as ‘non-recurring’ and ‘no-cash’ charges16. Since analysts’ EPS forecasts are based on the ‘Street’ EPS data, to construct market belief, we shall use the ‘Street’ EPS data rather than the GAAP EPS data although the latter data seems more transparent and has longer history of records.

Both actual and forecast EPS data provided by the I/B/E/S have different periodicities: quarterly, semi-annually, annually, etc. In this paper, we use quarterly EPS data due to the following reasons: first, the coverage of analysts is relatively high for quarterly EPS forecast; second, in the accounting literature, most time-series models have been developed to predict quarterly EPS.

Our empirical analysis will focus on the S&P500 Index stocks which are generally those of large publicly held firms, the reason is that small firms are covered with fewer financial analysts so that analysts’ earnings forecasts can not efficiently reflect the views of most investors in the financial markets and then market belief estimated from such earnings forecasts may be biased. In addition, the stocks issued by small firms are more likely to be affected by the problems of inventory risk and information asymmetry, for such stocks, it could be difficult to abstract the effects of market belief and its volatility from the effects of other two factors on stock liquidity, we want to avoid this problem.

The stocks to be included in the sample must meet two criteria: a) Quarterly continuous actual EPS data for the 1983-2008 period; b) Monthly continuous mean forecast data on EPS

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15The finance theory in the textbook tells us that the price of a stock equals the expected present value of discounted dividends which it receives in its lifetime, and the variables closely related to stock price such as price volatility and liquidity are therefore affected by the variations of DPS, from this aspect, it is better to use the DPS data in our empirical works. However, if the payout ratios of firms are stable over the sample period, the empirical results with either EPS or DPS should be similar.

16The difference between the ‘Street’ and GAAP earnings has been also discussed in Ciccone (2002) and Cote and Qi (2005).
for the 1993-2008 period. 114 S&P500 Index stocks well satisfy these two criteria, and Table 1 displays the industry decomposition of the sample along with the GICS two-digit industry codes: stocks from three industry sectors of ‘Industrials’, ‘Consumption Discretionary’, and ‘Consumption Staples’ comprise more than 50% of the sample while there are only few stocks from ‘Telecommunication Services’ and ‘Utilities’ sectors.

Common stock data, including prices, returns, trading volume, the number of outstanding shares, etc, are collected from the CRSP database.

3.2 Time-Series Models

By equation (8), to construct individual beliefs and then market belief, we need one-period-ahead EPS predicted with historical data. A simple time-series model that can perform this prediction is the seasonal random walk with drift model [SRWD] that takes a form as follows

\[ E(Q_t) = \delta + Q_{t-4} \]  

(34)

where \( E(Q_t) \) is the earnings forecast for quarter \( t \), \( \delta \) is a (typically positive) trend term, and \( Q_{t-4} \) is the actual earning for quarter \( t-4 \). The advantage of this model is that it can capture the seasonality characteristics in the quarterly earnings data documented by, for example, Lorek (1979) among others. But, such a model is generally not very accurate because seasonal differences in quarterly earnings, \( Q_t - Q_{t-4} \), are also affected by factors other than a trend term. In particular, seasonal differences for quarter \( t \) typically exhibit diminishing positive correlation with the three prior quarters’ seasonal differences (\( Q_{t-1} - Q_{t-5}, Q_{t-2} - Q_{t-6}, Q_{t-3} - Q_{t-7} \)) and a negative correlation with the seasonal difference four quarters prior (\( Q_{t-4} - Q_{t-8} \)). Brown and Rozeff (1979) [BR] suggest that the following model, which incorporates this autocorrelation structure, most accurately predicts the quarterly earnings per share

\[ E(Q_t) = \delta + Q_{t-4} + \phi (Q_{t-1} - Q_{t-5}) + \theta \epsilon_{t-4} \]  

(35)

where \( Q_{t-k} \) is the actual EPS for quarter \( t - k \), \( \epsilon_{t-4} \) is the EPS shock experienced at quarter \( t - 4 \), and, in general, \( \phi > 0 \) and \( \theta < 0 \). This model contains an autoregressive component \( Q_{t-1} - Q_{t-5} \) which reflects the positive autocorrelations in seasonal quarterly differences at the first three lags and a moving average component \( \epsilon_{t-4} \) which is responsible for the negative correlation in seasonal difference at the fourth lag\(^{17} \). In this paper, both these two time-series models are used to predict quarterly EPS while the first one is for the purpose of robustness test.

For each stock, the prediction of EPS for each quarter during the sample period between 1993 and 2008 is derived with the estimated coefficients from either a regression of \( Q_t \) on \( Q_{t-4} \)

\(^{17}\)From equation (35), we obtain \( \epsilon_{t-4} = Q_{t-4} - Q_{t-8} - \delta - \phi (Q_{t-5} - Q_{t-9}) - \theta \epsilon_{t-8} \). The last two terms are small compared to \( Q_{t-4} \) and \( Q_{t-8} \) and \( \delta \) is constant, this suggests that \( Q_{t-4} - Q_{t-8} \) can be considered as a reasonable proxy for \( \epsilon_{t-4} \).
or a regression of seasonal changes in actual EPS for quarter \( t \), \( Q_t - Q_{t-4} \), on \( Q_{t-1} - Q_{t-5} \) and \( \epsilon_{t-4} \) (depending on which time-series model is used), and each regression is estimated using 40 prior quarters historical EPS data. A time-series set of 64 estimates for each coefficient in both the seasonal random walk with drift model and the Brown and Rozell (1979) model are obtained for each stock, the cross-sectional statistics of time series means of the estimated coefficients are in Table 2. Consistent with previous findings in the literature, in general, the estimated coefficients \( \hat{\delta} \) and \( \hat{\phi} \) are positive while the estimated coefficient \( \hat{\theta} \) is negative.

### 3.3 Market State of Belief

Denote by \( E_t^i(\text{eps}_{t+1}) \) analyst \( i \)'s conditional forecast of quarter \( t+1 \) EPS and by \( E_t^m(\text{eps}_{t+1}) \) the prediction under the stationary empirical probability \( m \). Analyst \( i \)'s state of belief about EPS is defined as:\(^{18}\)

\[
g_t^{\text{eps}, i} = E_t^i(\text{eps}_{t+1}) - E_t^m(\text{eps}_{t+1}) \tag{36}
\]

A positive \( g_t^{\text{eps}, i} \) means that analyst \( i \) is more optimistic about the earnings for quarter \( t+1 \) than the market. Market belief is defined as the average of individual beliefs across analysts

\[
Z_t^{\text{eps}} = \frac{1}{N} \sum_{i=1}^{N} [E_t^i(\text{eps}_{t+1}) - E_t^m(\text{eps}_{t+1})] = \overline{E}_t(\text{eps}_{t+1}) - E_t^m(\text{eps}_{t+1}) \tag{37}
\]

where \( \overline{E}_t(\text{eps}_{t+1}) \) is the average forecast across analysts, and \( Z_t^{\text{eps}} \) reflects the market’s views about the earnings for quarter \( t + 1 \).

To construct \( Z_t^{\text{eps}} \), we need data on both \( \overline{E}_t(\text{eps}_{t+1}) \) and \( E_t^m(\text{eps}_{t+1}) \). The forecast mean of quarterly EPS provided by the I/B/E/S can be used as a proxy for the average forecast \( \overline{E}_t(\text{eps}_{t+1}) \), and as for \( E_t^m(\text{eps}_{t+1}) \), it can be predicted using the time-series models proposed in Section 3.2. It is worthful to notice that the frequencies of \( \overline{E}_t(\text{eps}_{t+1}) \) and \( E_t^m(\text{eps}_{t+1}) \) are different: the former is in month while the latter is in quarter. When constructing monthly market belief, \( E_t^m(\text{eps}_{t+1}) \) is subtracted from all monthly forecast means for quarter \( t + 1 \).\(^{19}\)

Figure 7 plots the graphs of market beliefs for all sample stocks over the 1993-2008 period while the upper graph traces market beliefs estimated with the Brown and Rozell (1979) model and the bottom graph traces market beliefs estimated with the seasonal random walk with drift model, these graphs show that market beliefs fluctuate dramatically over time.

Table 3 reports the cross-sectional statistics of time-series means, skewness, and kurtosis of market beliefs. It is clear that the mean of market beliefs is not zero over short time period

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\(^{18}\)This definition is inspired by equation (8)

\(^{19}\)Another way to construct \( Z_t^{\text{eps}} \) is to first translate quarterly forecasts \( E_t^m(\text{eps}_{t+1}) \) to monthly forecasts by using the cubic spline technique, and then to subtract the interpolated monthly forecasts from monthly analysts’ forecast means. The empirical results derived with such market belief are similar to what obtained in this paper.
although the theory requires to have a long-term time average equal to zero, the market is on average optimistic about firms’ future earnings. Moreover, market beliefs are distributed with heavy-tail, meaning that market beliefs can take extreme values.

3.4 Volatility of Market Belief

The volatility of market belief is unobservable, but there are many methods to estimate it in the finance literature. In this paper, as a benchmark case, we use rolling regression volatility estimators with window length $\tau$ for each stock $j$, namely,

$$
\hat{\text{var}}_t (Z_{j,t}^{ep}) = \sum_{k=1}^{\tau} \omega_k (Z_{j,t+1-k}^{ep} - \mu_{j,t})^2
$$

where $\mu_{j,t} = \sum_{k=1}^{\tau} \omega_k Z_{j,t+1-k}^{ep}$, the weights $\omega_k$ decline geometrically with $\sum_{k=1}^{\tau} \omega_k = 1$, and $\tau$ represents the window length. A type of geometrically declining weights $\omega_k = e^{-ak}$ proposed by Foster and Nelson (1996) is used in this paper, and the window length $\tau$ is randomly set equal to four months. The volatility of market belief could be estimated using the squared periodic market beliefs, however, since volatilities are generally persistent, as we have learned from the stochastic volatility literature, such an estimator of volatility could be biased and inefficient, rolling regression estimators can somehow avoid this problem. Another advantage of rolling regression estimators is that they do not need intensive computations. In addition to rolling regression method, as a robustness check, we also use a GARCH($p$, $q$) model with orders $p=q=1$ to estimate the volatility of market belief, which takes a form as follows:

$$
Z_t = \phi_0 + \phi_1 Z_{t-1} + \epsilon_{z,t}
$$

$$
\epsilon_{z,t} = \sigma_{z,t} v_{z,t}
$$

where $\{v_{z,t}\}$ is a i.i.d $N(0,1)$, and $\{\sigma_{z,t}\}$ satisfies the following recurrence equation

$$
\sigma_{z,t}^2 = \alpha_0 + \alpha_1 \epsilon_{z,t-1}^2 + \beta_1 \sigma_{z,t-1}^2
$$

Table 4 details the cross-sectional statistics of time series means of the estimated volatilities of market belief. It is shown in this table that the volatility of market belief estimated with GARCH(1,1) model is larger. Adding longer lagged market beliefs (i.e., $Z_{t-2}$, $Z_{t-3}$, ...) into equation (39) can reduce the estimated volatility of market belief, but this won’t change the conclusions to be made later in this paper. The magnitude of the volatility of market belief varies dramatically among stocks, as indicated by the fact that the cross-sectional standard deviation of the volatility of market belief is much larger than its mean.

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20 The subsequent empirical results obtained with the estimated volatility of market belief for $\tau = 2$ or 6 months are similar to what obtained for $\tau = 4$ months, but the results with longer $\tau$ are less significant, this may be caused by the possibility that the persistence of the volatility of market belief decline fast so that the volatility estimated with long time prior data are very noise.

21 The results with other reasonable orders $p$ and $q$ are similar.
3.5 Trading Volume, Price Volatility, and Liquidity

The data used to calculate the values of the variables of trading volume, price volatility, and liquidity of stocks are collected from the CRSP database.

- Trading Volume: we use turnover rate as a measure of trading volume, which is defined as the ratio of the number of shares traded in a month by the number of shares outstanding, and this measure is consistent with the definition of trading volume in Section 2;

- Price Volatility: defined as the variance of daily prices in a month;

- Liquidity: there exist many illiquidity measures in the finance literature, including bid-ask spread, effective bid-ask spread, depth, Kyle’s (1985) lambda, etc, each of them reflecting a different aspect of liquidity properties. As explained in Section 2.7, the illiquidity measure used in the theoretical model is price pressure. Among the existing illiquidity measures, the effective bid-ask spread is a good proxy for price pressure. In the following, we will use the effective bid-ask spread proposed by Roll (1984) as a benchmark illiquidity measure. The Roll’s spread can be estimated as follows: denote by \( p_t \) the transaction price for a trade at time \( t \), which may be expressed as

\[
p_t = m_t + q_t c
\]

\[
m_t = m_{t-1} + u_t
\]

where \( m_t \) denotes the efficient price and \( u_t \) are i.i.d zero mean random variables with variance \( \sigma_u^2 \), \( q_t \) is a trade direction indicator set to +1 if the customer is buying and -1 if the customer is selling, and \( c \) is transaction cost which is assumed not to be related to the dynamics of \( m_t \). It is clear from equation (42) that \( c \) measures the deviation of the transaction price \( p_t \) from the efficient price \( m_t \) and is exactly a measure of price pressure. Assume that buys and sells are equally likely and serially independent and that investors buy or sell independently of \( u_t \). Denote \( \Delta p_t = p_t - p_{t-1} \), it is easy to verify that \( E(\Delta p_t) = 0 \). The first order autocovariance of price changes is

\[
\gamma_1 = cov(\Delta p_{t-1}, \Delta p_t) = E(\Delta p_{t-1}\Delta p_t) = E[c^2(q_{t-2}q_{t-1} - q_{t-1}^2 - q_{t-2}q_t + q_{t-1}q_t) + c(u_{t-1}q_t - u_{t-1}q_{t-1} + q_{t-1}u_t - q_{t-2}u_t)]
\]

\[
= -c^2
\]

We obtain \( c = \sqrt{-\gamma_1} \). The first order autocovariance \( \gamma_1 \) is not always negative, \( c \) is set equal to 22

\[\text{The effective bid-ask spread, according to its definition, measures the absolute deviation of stock transaction price from its intrinsic value.}\]

\[\text{A reason for us to use this spread as well as the bid-ask spread proposed by Corwin and Schultz (2010) and the Amihud’s (2002) illiquidity measure later in the robustness test section is that different from other price pressure measures, they are easy to be computed.}\]

22

23
to zero when $\gamma_1 > 0$. For each stock, $c$ is assumed to be constant within a given month and is estimated using daily prices data in that month;

In a month, to compute turnover rate, price volatility, and the Roll’s spread, stocks are required to have at least 15 daily transactions data, otherwise the values of these variables are set equal to zero.

The columns 2 – 4 of Table 5 display the cross-sectional statistics of time-series means of turnover rate, price volatility, and the Roll’s spread. The monthly amount of traded shares on average represents around 10% of total outstanding shares, and the Roll’s spread is $0.23$ per share. The cross-sectional average of price volatilities is surprisingly large, it, as shown in Figure 8, is driven by the extreme price volatilities of some stocks. It is also obvious in Figure 8 that turnover rate has been increasing over the sample period, this is in line with the findings by other studies such as Lo and Wang (2001), and Chordia et al. (2007).

### 3.6 Regression Results

We start with univariate regressions in which the volatility of market belief about stock future quarterly earnings per share is the unique explanatory variable for the trading volume, price volatility, and liquidity of underlying stocks, and then continue with multivariate regressions to control for the effects of additional explanatory variables. Different methods of estimating market belief and its volatility as well as alternative illiquidity measures will be used to check whether our results are robust.

#### 3.6.1 Univariate Regressions

For each stock, the univariate regression takes the following form:

$$Y_t = \alpha + \beta vZ_t + \epsilon_t$$

where $Y_t$ is either turnover rate or price volatility or the Roll’s spread, $vZ_t$ is the volatility of market belief estimated using rolling regression method, and $\epsilon_{t=1,\ldots,T}$ are i.i.d normal random variables with zero mean and constant variance.

Table 6 reports the cross-sectional statistics of estimated betas $\hat{\beta}_i$. It can be summarized that the empirical results are generally consistent with the theoretical predictions derived in Section 2. We first discuss the results for the case in which $Z_i$ is estimated using the Brown and Rozeff (1979) model, these results are displayed in the second column of Table 6.

As shown in Panel B, for most of sample stocks, price volatility is significantly positively correlated with the volatility of market belief since around 84% of $\hat{\beta}_i$ are positive while 61% are greater than the 5% critical value in one-tailed test, and the cross-sectional average of $\hat{\beta}_i$.

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24Certain stocks have larger price volatilities since their prices are in higher levels.
is 662.7 with a t-statistics equal to 7.090. In addition, the explanatory power of the typical individual regression is impressive in that the average $R^2$ is more than 6.8%.

Stock liquidity generally declines when market belief becomes more volatile, this is shown in Panel C: over three quarters of $\hat{\beta}_i$ are positive while 42% exceed the 5% critical value in one-tailed test, and, as in the price volatility case, the cross-sectional average of $\hat{\beta}_i$ is significantly positive. On average, the volatility of market belief can approximately explain 3%-4% of the variation of liquidity.

As can be seen in Figure 8, there is obviously a time trend in stock turnover rate which has been increasing over the 1993-2008 period and is therefore nonstationary. For this reason, the empirical studies of trading volume will use some forms of detrending to induce stationarity. There exist several detrending skills, including linear detrending, log-linear detrending, first differencing, etc\textsuperscript{25}. We use the simple method of first differencing to detrend stock turnover rate, and the regression results for the detrended stock turnover rate are well summarized in Panel A. As expected, stock turnover rate declines with the volatility of market belief for a majority of sample stocks, but the correlation is rather weak because only less than 2% of $\hat{\beta}_i$ are significantly negative. Moreover, the explanatory power of the volatility of market belief on stock turnover rate is not strong in that the average $R^2$ is less than 0.5%. In contrast to the theoretical prediction in Section 2.5, the cross-sectional average of $\hat{\beta}_i$ is positive although insignificant. These evidences suggest that there should be a large component of noise and/or other influences in stock turnover rate.

It is possible that the above results may depend on the specification of time-series models used to predict quarterly EPS, different predicting models will lead to different conclusions. As a robustness test, the regression results for the case in which market belief is estimated using the seasonal random walk with drift model are displayed in the third column of Table 6. The results in this case are clearly very similar to what we obtained above. In fact, in this case, the empirical findings are even stronger to support the theoretically predicted relations between the volatility of market belief about stock future payoffs and the trading volume, price volatility, and liquidity of underlying stocks given that more $\hat{\beta}_i$ do exhibit theoretically predicted signs and are significant, and in particular, the cross-sectional average of $\hat{\beta}_i$ in the trading volume case turns to be negative and is significant at the 5% level.

### 3.6.2 Multivariate Regressions

The empirical findings in the univariate regressions are consistent with the theoretical predictions about the relations between the volatility of market belief about stock future payoffs and the trading volume, price volatility, and liquidity of underlying stocks, but these relations remain to be further examined since other factors than the volatility of market belief,\textsuperscript{25}Refer to Lo and Wang (2001) for more discussions about these skills.
which may affect trading volume, price volatility, and liquidity, are ignored.

A. Trading Volume
To study the behavior of trading volume is a crucial topic in the finance literature provided that trading volume is one of the fundamental building blocks of any theory of market interactions and is important in modeling asset markets. Previous empirical studies focusing on the time-series behavior of trading volume document the positive price/volume and volatility/volume relations. Different theoretical models, such as sequential arrival of information models, a mixture of distributions models, asymmetric information models, and differences in opinion models, have been developed to explain the relation between price and trading volume. Some studies also report bidirectional causality between price and trading volume.

The mixture of distribution models explains the positive relation between return volatility and trading volume as they jointly depend on a common factor, i.e., information innovation. The third control variable to be included is $Z_t$. How does market belief affect trading volume is not specified in the theoretical model, but, according to the experience learned from the financial markets, it seems reasonable to believe that trading volume will increase when the market expectation about stock future earnings is optimistic. Precisely, for each stock, the multivariate regression takes the following form:

$$Y_t = \alpha + \beta_1 vZ_t + \beta_2 Z_t + \beta_3 Price_t + \beta_4 vRet_t + \epsilon_t$$  \hspace{1cm} (46)

where $Y_t$ is the detrended turnover rate, $vZ_t$ is the volatility of market belief, $Z_t$ is market belief, $Price_t$ is the average of daily prices in month $t$, and $vRet_t$ is return volatility. $\epsilon_t=1,...,T$ are i.i.d normal random variables with zero mean and constant variance.

Table 7 reports the cross-sectional statistics of the multivariate regression results for the detrended stock turnover rate. It is clear that adding additional explanatory variables does not have a major effect on the negative correlation between trading volume and the volatility of market belief. In the multivariate regression, the estimated betas of the volatility of market belief are significant at the 5% level for more stocks and its cross-sectional average is negative although still insignificant. The average betas of stock price and return volatility exhibit the same signs as we expect, and in particular, trading volume strongly positively covaries with return volatility. Trading volume increases with market belief whose effect is however much less significant than those of stock price and return volatility. Adding additional explanatory variables makes the explanatory power of the typical individual regression increase from less than 0.5% to more than 8%, this is a huge improvement. These results tell us that while the volatility of market belief negatively affects trading volume, its effect, compared with those of other factors like stock price and return volatility, is limited.

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26See, for example, Hiemstra and Jones (1994); Chen, Firth, and Rui (2001); Ratner and Leal (2001)
27In the optimistic periods, more investors are expected to crowd into the financial markets and speculation is more active, this will likely drive trading volume up.
B. Price Volatility

There exists extensive evidence on the relation between price volatility and trading volume. Karpoff (1987), for example, cites many studies that document a positive relation between price volatility and trading volume in the financial markets, and this relation is robust to various time intervals and numerous financial markets. For this reason, stock turnover rate is added as a control variable when we run the regression of price volatility on the volatility of market belief. The second control variable is size which is defined as the market value (in billions of US dollars) of stock on the last day of the previous month. Size is included for the following reason: size is correlated with institutional ownership, Falkenstein (1996) finds that institutional investors display a revealed preference for larger firms. Dennis and Strickland (2004) record that firm-level price volatility is positively related to increased institutional ownership. Cheung and Ng (1998) find that size is positively correlated with price volatility. As in the trading volume case, market belief is also included as a control variable. Adding these control variables into equation (45) gives

\[ Y_t = \alpha + \beta_1 vZ_t + \beta_2 Z_t + \beta_3 Size_t + \beta_4 Volume_t + \epsilon_t \]  

(47)

where \( Y_t \) is price volatility, \( vZ_t \) is the volatility of market belief, \( Z_t \) is market belief, \( Size_t \) is the market value of a firm’s stock in the beginning of month \( t \), and \( Volume_t \) is stock turnover rate. \( \epsilon_{t=1,\ldots,T} \) are i.i.d normal random variables with zero mean and constant variance.

The results in Table 8 confirm the previous findings that price volatility increases with trading volume although the correlation is not significant. The results about the effect of size on price volatility are however mixed while the average betas of size are not different from zero. Price volatility is positively correlated with market belief as well, this result somehow contradicts our intuition: if bull market is accompanied with higher \( Z_t \) and bear market with lower \( Z_t \), then price volatility should decrease with market belief as volatility is higher in the recession periods. One of potential explanations for this result is that optimistic market belief, as shown in the results for trading volume, drives up trading volume which will in turn raise price volatility. Importantly, after controlling for the effects of additional explanatory variables, the positive correlation between price volatility and the volatility of market belief again holds for a majority of sample stocks in that more than 75% of estimated betas of the volatility of market belief are positive while 55% exceed the 5% one-tailed critical value. On average, the factors of the volatility of market belief, market belief, size and trading volume can approximately explain 15% of the variation of price volatility, with a minor half of the explanatory power from the volatility of market belief.

C. Liquidity

Market microstructure theory suggests that there are two potential affecting factors on stock liquidity, namely, inventory risk and information asymmetry. The inventory explanation for
liquidity argues that more trading should lead to tight spread because inventory balances and risk per trade can be maintained at lower levels. To account for the influence of inventory risk on liquidity, we add a measure of trading activity - stock turnover rate - as a control variable. We use the analyst forecast dispersion (available in the I/B/E/S database) as a variable to control the effect of asymmetric information on liquidity. Sadka and Scherbina (2009) show that investors disagree more when the problem of asymmetric information is more serious\textsuperscript{28}. Stock exchanges have made many efforts to strengthen the efficiency and transparency over the last two decades, as a result, liquidity in the financial markets has increased. A precise example is that the bid-ask spread dramatically declined after the reduction of tick size in 1997. To control for the effect of the time trend in the series of stock liquidity, time is added as a control variable in the regressions. Return volatility is a nuisance variable that possibly influences liquidity. As in the previous two cases, we also want to examine whether market belief itself affects liquidity. Thus, the multivariate regression for liquidity can be formulated as follows

\[ Y_t = \alpha + \beta_1 vZ_t + \beta_2 Z_t + \beta_3 t + \beta_4 Volume_t + \beta_5 Dispersion_t + \beta_6 vRet_t + \epsilon_t \]  

(48)

where \( Y_t \) is the Roll’s spread, \( vZ_t \) is the volatility of market belief, \( Z_t \) is market belief, \( t \) is time, \( Volume_t \) is stock turnover rate, \( Dispersion_t \) is analyst forecast dispersion, and \( vRet_t \) is return volatility. \( \epsilon_{t=1,...,T} \) are i.i.d normal random variables with zero mean and constant variance.

Table 9 reports the cross-sectional statistics of the multivariate regression results for the Roll’s spread. Including additional explanatory variables does not have a major effect on the negative correlation between liquidity and the volatility of market belief: the Roll’s spread is positively correlated with the volatility of market belief about stock future earnings for about two thirds of sample stocks, and the positive correlation is significant for more than one third of sample stocks. The Roll’s spread does widen with return volatility. However, the effects of trading volume and the analyst belief dispersion on liquidity are opposite to what expected. For example, larger trading volume widens the Roll’s spread. This result is indeed not so surprise as it looks like. As shown in Table 8, trading volume increases price volatility that will in turn increase price pressure which is what the Roll’s spread tries to measure. The same argument can be applied to explain the effect of market belief on the Roll’s spread. The Roll’s spread unexpectedly narrows with the analyst forecast dispersion, suggesting that either the analyst forecast dispersion is not a good proxy variable for information asymmetry

\textsuperscript{28}Previous work by Jones et al. (1994) suggests that the number of trades, not the dollar volume of trading, is an indicator of individual firm asymmetric information. The reason is that, as Barclay and Warner (1993) argue, informed traders do break up their orders and are most active in the medium-size trades. Another measure of asymmetric information often used in the literature is the PINs which gives the probability of informed trading. Unfortunately, we do not have the \textit{suitable} data of these two variables for our empirical works.
or the Roll’s spread is not greatly affected by information asymmetry. It is shown in table 9 that there is no significant time trend in the time-series of the Roll’s spread, this can be also found out in the bottom-left graph of Figure 8.

3.7 Robustness Tests

3.7.1 Volatility of Market Belief Estimated Using GARCH(1,1) Model

The empirical results about the relations between the volatility of market belief about stock future earnings per share and the trading volume, price volatility, and liquidity of underlying stocks obtained so far are based on the volatility of market belief estimated by using rolling regression method. To examine whether these results depend on the specification of volatility estimating models, this paper also reports in Table 10 the cross-sectional statistics of both univariate and multivariate regression results for the case in which the volatility of market belief is estimated using a GARCH(p, q) model with $p=q=1$ (as described in Section 3.4).

The main conclusions to be made from Table 10 can be summarized as follows:

First, the empirical results again support the theoretically predicted relations between the volatility of market belief about stock future payoffs and the trading volume, price volatility, and liquidity of underlying stocks: for a major number of sample stocks, when market belief is more volatile, trading volume and liquidity decline while price volatility increases. Especially, the negative correlation between trading volume and the newly estimated volatility of market belief holds and is significant for more stocks and the cross-sectional averages of the estimated betas of the volatility of market belief are significantly negative in all cases;

Second, the newly estimated volatility of market belief explains higher proportions of the variations of both price volatility and liquidity. Precisely, in the univariate regressions, the average $R^2$ approximately rises from 7% to 14% in the price volatility case and from 3.5% to 7% in the liquidity case, almost doubled in both cases;

Third, the average betas of the volatility of market belief in both the price volatility and liquidity cases are again significantly positive although with smaller t-statistic values. Note that, these average betas are several times larger but less significant than those obtained when the volatility of market belief estimated with rolling regression method is used as regressor, this is due to the fact (not shown in the table) that the magnitudes of the correlations between the newly estimated volatility of market belief and both price volatility and liquidity vary dramatically across stocks so that the variances of the respective betas are very large.

It is clear that using the volatility of market belief estimated with GARCH(1,1) model as explanatory variable does not change the empirical results about the relations between the volatility of market belief about stock future payoffs and the trading volume, price volatility, and liquidity of underlying stocks.
3.7.2 Alternative Illiquidity Measures

As mentioned in Section 2.6, liquidity is a complex concept, and there exist many illiquidity measures in the literature, each of which reflecting a different aspect of liquidity properties. In the above empirical analysis, we used the Roll’s spread estimated from stock daily prices as dependent variable to examine the negative correlation between liquidity and the volatility of market belief. One might wonder whether such a negative correlation also holds for other illiquidity measures. To answer this question, in the following robustness test, we use other two illiquidity measures which are the bid-ask spread proposed by Crowin and Schultz (2010) and the Amihud’s (2002) illiquidity measure respectively.

Similar as the Roll’s spread, the bid-ask spread developed by Corwin and Schultz (2010) also measures price pressure (i.e., the deviation of stock transaction price from its fundamental value), but it is estimated from daily high and low prices. Daily high (low) prices are almost always buy (sell) orders. Hence the high-low price ratio reflects both the stock’s variance and its bid-ask spread. Further, the variance component of the high-low ratio is proportional to the return interval, while the bid-ask spread component is not. This allows to derive a spread estimator as a function of high-low ratios over one-day and two-day intervals. Particularly, this spread, denoted by $CSS$, is assumed to be constant over a two-day interval and can be estimated as

$$CSS = \frac{2 (e^{\alpha} - 1)}{1 + e^{\alpha}}$$

$$\alpha = \frac{\sqrt{2\beta} - \sqrt{\beta}}{3 - 2\sqrt{2}} - \sqrt{\frac{\gamma}{3 - 2\sqrt{2}}}$$

$$\beta = \sum_{j=0}^{1} \left[ \ln \left( \frac{H_{t+j}^O}{L_{t+j}^O} \right) \right]^2$$

$$\gamma = \left[ \ln \left( \frac{H_{t,t+1}^O}{L_{t,t+1}^O} \right) \right]^2$$

where $H_{t+j}^O (L_{t+j}^O)$ is the observed high (low) price on day $t+j$ with $j = 0, 1$, and $H_{t,t+1}^O (L_{t,t+1}^O)$ is the observed interday high (low) price over days $t$ and $t+1$. The spread for a month is the average of the spread estimates from all overlapping two-day sub periods within the month.

The Amihud’s illiquidity measure on day $d$ for stock $j$ is defined as the ratio of its absolute daily return to the daily trading volume (in billions of dollars), and it is designed to capture the price impact of the order flow. The average illiquidity of stock $j$ in month $m$ denoted by $ILLIQ_{jm}$ can be formulated as follows

$$ILLIQ_{jm} = \frac{1}{D_{jm}} \sum_{d=1}^{D_{jm}} \frac{|R_{jmd}|}{VOL_{jmd}}$$

where $D_{jm}$ is the number of trading days in month $m$ for stock $j$, $R_{jmd}$ is the return on stock $j$ on day $d$ of month $m$, and $VOL_{jmd}$ is the respective daily trading volume (in hundred millions

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29 Appendix A.3 details how to derive this spread under certain assumptions. One thing we would like to emphasize here is that this spread, different from the Roll’s spread, measures the percentage of the deviation of stock transaction price from its intrinsic value.
of dollars). This illiquidity measure, as shown in Amihud (2002), is positively related to the measures of price impact and fixed transaction costs. To reduce the potential non-stationarity of $ILLIQ_{jm}$ caused by market capitalization growth over the sample period, $ILLIQ_{jm}$ will be scaled by multiplying a factor which equals the ratio of the market capitalization of stock $j$ in the beginning of month $m$ to its value in the last month of the sample period.

The 5-6 columns of Table 5 report the cross-sectional statistics of time-series means of CSS and ILLIQ. Compared with the Roll’s spread, CSS is much smaller while both of them similarly measure price pressure. Remember that, as mentioned in footnote (29), the unit of CSS is percentage while the Roll’s spread is expressed in dollars. The time-series of monthly CSS and ILLIQ are plotted in Figure 8. It is clear that ILLIQ has been constantly declining over the period from 1993 to 2008, this declining trend will be also observed in the following regression results. CSS, like the Roll’s spread, does not change monotonically over time.

The cross-sectional statistics of the multivariate regression results for $CSS$ and $ILLIQ$ are reported in Table 11. Just as in the Roll’s spread case, we use time, market belief, trading volume, analyst belief dispersion, and return volatility as control variables in the multivariate regressions. Let us first discuss the results for $CSS$: again, the negative correlation between liquidity and the volatility of market belief holds for a majority of sample stocks, and it is significant for a large proportion of sample stocks. Precisely, CSS tends to widen for about $2/3$ of sample stocks when market belief is more volatile, this number is quite close to that one obtained in the Roll’s spread case, and the negative correlation is significant for about 23% of sample stocks – a fairly high proportion although lower than the proportion obtained with the Roll’s spread as an illiquidity measure. Further, the average betas of the volatility of market belief are positive with t-statistics greater than the 5% critical value in one-tailed test in three out of four cases. The effects of other explanatory variables on $CSS$ are similar as their effects on the Roll’s spread except that $CSS$ narrows in the periods when the market is optimistic. Given that both the Roll’s spread and $CSS$ measure price pressure, these results are not so surprising.

The volatility of market belief affects $ILLIQ$ in the predicted way although the effect is less strong in the sense that the proportions of the stocks for which the negative correlation between $ILLIQ$ and the volatility of market belief holds and is significant are lower and that the average betas of the volatility of market belief are insignificantly positive. The different results obtained in the $ILLIQ$ case are that: first, as implied in the inventory theory, $ILLIQ$ decreases with trading volume; second, in contrast to the Roll’s spread and $CSS$, $ILLIQ$ has constantly declined over the sample period, this is in harmony with the findings in previous studies.

We can conclude from these results that the negative correlation between stock liquidity and the volatility of market belief is robust to alternative illiquidity measures.
4 Conclusions

This paper studies the effects of investors’ heterogeneous beliefs on the trading volume, price volatility, and liquidity of stocks. Following Kurz and Motolesse (2008), we propose a simple theoretical model to show that the equilibrium stock price is linearly and positively correlated with market belief about stock future payoffs. In this paper, it is further shown that when market belief is more volatile, trading volume and liquidity decline while price volatility increases. Using the analyst forecast data on quarterly EPS provided by the I/B/E/S, we obtain empirical results supporting the theoretically predicted relations between the volatility of market belief about stock future payoffs and the trading volume, price volatility, and liquidity of underlying stocks, and these results are robust to various methods of estimating market belief and its volatility and to alternative illiquidity measures.

The sample in this paper consists of the S&P500 Index stocks which are generally issued by large publicly held companies. It will be very natural for us to extend the sample to include more stocks, particularly those of small and medium-sized companies, to check whether the conclusions in this paper will hold for a wider range of stocks. Having more stocks also makes it possible to study whether the relations between the volatility of market belief about stock future payoffs and the trading volume, price volatility and liquidity of underlying stocks vary across industry categories and different-sized firms.

We have seen that market belief, in addition to its volatility, also empirically affects the trading volume, price volatility and liquidity of stocks, but theoretically the relations among these variables are not justified. This issue needs more works in the future research.
References


A APPENDIX

A.1 Proof of Proposition II

For simplicity, we ignore in this appendix index $i$ identifying the investor who carries out the optimization. The dynamic programming problem is as follows: given initial values $(\theta_0, W_0)$, maximize

$$U_t = E_{\theta,c} \left[ \sum_{s=0}^{\infty} -\beta^{t+s-1} e^{-\frac{1}{2} \Sigma_{t+s} \mathbf{3}_t, g_t} \right]$$

subject to the following constraints

$$W_{t+1} = (W_t - C_t) R + \theta_t Q_{t+1}$$
$$Q_{t+1} = p_{t+1} + (d_{t+1} + \mu) - p_t R$$

and $\psi_t = (1, d_t, Z_t, g_t)^T$. The stochastic transition functions for $d_t, Z_t$ and $g_t$ are defined as

$$d_{t+1} = \lambda_d d_t + \lambda^d_{g} g_t + \epsilon^d_{t+1}$$
$$Z_{t+1} = \lambda_Z Z_t + \lambda^Z_{g} g_t + \epsilon^Z_{t+1}$$
$$g_{t+1} = \lambda_Z g_t + \epsilon^g_{t+1}$$

In the following, we define that

$$\Lambda_\psi = \begin{pmatrix} 1 & 0^T \\ 0 & \Lambda \end{pmatrix}, \quad \epsilon_t = \begin{pmatrix} 1 \\ \hat{\epsilon}_t \end{pmatrix}, \quad \hat{\epsilon}_t = \begin{pmatrix} \epsilon^d_t \\ \epsilon^Z_t \\ \epsilon^g_t \end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \hat{\sigma}_d^2 & \hat{\sigma}_{dz} & 0 \\ \hat{\sigma}_{dz} & \hat{\sigma}_z^2 & 0 \\ 0 & 0 & \sigma_g^2 \end{bmatrix} \right)$$

**Step One: Simplification** We define that, for an unknown symmetric matrix $V$

$$\Lambda = \begin{pmatrix} \lambda_d & 0 & \lambda^d_{g} \\ 0 & \lambda_z & \lambda^z_{g} \\ 0 & 0 & \lambda_z \end{pmatrix}, \quad V = \begin{pmatrix} v_{00} & v_0^T \\ v_0 & V_{11} \end{pmatrix}$$

Hence $\psi_{t+1} = \Lambda_\psi \psi_t + \Lambda_\epsilon \epsilon_{t+1}$, $\Lambda_\epsilon$ being a $4 \times 4$ matrix with a $I_3 \times 3$ matrix in the right bottom and zeros otherwise.

Assume that $p_t = P_0 + a_dd_t + a_z Z_t$ and we will verify this formula later when we solve for the equilibrium. Using this price formula, we can compute the excess share return in terms of the state variables

$$Q_{t+1} = a_dd_{t+1} + a_z Z_{t+1} + P_0 + d_{t+1} + \mu - (a_dd_t + a_z Z_t + P_0) R$$
$$= a^T \psi_t + b^T \epsilon_{t+1}$$
where \( a \) and \( b \) are defined as follows

\[
a = (P_0 (1 - R) + \mu, (a_d + 1) \lambda_d - R a_d, a_z \lambda_z - R a_z, (a_d + 1) \lambda'_d + a_z \lambda'_z)^T
\]

\[
b = (0, a_d + 1, a_z, 0)^T
\]

Hence, we have that \( E_t(Q_{t+1}) = a^T \psi_t \). Also, we use the notation \( \hat{b} = (a_d + 1, a_z, 0)^T \). Consider the following trial solution for the value function of the dynamic programming problem

\[
J(W_t; \psi_t; t) = -\beta^{t-1} \exp \left\{ -\frac{1}{2} \psi_t^T V \psi_t \right\}
\]

Compute the expression

\[
-\alpha W_{t+1} - \frac{1}{2} \psi_{t+1}^T V \psi_{t+1} = -\alpha (W_t - C_t) R + \alpha \theta_t [a^T \psi_t + b^T \epsilon_{t+1}]
\]

\[
= -\frac{1}{2} \psi_t^T \Lambda^T \psi \Lambda \psi_t - \psi_t^T \Lambda^T \psi \Lambda^T \psi_{t+1} - \frac{1}{2} \epsilon_{t+1} \Lambda^T \psi \Lambda \psi_{t+1}
\]

\[
= -A_t - e_t^T \epsilon_{t+1} - \frac{1}{2} \epsilon_{t+1}^T V_{11} \hat{\epsilon}_{t+1}
\]

where \( A_t \) and \( e_t \) are defined as

\[
A_t = \alpha (W_t - C_t) R + \alpha \theta_t a^T \psi_t + \frac{1}{2} \psi_t^T \Lambda^T \psi \Lambda \psi_t
\]

\[
e_t = \left( \alpha \theta_t \hat{b}^T + \psi_t^T \Lambda \right)^T
\]

with \( \Lambda_0 = (v_0, V_{11} \Lambda) \), which is a (3x4) matrix.

**Step Two: The Bellman Equation** The Bellman equation for this problem with \( \gamma = \frac{1}{\tau} \) can be written in the form

\[
J_t = \max_{\theta_t, C_t} \left\{ -\beta^{t-1} \exp\{ -\gamma C_t \} + E(J_{t+1}|3_t, g_t) \right\}
\]

\[
= \max_{\theta_t, C_t} \left\{ -\beta^{t-1} \exp\{ -\gamma C_t \} - \beta^t E_t \left( \exp \left\{ -A_t - e_t^T \hat{\epsilon}_{t+1} - \frac{1}{2} \epsilon_{t+1}^T V_{11} \hat{\epsilon}_{t+1} \right\} \right) \right\}
\]

It can be shown that

\[
E_t \left( \exp \left\{ -A_t - e_t^T \hat{\epsilon}_{t+1} - \frac{1}{2} \epsilon_{t+1}^T V_{11} \hat{\epsilon}_{t+1} \right\} \right) = \left| 1 + \sum V_{11} \right|^{-\frac{1}{2}} \exp \left[ \frac{1}{2} e_t^T (1 + \sum V_{11})^{-1} \sum e_t - A_t \right]
\]

Also

\[
\frac{1}{2} e_t^T (1 + \sum V_{11})^{-1} \sum e_t = \frac{1}{2} \left( \alpha \theta_t \hat{b}^T + \psi_t^T \Lambda^T \right) (1 + \sum V_{11})^{-1} \sum \left( \alpha \theta_t \hat{b}^T + \psi_t^T \Lambda^T \right)^T
\]

\[
= \frac{1}{2} \alpha^2 \theta_t^2 \hat{b}^T \Omega^2 + \alpha \theta_t \hat{b}^T \Omega \Lambda \psi_t + \frac{1}{2} \psi_t^T \Lambda^T \Lambda \psi_t
\]

where \( \Omega = (1 + \sum V_{11})^{-1} \sum \).

Hence, we have an expression for the exponent term

\[
\frac{1}{2} e_t^T (1 + \sum V_{11})^{-1} \sum e_t - A_t = -\alpha (W_t - C_t) R - \alpha \theta_t \left( a^T - \hat{b}^T \Omega \right) \psi_t
\]

\[
+ \frac{1}{2} \alpha^2 \theta_t^2 \hat{b}^T \Omega^2 - \frac{1}{2} \psi_t^T \left( \Lambda^T V \Lambda - \Lambda^T \Omega \Lambda \right) \psi_t
\]

37
The first order condition with respect to \( \theta \) leads to
\[
- \alpha \left[ a^T - \hat{b}^T \Omega_0 \right] \psi_t + \alpha^2 \theta \hat{b}^T \Omega \hat{b} = 0
\]

The demand of each investor thus equals (since \( E_t (Q_{t+1}) = a^T \psi_t \))
\[
\theta_t = \frac{1}{\alpha \hat{b}^T \Omega \hat{b}} \left[ (a^T - \hat{b}^T \Omega_0) \psi_t \right] = \frac{1}{\alpha \hat{b}^T \Omega \hat{b}} \left[ E_t (Q_{t+1}) + u^T \psi_t \right]
\]
with \( u^T = - \hat{b}^T \Omega \Lambda_0 \).

Note that the vector \( u \) is the same for all investors since the assumption made in the text is that all investors are identically the same except for their belief states \( g_t \) and the last equation shows that the vector \( u \) depends only upon parameters of the stochastic structure.

**Step Three: The Adjusted Variance and Constants**

Define that
\[
\hat{\sigma}_Q^2 = \hat{b}^T \hat{\Omega} \hat{b}
\]

which is the variance of the excess return function where the covariance matrix used is not \( \sum \) but rather \( \Omega \). We now have
\[
\alpha^2 \theta_t \hat{b}^T \Omega \hat{b} = \frac{1}{\hat{b}^T \Omega \hat{b}} \left\{ \psi_t^T \left[ a^T - \hat{b}^T \Omega_0 \right]^T \left[ a^T - \hat{b}^T \Omega_0 \right] \psi_t \right\}
\]

Hence the optimized value of the exponent is simply
\[
\frac{1}{2} \hat{e}_t^T (1 + \sum V_{11})^{-1} \sum e_t - A_t = - \alpha (W_t - C_t) R - \frac{1}{2} \psi_t^T M \psi_t
\]

where
\[
M = \frac{1}{\hat{b}^T \Omega \hat{b}} \left[ a^T - \hat{b}^T \Omega_0 \right]^T \left[ a^T - \hat{b}^T \Omega_0 \right] + \left[ \Lambda_{\psi}^T V \Lambda_{\psi} - \Lambda_{0}^T \Omega \Lambda_0 \right]
\]

Now take the derivative with respect to \( C_t \) and equate to zero to obtain
\[
\gamma \exp \{ - \gamma C_t \} = \alpha R \beta \left[ 1 + \sum V_{11} \right]^{-\frac{1}{2}} \exp \left\{ - \alpha (W_t - C_t) R - \frac{1}{2} \psi_t^T M \psi_t \right\}
\]

Let \( G = \left[ 1 + \sum V_{11} \right]^{-\frac{1}{2}} \). Hence, the solution for \( C_t \) must satisfy
\[
\gamma C_t = - \ln \left[ \frac{\alpha R \beta G}{\gamma} \right] + \alpha (W_t - C_t) R + \frac{1}{2} \psi_t^T M \psi_t
\]

We finally have
\[
C_t = -\frac{1}{\gamma + \alpha R} \ln \left[ \frac{\alpha R \beta G}{\gamma} \right] + \frac{\alpha R}{\gamma + \alpha R} W_t + \frac{1}{2 (\gamma + \alpha R)} \psi_t^T M \psi_t
\]
Substituting the optimal consumption-investment policy back into the Bellman equation, we obtain

$$\alpha = \frac{r \gamma}{R}$$

$$\exp \left\{ -\frac{1}{2} \psi^T \left[ \frac{M}{R} - V + 2 \left( \gamma \mathcal{C} + \ln \left( \frac{r}{R} \right) \right) i_{11} \right] \psi \right\} = 1$$

where $i_{11}$ is a $4 \times 4$ matrix with the element $(1, 1)$ being one and all other elements being zero and $\mathcal{C} = -\frac{1}{\gamma R} \ln (r \beta G)$. This leads to the following equation for $V^{30}$

$$\frac{M}{R} - V + 2 \left( \gamma \mathcal{C} + \ln \left( \frac{r}{R} \right) \right) i_{11} = 0$$

**Step Four: The Equilibrium Pricing** Average $\theta_t$ over all investors, given that the total supply of stock is one, we obtain

$$\frac{r \hat{\sigma}^2_Q}{RN_T} = \left[ E_t \left( p_{t+1} + d_{t+1} + \mu \right) - Rp_t + (u_0 + u_1d_t + (u_2 + u_3)Z_t) \right]$$

Use the relationships (19) and (20) to deduce a linear difference equation for $p_t$

$$E_t (p_{t+1}) = Rp_t - (\lambda_d + u_1) d_t - (\lambda^q_d + u_2 + u_3) Z_t + \frac{r \hat{\sigma}^2_Q}{RN_T} - (\mu + u_0)$$

This is a typical linear difference problem. The parameters and the market average processes of $d_t$ and $Z_t$ satisfy the conditions, as specified in Blanchard and Kahn (1980), with which there exists an unique equilibrium solution for stock price which, like the equation (3) in Blanchard and Kahn (1980), takes a form as follows

$$p_t = \sum_{i=0}^{\infty} R^{-i-1} \left\{ \gamma_1^T \mathbb{E} (X_{t+i} | \mathcal{F}_t) \right\}$$

(52)

where

$$\gamma_1 = \begin{bmatrix} (\mu + u_0) - \frac{r \hat{\sigma}^2_Q}{RN_T}, \lambda_d + u_1, \lambda^q_d + u_2 + u_3 \end{bmatrix}^T$$

$$X_t = [1, d_t, Z_t]^T$$

The price function in the equation (22) can be obtained by simplifying the price formula (27).

---

30 The equation (ii) determining $V$ in the appendix A of Kurz and Motolesse (2008) seems incorrect.
A.2 Some Moments of Absolute Normal Random Variables

Let $Y$ be a normally distributed random variable with mean $\mu$ and $\sigma^2$, that is, $Y \sim N(\mu, \sigma^2)$. The expectation of the absolute value of $Y$ is:

$$ E(|Y|) = \int_{-\infty}^{+\infty} \frac{|y|}{\sqrt{2\pi\sigma}} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right] dy $$

$$ = \int_{-\mu}^{+\infty} \frac{x+\mu}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx - \int_{-\infty}^{-\mu} \frac{x+\mu}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx $$

$$ = 2 \int_{-\mu}^{+\infty} \frac{x}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx + \mu \left[1 - 2 \int_{-\infty}^{-\mu} \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx\right] $$

It is easy to show that

$$ \int_{-\mu}^{+\infty} \frac{x}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = \sigma \sqrt{\frac{2}{\pi}} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) $$

Thus, we have that the expectation of the absolute value of $Y$ is given by

$$ E(|Y|) = \sigma \sqrt{\frac{2}{\pi}} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) + \mu \left[1 - 2 \Phi\left(-\frac{\mu}{\sigma}\right)\right] $$

The variance of $|Y|$ is easy to calculate with $E(|Y|)$

$$ \text{var}(|Y|) = E(|Y|^2) - E(|Y|)^2 = \text{var}(Y^2) + E(Y^2) - E(|Y|)^2 $$

$$ = \mu^2 + \sigma^2 - \left[\sigma \sqrt{\frac{2}{\pi}} \exp\left(-\frac{\mu^2}{2\sigma^2}\right) + \mu \left(1 - 2 \Phi\left(-\frac{\mu}{\sigma}\right)\right)\right]^2 $$

These results can be used to calculate the expectations of $V_t^i$ and $|\Lambda_t|$ defined in Section 2.

A.3 The Bid-Ask Spread of Corwin and Schultz (2010)

Assume that the real value of stock price follows a diffusion process and there is a spread of $S\%$ which is constant over a two-day interval. Because of the spread, observed prices deviate from the actual values by $(S/2)\%$. We further assume that the high price of the day is a buy order and is grossed up by half of the spread while the low price of the day is a sell order and is discounted by one half of the spread. Hence the observed high-low range contains both the range of the actual prices and the bid-ask spread. Denote $H_t^A(L_t^A)$ as the actual high (low) stock price on day $t$ and $H_t^O(L_t^O)$ as the observed high (low) stock price for day $t$, we can write

$$ \left[\ln\left(\frac{H_t^O}{L_t^O}\right)\right]^2 = \left[\ln\left(\frac{H_t^A (1+S/2)}{L_t^A (1-S/2)}\right)\right]^2 $$

Rearranging this equation gives

$$ \left[\ln\left(\frac{H_t^O}{L_t^O}\right)\right]^2 = \left[\ln\left(\frac{H_t^A}{L_t^A}\right)\right]^2 + 2 \left[\ln\left(\frac{H_t^A}{L_t^A}\right)\right] \left[\ln\left(\frac{2+S}{2-S}\right)\right] + \left[\ln\left(\frac{2+S}{2-S}\right)\right]^2 $$

(53)
Under the assumptions that stock prices follow the usual geometric Brownian motion and the price is observed continuously, Parkinson (1980) and Garman and Klass (1980) show that

\[ E \left\{ \frac{1}{T} \sum_{t=1}^{T} \left[ \ln \left( \frac{H_t^A}{L_t} \right) \right]^2 \right\} = k_1 \sigma_{HL}^2 \]  \hspace{1cm} (55)

\[ E \left\{ \frac{1}{T} \sum_{t=1}^{T} \left[ \ln \left( \frac{H_t^A}{L_t} \right) \right] \right\} = k_2 \sigma_{HL} \]  \hspace{1cm} (56)

where \( k_1 = 4\ln(2) \) and \( k_2 = \sqrt{8/\pi} \). Taking expectations of (51) and substituting (52) and (53) yields

\[ E \left\{ \left[ \ln \left( \frac{H_t^O}{L_t^O} \right) \right]^2 \right\} = k_1 \sigma_{HL}^2 + 2k_2 \sigma_{HL} \left[ \ln \left( \frac{2 + S}{2 - S} \right) \right] + \left[ \ln \left( \frac{2 + S}{2 - S} \right) \right]^2 \]  \hspace{1cm} (57)

The expectation of the sum of (54) over two single days is

\[ E \left\{ \sum_{j=0}^{1} \left[ \ln \left( \frac{H_{t+j}^O}{L_{t+j}^O} \right) \right]^2 \right\} = 2k_1 \sigma_{HL}^2 + 4k_2 \sigma_{HL} \left[ \ln \left( \frac{2 + S}{2 - S} \right) \right] + 2 \left[ \ln \left( \frac{2 + S}{2 - S} \right) \right]^2 \]  \hspace{1cm} (58)

To simplify, we set

\[ \alpha = \left[ \ln \left( \frac{2 + S}{2 - S} \right) \right], \quad \beta = \sum_{j=0}^{1} \left[ \ln \left( \frac{H_{t+j}^O}{L_{t+j}^O} \right) \right]^2 \]  \hspace{1cm} (59)

This allows us to rewrite (55) as

\[ 2k_1 \sigma_{HL}^2 + 4k_2 \sigma_{HL} \alpha + 2\alpha^2 - \beta = 0 \]  \hspace{1cm} (60)

Similarly, squaring the log price range over a two-day period yields

\[ \left[ \ln \left( \frac{H_{t,t+1}^O}{L_{t,t+1}^O} \right) \right]^2 = \left[ \ln \left( \frac{H_{t,t+1}^A}{L_{t,t+1}^A} \right) \right]^2 + 2 \left[ \ln \left( \frac{H_{t,t+1}^A}{L_{t,t+1}^A} \right) \right] \left[ \ln \left( \frac{2 + S}{2 - S} \right) \right] + \left[ \ln \left( \frac{2 + S}{2 - S} \right) \right]^2 \]  \hspace{1cm} (61)

where \( H_{t,t+1}(L_{t,t+1}) \) is the high (low) price over the two days \( t \) and \( t + 1 \). Set

\[ \gamma = \left[ \ln \left( \frac{H_{t,t+1}^O}{L_{t,t+1}^O} \right) \right]^2 \]  \hspace{1cm} (62)

if the variance and the spread are constant over two-day periods and true returns are serially uncorrelated, the variance component of the high-low price is twice as large for a two-day period as it is for a single day, but the spread component of the price range is unaffected by the time interval. Using the notation \( \gamma \) and taking expectations in (58) yields

\[ 2k_1 \sigma_{HL}^2 + 2\sqrt{2}k_2 \sigma_{HL} \alpha + 2\alpha^2 - \gamma = 0 \]  \hspace{1cm} (63)
Now we have two equations (57) and (60) with two unknown parameters $\sigma_{HL}$ and $\alpha$. Generally, there are no closed-form solutions for $\sigma_{HL}$ and $\alpha$, but a further simplifying assumption allows us to achieve this. We ignore Jensen’s inequality in (53) and assume that

$$E\left\{ \frac{1}{T} \sum_{t=1}^{T} \left[ \ln \left( \frac{H_t}{L_t} \right) \right] \right\} = \sqrt{E\left\{ \frac{1}{T} \sum_{t=1}^{T} \left[ \ln \left( \frac{H_t}{L_t} \right) \right]^2 \right\}} = \sqrt{k_1 \sigma_{HL}^2} = \sqrt{k_1 \sigma_{HL}} \quad (64)$$

then $k_1 = k_2^2$. Using this assumption and solving equations (57) and (60) yields a closed-form solution for $\alpha$, which is exactly the one in (47). The spread $S$ can be obtained by a simple transformation of $\alpha$ in (56). As $\beta$ and $\gamma$ are observed, then $\alpha$ is known, so is the spread $S$. 
Figure 1A: This figure plots the numerically computed values for the coefficients $u_0^*, u_1^*, P_0^*$ and $\sigma_d^*$ against $\lambda_d$ with $r=0.1, \mu=2, \beta=0.9, \tau=2.0, \sigma_d^2=1.0, N=2$.

Figure 1B: This figure plots the numerically computed values for the coefficients $u_0^*, u_1^*, P_0^*$ and $\sigma_d^*$ against $\sigma_d^2$ with $r=0.1, \mu=2, \beta=0.9, \tau=2.0, \lambda_d=0.5, N=2$. 
Figure 2A: This figure plots the numerically computed values for the coefficients \(u_0\), \(u_1\), \(u_2\), \(u_3\), \(P_0\), \(a_d\), \(a_Z\) and \((a_d + 1)\lambda_d^0 + a_Z \lambda_Z^0 + u_3\) against both \(\lambda_d\) and \(\lambda_d + \lambda_d^0\) with \(r=0.1\), \(\mu=2\), \(\beta=0.9\), \(\tau=2.0\), \(\lambda_d^0=0.3\), \(\sigma_d^2=1.0\), \(\sigma_Z^2=0.6\), \(\sigma_g^2=1.0\), \(\hat{\sigma}_d=\hat{\sigma}_d=0\), \(N=2\).
Figure 2B: This figure plots the numerically computed values for the coefficients $u_0$, $u_1$, $u_2$, $u_3$, $P_0$, $a_d$, $a_Z$ and $(a_d + 1)\lambda^0_d + a_Z\lambda^0_Z + u_3$ against both $\sigma_d^2$ and $\sigma_Z^2$ with $r=0.1$, $\mu=2$, $\beta=0.9$, $\tau=2.0$, $\lambda_d=0.5$, $\lambda^0_d=0.3$, $\lambda_Z=0.5$, $\lambda^0_Z=0.3$, $\sigma^2_g=1.0$, $\hat{\sigma}_{dz}=\hat{\sigma}_{dg}=0$, $N=2$. 
Figure 3: This figure plots the numerically computed expected trading volume $\bar{V}$ against $\sigma^2_Z$ with $r=0.1$, $\mu=2$, $\beta=0.9$, $\tau=2.0$, $\lambda_d=0.5$, $\lambda^g_d=0.3$, $\lambda_Z=0.5$, $\lambda^g_Z=0.3$, $\sigma_d^2=1.0$, $\sigma^2_g=1.0$, $\hat{\sigma}_{dz}=\hat{\sigma}_{dg}=0$, $N=2$. 
Figure 4A: This figure plots the numerically computed price volatility $\sigma_p^2$ against $\sigma_d^2$ with $r$ =0.1, $\mu=2$, $\beta=0.9$, $\tau=2.0$, $\lambda_d=0.5$, $N=2$.

Figure 4B: This figure plots the numerically computed price volatility $\sigma_p^2$ against both $\sigma_d^2$ and $\sigma_Z^2$ with $r=0.1$, $\mu=2$, $\beta=0.9$, $\tau=2.0$, $\lambda_d=0.5$, $\lambda_d^0=0.3$, $\lambda_Z=0.5$, $\lambda_Z^0=0.3$, $\sigma_g^2=1.0$, $\hat{\sigma}_{dZ}=\hat{\sigma}_{dY}=0$, $N=2$. 
Figure 5: This figure plots the numerically computed expected illiquidity measure $|\Lambda_i|$ against $\sigma_Z^2$ with $r=0.1$, $\mu=2$, $\beta=0.9$, $\tau=2.0$, $\lambda_d=0.5$, $\lambda_d^0=0.3$, $\lambda_Z=0.5$, $\lambda_Z^0=0.3$, $\sigma_d^2=1.0$, $\sigma_Z^2=1.0$, $\hat{\sigma}_{dz} = \hat{\sigma}_{dz^0}=0$, $N=2$.

Figure 6: This figure plots, for 114 S&P500 Index stocks, the time series of monthly cross-sectional mean and median of financial analysts over the period from 1984 to 2008.
Figure 7: This figure simultaneously plots, for 114 S&P500 Index stocks, the time series of monthly market beliefs estimated with both the Brown and Rozell (1979) model and the seasonal random walk with drift model, over the period from 1993 to 2008.
Figure 8: This figure simultaneously plots, for 114 S&P500 Index stocks, the time series of monthly dependent variables: turnover rate, price volatility, the Roll’s (1984) effective bid-ask spread, the Corwin and Schultz’s (CS) (2010) bid-ask spread, and the Amihud’s (2002) illiquidity measure, over the period from 1993 to 2008.
Table 1
G.I.C.S. Industry Breakdown of the Sample.

<table>
<thead>
<tr>
<th>GICS Two-Digit Industry Code</th>
<th>Sector</th>
<th>Number of Stocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>Energy</td>
<td>10</td>
</tr>
<tr>
<td>15</td>
<td>Materials</td>
<td>14</td>
</tr>
<tr>
<td>20</td>
<td>Industrials</td>
<td>24</td>
</tr>
<tr>
<td>25</td>
<td>Consumer Discretionary</td>
<td>19</td>
</tr>
<tr>
<td>30</td>
<td>Consumer Staples</td>
<td>14</td>
</tr>
<tr>
<td>35</td>
<td>Health Care</td>
<td>11</td>
</tr>
<tr>
<td>40</td>
<td>Financials</td>
<td>8</td>
</tr>
<tr>
<td>45</td>
<td>Information Technology</td>
<td>12</td>
</tr>
<tr>
<td>50</td>
<td>Telecommunication Services</td>
<td>0</td>
</tr>
<tr>
<td>55</td>
<td>Utilities</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>114</td>
</tr>
</tbody>
</table>
Table 2
The cross-sectional statistics of time series means of the estimated coefficients for the Brown and Rozeff (1979) model (BR) and the seasonal random walk with drift model (SRWD).

This table reports the cross-sectional statistics of time series means of the estimated coefficients for both the Brown and Rozeff (1979) model and the seasonal random walk with drift model. The seasonal random walk with drift model takes a form as follows:

\[ E(Q_t) = \delta + Q_{t-4} \]

where \( E(Q_t) \) is the earnings forecast for quarter \( t \), \( \delta \) is a (typically) positive trend, and \( Q_{t-4} \) is the actual earnings for quarter \( t-4 \). The Brown and Rozeff (1979) model is formulated as:

\[ E(Q_t) = \delta + Q_{t-4} + \phi (Q_{t-1} - Q_{t-5}) + \theta \epsilon_{t-4} \]

where \( Q_{t-k} \) is the actual earnings for quarter \( t-k \) and \( \epsilon_{t-4} \) is the white noise earnings shock experienced at quarter \( t-4 \). The sample consists of 114 S&P500 Index stocks.

<table>
<thead>
<tr>
<th></th>
<th>SRWD</th>
<th>BR</th>
</tr>
</thead>
<tbody>
<tr>
<td># of negative medians</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>Mean</td>
<td>0.027</td>
<td>0.017</td>
</tr>
<tr>
<td>Median</td>
<td>0.024</td>
<td>0.015</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.024</td>
<td>0.018</td>
</tr>
</tbody>
</table>

Table 3
The cross-sectional statistics of time series means, skewness, and kurtosis of market belief.

This table reports the cross-sectional statistics of time series means, skewness and kurtosis of market belief \( Z_t \) estimated by subtracting the quarterly EPS predicted with both the Brown and Rozeff (1979) model and the seasonal random walk with drift model from the mean of analysts’ earnings forecasts, and the estimated beliefs are respectively denoted by \( Z_{br} \) and \( Z_{srwd} \). The sample consists of 114 S&P500 Index stocks.

<table>
<thead>
<tr>
<th></th>
<th>( Z_{br} )</th>
<th>( Z_{srwd} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.021</td>
<td>0.035</td>
</tr>
<tr>
<td>Median</td>
<td>0.011</td>
<td>0.024</td>
</tr>
</tbody>
</table>
Table 4

The cross-sectional statistics of time series means of the volatility of market belief.

This table reports the cross-sectional statistics of time series means of the volatility of market belief estimated using either rolling regression method or GARCH(1,1) model as suggested in Section 3.4. The sample consists of 114 S&P500 Index stocks.

<table>
<thead>
<tr>
<th></th>
<th>Volatility_{zbr}</th>
<th></th>
<th>Volatility_{zsrwd}</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RR Method</td>
<td>GARCH(1,1)</td>
<td>RR Method</td>
</tr>
<tr>
<td>Mean</td>
<td>0.033</td>
<td>0.061</td>
<td>0.020</td>
</tr>
<tr>
<td>Median</td>
<td>0.006</td>
<td>0.012</td>
<td>0.005</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.089</td>
<td>0.161</td>
<td>0.048</td>
</tr>
</tbody>
</table>

Table 5

The cross-sectional statistics of time series means of dependent variables.

This table reports the cross-sectional statistics of time series means of the following dependent variables: turnover rate (Turnover), price volatility (Volatility), the Roll’s (1984) effective bid-ask spread (Roll), the Corwin and Schultz’s (2010) bid-ask spread (CSS), the Amihud’s (2002) illiquidity measure (ILLIQ). The sample consists of 114 S&P500 Index stocks.

<table>
<thead>
<tr>
<th></th>
<th>Turnover</th>
<th>Volatility</th>
<th>Roll</th>
<th>CSS</th>
<th>ILLIQ</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.124</td>
<td>9.477</td>
<td>0.232</td>
<td>0.0011</td>
<td>0.102</td>
</tr>
<tr>
<td>Median</td>
<td>0.100</td>
<td>6.019</td>
<td>0.207</td>
<td>0.0002</td>
<td>0.038</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>0.084</td>
<td>18.208</td>
<td>0.145</td>
<td>0.0022</td>
<td>0.307</td>
</tr>
</tbody>
</table>
Table 6
Explaining the time-series variations of the trading volume, price volatility, and liquidity of stocks with the volatility of market belief estimated using rolling regression method.

This table reports the cross-sectional statistics of the following OLS regression results:

\[ Y_{i,t} = \alpha_i + \beta_i vZ_{i,t}^j + \epsilon_{i,t} \ (i = 1, ..., 114) \]

where superscript index \( j = \text{BR or SRWD} \), referring to the time-series model used to predict quarterly earnings per share, \( Y_i \) is either the detrended stock turnover rate or price volatility or the Roll’s spread, \( vZ_t \) is the volatility of market belief estimated using rolling regression method. Cross-sectional averages of estimated betas are reported with t-statistics in parentheses. ‘%positive’ (‘%negative’) means the percentage of positive (negative) betas, while ‘%+significance’ (‘%-significance’) gives the percentage with t-statistics greater than +1.645 (smaller than -1.645) (the 5% critical value in one-tailed test). The sample consists of 114 S&P500 Index stocks.

<table>
<thead>
<tr>
<th></th>
<th>Brown and Rozeff (1979) Model</th>
<th>Seasonal Random Walk with Drift Model</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Trading Volume</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( vZ_t )</td>
<td>2.777</td>
<td>-5.928</td>
</tr>
<tr>
<td></td>
<td>(0.76)</td>
<td>(-1.66)</td>
</tr>
<tr>
<td>% negative</td>
<td>58.77</td>
<td>68.42</td>
</tr>
<tr>
<td>% -significant</td>
<td>1.754</td>
<td>0.877</td>
</tr>
<tr>
<td>( R^2(% \ \text{mean}) )</td>
<td>0.359</td>
<td>0.336</td>
</tr>
<tr>
<td></td>
<td>median</td>
<td>0.129</td>
</tr>
<tr>
<td><strong>Panel B: Price Volatility</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( vZ_t/(%100) )</td>
<td>6.627</td>
<td>7.624</td>
</tr>
<tr>
<td></td>
<td>(7.09)</td>
<td>(6.59)</td>
</tr>
<tr>
<td>% positive</td>
<td>84.21</td>
<td>85.97</td>
</tr>
<tr>
<td>% +significant</td>
<td>60.53</td>
<td>64.91</td>
</tr>
<tr>
<td>( R^2(% \ \text{mean}) )</td>
<td>6.820</td>
<td>7.083</td>
</tr>
<tr>
<td></td>
<td>median</td>
<td>2.989</td>
</tr>
<tr>
<td><strong>Panel C: Liquidity</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( vZ_t )</td>
<td>3.194</td>
<td>3.642</td>
</tr>
<tr>
<td></td>
<td>(5.53)</td>
<td>(5.49)</td>
</tr>
<tr>
<td>% positive</td>
<td>75.44</td>
<td>77.19</td>
</tr>
<tr>
<td>% +significant</td>
<td>42.11</td>
<td>48.25</td>
</tr>
<tr>
<td>( R^2(% \ \text{mean}) )</td>
<td>3.591</td>
<td>3.677</td>
</tr>
<tr>
<td></td>
<td>median</td>
<td>1.206</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.516</td>
</tr>
</tbody>
</table>
### Table 7

**Multivariate Regression: Trading Volume**

This table reports the cross-sectional statistics of the following OLS regression results:

\[
Y_{i,t} = \alpha_i + \beta_{i,1}vZ_{i,t} + \beta_{i,2}Z_{i,t}^j + \beta_{i,3}Price_{i,t} + \beta_{i,4}vRet_{i,t} + \epsilon_{i,t} \quad (i = 1, \ldots, 114)
\]

where superscript index \( j = BR \) or SRWD, referring to the time-series model used to predict quarterly earnings per share, \( Y_i \) is the detrended stock turnover rate, \( vZ_t \) is the volatility of market belief estimated using rolling regression method, \( Z_t \) is market belief, \( Price_t \) is the average of daily prices in month \( t \), and \( vRet_t \) is return volatility. Cross-sectional averages of estimated betas are reported with t-statistics in parentheses. '%negative' means the percentage of negative betas, while '%-significance' gives the percentage with t-statistics smaller than -1.645 (the 5% critical value in one-tailed test). The sample consists of 114 S&P500 Index stocks.

<table>
<thead>
<tr>
<th></th>
<th>Brown and Rozeff (1979) Model</th>
<th>Seasonal Random Walk with Drift Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( vZ_{i,t} )</td>
<td>-0.682</td>
<td>-8.411</td>
</tr>
<tr>
<td></td>
<td>(-0.19)</td>
<td>(-1.40)</td>
</tr>
<tr>
<td>% negative</td>
<td>60.53</td>
<td>70.18</td>
</tr>
<tr>
<td>% -significant</td>
<td>3.509</td>
<td>5.263</td>
</tr>
<tr>
<td>( Z_t )</td>
<td>0.469</td>
<td>0.736</td>
</tr>
<tr>
<td></td>
<td>(1.27)</td>
<td>(2.28)</td>
</tr>
<tr>
<td>( Price_t )</td>
<td>0.017</td>
<td>0.016</td>
</tr>
<tr>
<td></td>
<td>(7.71)</td>
<td>(6.82)</td>
</tr>
<tr>
<td>( vRet_t(\times 100) )</td>
<td>23.57</td>
<td>22.99</td>
</tr>
<tr>
<td></td>
<td>(12.7)</td>
<td>(11.8)</td>
</tr>
<tr>
<td>( R^2(% \text{ mean}) )</td>
<td>7.988</td>
<td>7.902</td>
</tr>
<tr>
<td>median</td>
<td>6.951</td>
<td>7.362</td>
</tr>
</tbody>
</table>
Table 8
Multivariate Regression: Price Volatility

This table reports the cross-sectional statistics of the following OLS regression results:

\[ Y_{i,t} = \alpha_i + \beta_{i,1}vZ_{i,t}^j + \beta_{i,2}Z_{i,t}^j + \beta_{i,3}Size_{i,t} + \beta_{i,4}Volume_{i,t} + \epsilon_{i,t} \quad (i = 1, ..., 114) \]

where superscript index \( j = \) BR or SRWD, referring to the time-series model used to predict quarterly earnings per share, \( Y_t \) is price volatility, \( vZ_t \) is the volatility of market belief estimated using rolling regression method, \( Z_t \) is market belief, \( Size_t \) is the market capitalization of a firm’s stock in the beginning of month \( t \), and \( Volume_t \) is stock turnover rate. Cross-sectional averages of estimated betas are reported with t-statistics in parentheses. ‘%positive’ means the percentage of positive betas, while ‘%+significance’ gives the percentage with t-statistics greater than +1.645 (the 5% critical value in one-tailed test). The sample consists of 114 S&P500 Index stocks.

<table>
<thead>
<tr>
<th></th>
<th>Brown and Rozeff (1979) Model</th>
<th>Seasonal Random Walk with Drift Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( vZ_t (/100) )</td>
<td>6.244</td>
<td>6.610</td>
</tr>
<tr>
<td></td>
<td>(6.85)</td>
<td>(6.97)</td>
</tr>
<tr>
<td>% positive</td>
<td>75.44</td>
<td>79.83</td>
</tr>
<tr>
<td>% + significant</td>
<td>54.39</td>
<td>55.26</td>
</tr>
<tr>
<td>( Z_t )</td>
<td>32.25</td>
<td>30.43</td>
</tr>
<tr>
<td></td>
<td>(3.44)</td>
<td>(3.44)</td>
</tr>
<tr>
<td>( Size_t )</td>
<td>0.014</td>
<td>-0.055</td>
</tr>
<tr>
<td></td>
<td>(0.04)</td>
<td>(-0.16)</td>
</tr>
<tr>
<td>( Volume_t )</td>
<td>1.497</td>
<td>1.487</td>
</tr>
<tr>
<td></td>
<td>(1.31)</td>
<td>(1.32)</td>
</tr>
<tr>
<td>( R^2(%) ) mean</td>
<td>16.18</td>
<td>15.94</td>
</tr>
<tr>
<td>median</td>
<td>14.17</td>
<td>14.03</td>
</tr>
</tbody>
</table>
Table 9
Multivariate Regression: Liquidity

This table reports the cross-sectional statistics of the following OLS regression results:

\[ Y_{i,t} = \alpha_i + \beta_{i,1}vZ^2_{i,t} + \beta_{i,2}Z^3_{i,t} + \beta_{i,3}t + \beta_{i,4}Volume_{i,t} \\
+ \beta_{i,5}Dispersion_{i,t} + \beta_{i,6}vRet_{i,t} + \epsilon_{i,t} \quad (i = 1, \ldots, 114) \]

where superscript index \( j = \text{BR or SRWD} \), referring to the time-series model used to predict quarterly earnings per share, \( Y_t \) is the Roll’s spread, \( vZ_t \) is the volatility of market belief estimated using rolling regression method, \( Z_t \) is market belief, \( t \) is time, \( Volume_t \) is stock turnover rate, \( Dispersion_t \) is analyst forecast dispersion, and \( vRet_t \) is return volatility. Cross-sectional averages of estimated betas are reported with t-statistics in parentheses. ‘%positive’ means the percentage of positive betas, while ‘%+significance’ gives the percentage with t-statistics greater than +1.645 (the 5% critical value in one-tailed test). The sample consists of 114 S&P500 Index stocks.

<table>
<thead>
<tr>
<th></th>
<th>Brown and Rozef (1979) Model</th>
<th>Seasonal Random Walk with Drift Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>( vZ_t )</td>
<td>2.347</td>
<td>2.732</td>
</tr>
<tr>
<td></td>
<td>(4.77)</td>
<td>(5.03)</td>
</tr>
<tr>
<td>% positive</td>
<td>70.18</td>
<td>68.67</td>
</tr>
<tr>
<td>% +significant</td>
<td>32.46</td>
<td>37.72</td>
</tr>
<tr>
<td>( Z_t )</td>
<td>0.210</td>
<td>0.236</td>
</tr>
<tr>
<td></td>
<td>(4.90)</td>
<td>(5.32)</td>
</tr>
<tr>
<td>( t )</td>
<td>0.0001</td>
<td>-0.395</td>
</tr>
<tr>
<td></td>
<td>(0.69)</td>
<td>(-1.49)</td>
</tr>
<tr>
<td>( Volume_t )</td>
<td>0.002</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>(1.56)</td>
<td>(1.15)</td>
</tr>
<tr>
<td>( Dispersion_t )</td>
<td>-0.474</td>
<td>-0.395</td>
</tr>
<tr>
<td></td>
<td>(-1.87)</td>
<td>(-1.49)</td>
</tr>
<tr>
<td>( vRet_t(/100) )</td>
<td>1.440</td>
<td>1.470</td>
</tr>
<tr>
<td></td>
<td>(12.2)</td>
<td>(12.4)</td>
</tr>
<tr>
<td>( R^2(%) mean )</td>
<td>16.82</td>
<td>16.91</td>
</tr>
<tr>
<td>Median</td>
<td>15.43</td>
<td>15.67</td>
</tr>
</tbody>
</table>
Table 10
Explaining the time series variations of the trading volume, price volatility, and liquidity of stocks with the volatility of market belief estimated using GARCH(1,1) model

This table reports the cross-sectional statistics of the following OLS regression results:

\[ Y_{i,t} = \alpha_i + \beta_{i,1}vZ_{i,t} + \beta_i Vector_{i,t} + \epsilon_{i,t} \quad (i = 1, \ldots, 114) \]

where superscript index \( j = BR \) or SRWD, referring to the time-series model used to predict quarterly earnings per share, \( Y_t \) is either the detrended stock turnover rate or price volatility or the Roll’s spread, \( vZ_t \) is the volatility of market belief estimated using GARCH(1,1) model, Vector \( i \) is a vector containing different regressors for different dependent variables, and \( \beta_i \) is a coefficient vector. ‘Basic’ means the case in which the volatility of market belief is the unique explanatory variable while ‘Robust’ means the robustness test, that is, \( \beta_i \) is set equal to zero in the ‘Basic’ case. Cross-sectional averages of estimated betas are reported with t-statistics in parentheses. ‘%positive’ (‘%negative’) means the percentage of positive (negative) betas, while ‘%+significance’ (‘%-significance’) gives the percentage with t-statistics greater than +1.645 (smaller than -1.645) (the 5% critical value in one-tailed test). For the parsimonious reason, only are the summary statistics for the estimated betas of \( vZ_t \) reported in this table. The sample consists of 114 S&P500 Index stocks.

<table>
<thead>
<tr>
<th></th>
<th>Brown and Rozef (1979) Model</th>
<th>Seasonal Random Walk with Drift Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Basic</td>
<td>Robust</td>
</tr>
<tr>
<td><strong>Panel A: Trading Volume</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( vZ_t )</td>
<td>-20.59</td>
<td>-22.78</td>
</tr>
<tr>
<td></td>
<td>(-3.29)</td>
<td>(-4.00)</td>
</tr>
<tr>
<td>% negative</td>
<td>74.11</td>
<td>72.32</td>
</tr>
<tr>
<td>% -significant</td>
<td>3.571</td>
<td>6.250</td>
</tr>
<tr>
<td>( R^2(%) ) Mean</td>
<td>0.297</td>
<td>8.118</td>
</tr>
<tr>
<td></td>
<td>0.125</td>
<td>7.180</td>
</tr>
<tr>
<td><strong>Panel B: Price Volatility</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( vZ_t/(/100) )</td>
<td>29.55</td>
<td>29.23</td>
</tr>
<tr>
<td></td>
<td>(2.38)</td>
<td>(2.35)</td>
</tr>
<tr>
<td>% positive</td>
<td>78.57</td>
<td>75.00</td>
</tr>
<tr>
<td>% +significant</td>
<td>60.71</td>
<td>58.04</td>
</tr>
<tr>
<td>( R^2(%) ) Mean</td>
<td>14.26</td>
<td>22.54</td>
</tr>
<tr>
<td></td>
<td>4.203</td>
<td>15.78</td>
</tr>
<tr>
<td><strong>Panel C: Liquidity</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( vZ_t )</td>
<td>14.51</td>
<td>13.71</td>
</tr>
<tr>
<td></td>
<td>(2.32)</td>
<td>(2.12)</td>
</tr>
<tr>
<td>% positive</td>
<td>76.79</td>
<td>71.43</td>
</tr>
<tr>
<td>% +significant</td>
<td>48.21</td>
<td>41.96</td>
</tr>
<tr>
<td>( R^2(%) ) Mean</td>
<td>7.284</td>
<td>19.68</td>
</tr>
<tr>
<td></td>
<td>1.854</td>
<td>16.61</td>
</tr>
</tbody>
</table>

58
This table reports the cross-sectional statistics of the following OLS regression results:

\[ Y_{it} = \alpha_i + \beta_{i,1} vZ_{i,t} + \beta_{i,2} Z_{i,t}^2 + \beta_{i,3} t + \beta_{i,4} Volume_{it} + \beta_{i,5} Dispersion_{i,t} + \beta_{i,6} vRet_{i,t} + \epsilon_{i,t} \quad (i = 1, \ldots, 114) \]

where superscript index \( j \) = BR or SRWD, referring to the time-series model used to predict quarterly earnings per share, \( Y_t \) is either the bid-ask spread proposed by Corwin and Schultz (2010) (CSS) or the Amihud’s (2002) illiquidity measure (ILLIQ), \( vZ_t \) is the volatility of market belief estimated using either rolling regression method (RR Method) or GARCH(1,1) model, \( Z_t \) is market belief, \( t \) is time, \( Volume_t \) is stock turnover rate, \( Dispersion_t \) is analyst forecast dispersion, and \( vRet_t \) is return volatility. Cross-sectional averages of estimated betas are reported with t-statistics in parentheses. ‘%positive’ means the percentage of positive betas, while ‘%+significance’ gives the percentage with t-statistics greater than +1.645 (the 5% critical value in one-tailed test). The sample consists of 114 S&P500 index stocks.

<table>
<thead>
<tr>
<th></th>
<th>CSS</th>
<th>ILLIQ</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>RR Method</td>
<td>GARCH Model</td>
</tr>
<tr>
<td>( vZ_t )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>BR</td>
<td>SRWD</td>
</tr>
<tr>
<td></td>
<td>0.007</td>
<td>0.006</td>
</tr>
<tr>
<td></td>
<td>(2.34)</td>
<td>(1.51)</td>
</tr>
<tr>
<td>% positive</td>
<td>65.79</td>
<td>66.67</td>
</tr>
<tr>
<td>% +significant</td>
<td>22.81</td>
<td>22.81</td>
</tr>
<tr>
<td>( Z_t \times 100 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>BR</td>
<td>SRWD</td>
</tr>
<tr>
<td></td>
<td>-0.093</td>
<td>-0.129</td>
</tr>
<tr>
<td></td>
<td>(-4.04)</td>
<td>(-5.41)</td>
</tr>
<tr>
<td>( t \times 10000 )</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>BR</td>
<td>SRWD</td>
</tr>
<tr>
<td></td>
<td>0.049</td>
<td>0.048</td>
</tr>
<tr>
<td></td>
<td>(4.94)</td>
<td>(4.76)</td>
</tr>
<tr>
<td>( Volume_t \times 1000 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>BR</td>
<td>SRWD</td>
</tr>
<tr>
<td></td>
<td>0.017</td>
<td>0.017</td>
</tr>
<tr>
<td></td>
<td>(1.54)</td>
<td>(1.59)</td>
</tr>
<tr>
<td>( Dispersion_t )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>BR</td>
<td>SRWD</td>
</tr>
<tr>
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<td>-0.007</td>
<td>-0.008</td>
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<tr>
<td></td>
<td>(-4.61)</td>
<td>(-4.91)</td>
</tr>
<tr>
<td>( vRet_t )</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>BR</td>
<td>SRWD</td>
</tr>
<tr>
<td></td>
<td>3.746</td>
<td>3.716</td>
</tr>
<tr>
<td></td>
<td>(26.7)</td>
<td>(26.6)</td>
</tr>
<tr>
<td>( R^2(%) ) mean</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>48.78</td>
<td>49.08</td>
</tr>
<tr>
<td>( R^2(%) ) median</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>48.48</td>
<td>49.19</td>
</tr>
</tbody>
</table>