Conditional Density Models for Asset Pricing

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Conditional Density Models for Asset Pricing*

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Abstract

We model the dynamics of asset prices and associated derivatives by consideration of the dynamics of the conditional probability density process for the value of an asset at some specified time in the future. In the case where the asset is driven by Brownian motion, an associated “master equation” for the dynamics of the conditional probability density is derived and expressed in integral form. By a “model” for the conditional density process we mean a solution to the master equation along with the specification of (a) the initial density, and (b) the volatility structure of the density. The volatility structure is assumed at any time and for each value of the argument of the density to be a functional of the history of the density up to that time. This functional determines the model for the conditional density. In practice one specifies the functional modulo sufficient parametric freedom to allow for the input of additional option data apart from that implicit in the initial density. The scheme is sufficiently flexible to allow for the input of various types of data depending on the nature of the options market and the class of valuation problem being undertaken. Various examples are studied in detail, with exact solutions provided in some cases.

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1 Introduction

To begin, let us make a statement of the problem, and fix some notation and terminology. This paper is concerned with the problem of modelling the dynamics of volatility surfaces. The problem has an extensive literature associated with it which we shall not attempt to summarise; see Carmona & Nadtochiy (2009), Gatheral (2006), and Schweizer & Wissel (2008a, 2008b), where a number of key papers are mentioned. We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t \geq 0}$, where $\mathbb{P}$ is the probability measure governing market events, and $\{\mathcal{F}_t\}$ determines the flow of information to the market. We assume that $\{\mathcal{F}_t\}$ is $\mathbb{P}$-augmented, and that price processes are adapted to $\{\mathcal{F}_t\}$. We assume the absence of arbitrage, and the existence of a preferred pricing kernel $\{\pi_t\}$ associated with some choice of base currency as numeraire. Thus $\{\pi_t\}$ might be the “dollar” pricing kernel. We work in the setting of a multi-asset market, and do not assume that the market is complete. To avoid technicalities, we shall assume in what follows that the functions and processes under consideration satisfy measurability and integrability conditions sufficient to ensure that the indicated manipulations and statements are indeed meaningful. These conditions have to be checked in detail from case to case.

Let us write $\{W^\alpha_t\}_{\alpha=1,\ldots,n}$ for the components of a multi-dimensional Brownian motion $\{W_t\}$, and $\{A^i_t\}_{i=0,1,\ldots,N}$ for the price processes of a collection of non-dividend-paying assets. Prices are expressed in units of the numeraire (e.g. “dollars”). We refer to asset $i$ as $A^i$. We model the $\{A^i_t\}$ as Itô processes, and for each $A^i$ we require that

$$\pi_s A^i_s = \mathbb{E}_s \left[ \pi_t A^i_t \right], \quad (1.1)$$

for $s \leq t$, where $\mathbb{E}_s[-]$ denotes conditional expectation with respect to $\mathcal{F}_s$. We assume that $\{A^0_t\}$ represents a standard money-market account in the base currency, initialised to unity. If we set $\rho_t = \pi_t A^0_t$ for $t \geq 0$, it follows that $\{\rho_t\}$ is a $\mathbb{P}$-martingale. A standard argument shows that $\{\rho_t\}$ can be used to make a change of measure. The resulting measure $\mathbb{Q}^0$ is the “risk-neutral” measure associated with the base currency, and has the property that if the price of any non-dividend-paying asset is expressed in units of the money-market account, then the resulting process is a $\mathbb{Q}^0$-martingale for each $A^i$. That is to say, for each $i$ we have

$$A^i_s = A^i_0 \mathbb{E}_s^\mathbb{Q}^0 \left[ \frac{A^i_t}{A^0_t} \right]. \quad (1.2)$$

A similar situation arises with other choices of numeraire. Specifically, for any choice of non-dividend-paying asset $A^i$ of limited liability ($A^i_t > 0$ for all $t$), with price process $\{A^i_t\}$, there is an associated measure $\mathbb{Q}^i$ with the property that if the price of any other non-dividend-paying asset (whether or not of limited liability) is expressed in units of asset $i$ then the resulting process is a $\mathbb{Q}^i$-martingale. Thus for $0 \leq s \leq t$ and for all $i, j$ for which the price of $A_i$ is strictly positive we have:

$$A^j_s = A^j_0 \mathbb{E}_s^{\mathbb{Q}^i} \left[ \frac{A^j_t}{A^i_t} \right]. \quad (1.3)$$

Bearing these points in mind, we take the view that the option pricing problem is most naturally formulated in the following context. We consider European-style options of a
“Margrabe” type, for a pair of non-dividend-paying assets $A^i$ and $A^j$, where the option-holder has the right at time $t$ to exchange $K$ units of asset $i$ for one unit of asset $j$. The payoff $H_{ij}^t$ of such an option, in units of the base currency, is of the form

$$H_{ij}^t(K) = (A^j_t - KA^i_t)^+.$$  \hspace{1cm} (1.4)

The value of the option at $s \leq t$, expressed in units of the base currency, is given by

$$C_{ij}^{st}(K) = \frac{1}{\pi_s} \mathbb{E}_s \left[ \pi_t (A^j_t - KA^i_t)^+ \right].$$  \hspace{1cm} (1.5)

If $A^i$ is of limited liability, then the value of such an option, when expressed in units of $A^i$, is a $\mathbb{Q}^i$-martingale:

$$\frac{C_{ij}^{st}(K)}{A^i_s} = \mathbb{E}^{\mathbb{Q}^i}_s \left[ \left( \frac{A^j_t}{A^i_t} - K \right)^+ \right].$$  \hspace{1cm} (1.6)

Suppose we fix the choice of $A^i$ and $A^j$. Thus $A^j$ denotes a generic non-dividend-paying asset, and $A^i$ denotes a choice of numeraire from the positive non-dividend-paying assets. Let us write $A^i_s$ for the price at time $s \leq t$ of the generic asset expressed in units of the numeraire, and $C_{st}(K)$ for the price, in units of the numeraire, at time $s \leq t$, of a $t$-maturity $K$-strike option. Then the option payoff at time $t$ is given (in numeraire units) by

$$h_t(K) = (A^i_t - K)^+,$$  \hspace{1cm} (1.7)

and the value of the option at time $s \leq t$ is

$$C_{st}(K) = \mathbb{E}^\mathbb{Q}_s \left[ (A^i_t - K)^+ \right],$$  \hspace{1cm} (1.8)

where $\mathbb{Q}$ is the martingale measure associated with the numeraire.

We observe that “standard” options are not included in the category discussed above. A standard call option has the payoff $H_t = (A^i_t - K)^+$ where $A^i_t$ is the price of the underlying asset at time $t$ in currency units, and $K$ is a fixed strike price in currency units. But such an option is not an option to exchange $K$ units of a non-dividend-paying asset for one unit of another non-dividend-paying asset. Rather, one can think of the fixed strike as being $K$ units of a unit-valued floating rate note. The point is that the only asset that maintains a constant value (in base currency units) is a floating rate note—and such an asset pays a dividend. The dividend is the interest rate. In a given currency, by a “floating rate note” we mean an idealised note that pays interest continuously (rather than in lumps). It is elementary to see that the associated dividend is the short rate of interest. One might try to argue that the strike on a standard option is cash, and that cash is a non-dividend-paying asset; but this point of view leads to paradoxes. In the standard theory we regard cash as paying an “implicit” dividend, like a convenience yield, in the form of a liquidity benefit equivalent to the interest rate.

Thus a “standard” option is a rather complicated entity—it is an option to exchange a certain number of units of a dividend-paying asset for one unit of a non-dividend-paying asset. Our view is that it is more logical first to examine the case of an option based on a pair of non-dividend-paying assets. In the literature this approach is often implicitly
adopted through the device of “setting the interest rate equal to zero”. In that situation the floating rate note is a non-dividend-paying asset; thus, the setting we propose to operate within includes the zero-interest case. It would indeed be interesting to tackle the general problem of an option to exchange $K$ units of one dividend-paying asset for one unit of another dividend-paying asset (a “standard” foreign exchange option falls into that category); but, unless the dividend rate (or the interest rate) systems are deterministic, this is a more difficult problem than the one we propose to consider.

2 Conditional density processes

The market will be understood as having the setup described in the previous section. Thus we have a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ equipped with a filtration $\{\mathcal{F}_t\}$. A positive non-dividend-paying asset is chosen as numeraire, and henceforth all prices are expressed in units of that numeraire. The measure $\mathbb{Q}$ has the property that the price processes of non-dividend-paying assets, when expressed in units of the numeraire, are martingales. We refer to $\mathbb{Q}$ as the “martingale measure” associated with this numeraire.

We let $\{A_t\}$ denote the price process of a generic non-dividend-paying asset, expressed in units of the numeraire. It follows that $\{A_t\}$ is a $\mathbb{Q}$-martingale, and that the $\mathbb{Q}$-dynamics of $\{A_t\}$ are of the form

$$dA_t = \sigma_t^A dW_t$$ (2.1)

for some $\{\mathcal{F}_t\}$-adapted absolute volatility $\{\sigma_t^A\}$. It should be evident that $\{\sigma_t^A\}$ is vector-valued. In (2.1), and in what follows, we adopt a compact notation, writing, e.g.,

$$\sigma_t^A dW_t = \sum_{\alpha=1}^n \sigma_t^{A\alpha} dW_t^\alpha.$$ (2.2)

We fix a time $T > 0$, and assume for $0 \leq t < T$ the existence of a $\mathcal{F}_t$-conditional $\mathbb{Q}$-density $f_t(x)$ for the value of $A_T$. In our applications we typically have in mind the cases $x \in \mathbb{R}$ and $x \in \mathbb{R}^+$, but it is useful to keep the formalism flexible as regards the choice of the domain of the density function. In what follows we treat the case $x \in \mathbb{R}$, and leave it to the reader to supply the necessary adjustments for other domains. We require that $f_t(x)$, $x \in \mathbb{R}$, should satisfy the following: (1) $f_t(x) > 0$ for $x \in \mathbb{R}$ and for $t \in [0, T)$; (2) for any bounded, measurable function $g(x)$, $x \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} g(x) f_t(x) dx = \mathbb{E}[g(A_T)|\mathcal{F}_t];$$ (2.3)

and (3) for each $x \in \mathbb{R}$ the process $\{f_t(x)\}$ is an $\{\mathcal{F}_t\}$-martingale, and hence

$$\mathbb{E}[f_t(x)|\mathcal{F}_s] = f_s(x),$$ (2.4)

for $0 \leq s \leq t < T$. We can express the price of the asset in terms of the density by

$$A_t = \int_{\mathbb{R}} x f_t(x) dx,$$ (2.5)
which follows from the martingale condition
\[ A_t = \mathbb{E}[A_T | \mathcal{F}_t]. \tag{2.6} \]

Expressions analogous to (2.5) can be written for claims based on \( A_T \). For example, if \( C_t(K) \) denotes the price at \( t \) of a \( T \)-maturity, \( K \)-strike European call option, then
\[ C_t(K) = \int_{\mathbb{R}} (x - K)^+ f_t(x)dx. \tag{2.7} \]

Our goal is to model the dynamics of \( \{f_t(x)\} \). Since \( \{f_t\} \) is a positive \( \mathbb{Q} \)-martingale, for each \( x \in \mathbb{R} \) the associated dynamical equation takes the form
\[ f_t(x) = f_0(x) + \int_0^t \sigma_s^f(x) f_s(x) dW_s, \tag{2.8} \]
where \( \sigma_s^f(x) \) is the volatility of the density at \( x \). It is then a straightforward matter to establish the following:

**Lemma 2.1.** The normalisation condition
\[ \int_{\mathbb{R}} f_t(x)dx = 1 \tag{2.9} \]
holds for all \( t \in [0, T) \) if and only if \( \sigma_t^f(x) \) is of the form
\[ \sigma_t^f(x) = \sigma_t(x) \int_{\mathbb{R}} f_t(y)dy - \int_{\mathbb{R}} \sigma_t(y)f_t(y)dy \tag{2.10} \]
for some process \( \{\sigma_t(x)\} \), and
\[ \int_{\mathbb{R}} f_0(x)dx = 1. \tag{2.11} \]

Thus, once we specify \( \{\sigma_t(x)\} \) and the initial density \( f_0(x) \), the dynamical equation for the density—the so-called “master equation”—takes the form
\[ f_t(x) = f_0(x) + \int_0^t \left[ \sigma_s(x) - \int_{\mathbb{R}} \sigma_s(y)f_s(y)dy \right] f_s(x) dW_s. \tag{2.12} \]

**Lemma 2.2.** \( \{f_t(x)\} \) satisfies the master equation (2.12) with initial density \( f_0(x) \) and volatility structure \( \sigma_t(x) \) if and only if
\[ f_t(x) = \frac{f_0(x) \exp \left( \int_0^t \sigma_s(x)dz_s - \frac{1}{2} \int_0^t \sigma_s^2(x)ds \right)}{\int_{\mathbb{R}} f_0(y) \exp \left( \int_0^t \sigma_s(y)dz_s - \frac{1}{2} \int_0^t \sigma_s^2(y)ds \right) dy}, \tag{2.13} \]
where
\[ Z_t = W_t + \int_0^t \int_{\mathbb{R}} \sigma_s(y)f_s(y)dy ds. \tag{2.14} \]
Proof. Writing (2.12) in differential form, we have
\[ df_t(x) = f_t(x) [\sigma_t(x) - \langle \sigma_t \rangle] \, dW_t, \] (2.15)
where for convenience we introduce the bracket notation
\[ \langle \sigma_t \rangle = \int_\mathbb{R} \sigma_t(x) f_t(x) \, dx \] (2.16)
for the conditional “mean” of the volatility. We integrate (2.15) to obtain
\[ f_t(x) = f_0(x) \exp \left[ \int_0^t (\sigma_s(x) - \langle \sigma_s \rangle) \, dW_s - \frac{1}{2} \int_0^t (\sigma_s(x) - \langle \sigma_s \rangle)^2 \, ds \right]. \] (2.17)
Expanding the exponent, we obtain the following:
\[ f_t(x) = \bar{f}_0(x) \frac{\exp \left[ \int_0^t \sigma_s(x) \, dW_s + \langle \sigma_s \rangle \, ds - \frac{1}{2} \int_0^t \sigma_s^2(x) \, ds \right]}{\exp \left[ \int_0^t \langle \sigma_s \rangle \, dW_s + \langle \sigma_s \rangle \, ds - \frac{1}{2} \int_0^t \langle \sigma_s \rangle^2 \, ds \right]}. \] (2.18)
Then we introduce a process \( \{Z_t \} \) by writing
\[ Z_t = W_t + \int_0^t \langle \sigma_s \rangle \, ds, \] (2.19)
and it follows that
\[ f_t(x) = \bar{f}_0(x) \frac{\exp \left[ \int_0^t \sigma_s(x) \, dZ_s - \frac{1}{2} \int_0^t \sigma_s^2(x) \, ds \right]}{\exp \left[ \int_0^t \langle \sigma_s \rangle \, dZ_s - \frac{1}{2} \int_0^t \langle \sigma_s \rangle^2 \, ds \right]}. \] (2.20)
Next we derive an alternative expression for the term appearing in the denominator. Applying the normalization condition (2.9) to equation (2.20), we see that
\[ \exp \left( \int_0^t \langle \sigma_s \rangle \, dZ_s - \frac{1}{2} \int_0^t \langle \sigma_s \rangle^2 \, ds \right) = \int_\mathbb{R} f_0(x) \exp \left( \int_0^t \sigma_s(x) \, dZ_s - \frac{1}{2} \int_0^t \sigma_s^2(x) \, ds \right) \, dx. \] (2.21)
As a consequence we deduce that (2.20) reduces to (2.13). Conversely, suppose now that \( f_t(x) \) is given by (2.13). Denote the numerator in (2.13) by \( N_t(x) \) and the denominator by \( D_t \). Then it is straightforward to show that \( dN_t(x) = N_t(x) \sigma_t(x) \, dZ_t \) and \( dD_t = D_t \langle \sigma_t \rangle \, dZ_t \). The Ito quotient rule then gives
\[ df_t(x) = N_t(x) \frac{d}{dt} + \frac{1}{D_t} \, dN_t(x) + dN_t(x) \frac{1}{D_t} \]
\[ = \frac{N_t(x)}{D_t} \, (\sigma_t(x) - \langle \sigma_t \rangle) \, (dZ_t - \langle \sigma_t \rangle \, dt) \]
\[ = f_t(x) (\sigma_t(x) - \langle \sigma_t \rangle) \, dW_t. \] (2.22)
Clearly, both the initial condition and the normalization condition are satisfied. Of course, one has not solved the master equation yet, since \( \{Z_t\} \) implicitly involves the density, via (2.19). Nevertheless, we can use (2.13) as a starting point for obtaining solutions.

In concluding this section we show how the \( \mathbb{P} \)-dynamics of the density can be recovered. Let us consider the Radon-Nikodym derivative

\[
\frac{d\mathbb{P}}{d\mathbb{Q}}
\big|_{\mathcal{F}_t} = \exp \left[ \int_0^t \alpha_s dW_s + \frac{1}{2} \int_0^t \alpha_s^2 ds \right].
\]

Making use of Girsanov's theorem, we define a \( \mathbb{P} \)-Brownian motion by

\[
dW^\mathbb{P}_t = dW_t + \alpha_t dt.
\]

Given the \( \mathbb{Q} \)-dynamics (2.15), the \( \mathbb{P} \)-dynamics of the density is given by

\[
\frac{df_t(x)}{f_t(x)} = \alpha_t \left[ \sigma_t(x) - \langle \sigma_t \rangle \right] dt + \left[ \sigma_t(x) - \langle \sigma_t \rangle \right] dW^\mathbb{P}_t.
\]

Notice that the drift of the density process under \( \mathbb{P} \) is given by

\[
\mu^\mathbb{P}_t(x) = \alpha_t \left[ \sigma_t(x) - \int_{\mathbb{R}} \sigma_t(y)f_t(y)dy \right].
\]

## 3 Conditional density models

We are in a position now to say more precisely what we mean by a “conditional density model”. In doing so, we are motivated in part by recent advances in the study of infinite-dimensional stochastic differential equations. By a “model” for the density process we understand the following. We consider solutions of the master equation (2.12) satisfying the normalization condition (2.9), in conjunction with the specification of: (a) the initial density \( f_0(x) \); and (b) the volatility structure \( \{\sigma_t(x)\} \) in the form of a functional

\[
\sigma_t(x) = \Phi[f_t(\cdot), t, x].
\]

For each \( x \) and \( t \) the volatility \( \sigma_t(x) \) depends on \( f_t(y) \) for all \( y \in \mathbb{R} \). Hence (2.12), thus specified, determines the dynamics of an infinite-dimensional Markov process.

The initial density \( f_0(x) \) is fixed if one supplies initial option price data for the maturity date \( T \) and for all strikes \( K \in \mathbb{R} \). In particular, we have

\[
C_0(K) = \mathbb{E}[(A_T - K)^+]
= \int_{\mathbb{R}} (x - K)^+ f_0(x) dx.
\]

By use of the idea of Breeden & Litzenberger (1978) we see, in the present context, that for each value of \( x \) one has

\[
f_0(x) = \frac{\partial^2 C_0(x)}{\partial x^2}.
\]
Here \( f_0(x) \) is not the “risk-neutral” density, but rather the \( Q \)-density of the value at time \( T \) of the asset in units of the chosen numeraire. Once \( f_0(x) \) has been supplied, the choice of the functional \( \Phi \) determines the model for the conditional density: we give some examples later in the paper. In practice, one would like to specify \( \Phi \) modulo sufficient parametric freedom to allow the input of additional option price data. What form this additional data might take depends on the nature of the market and the class of valuation problems being pursued. For example, a standard problem would be to look at a limited-liability asset and consider additional data in the form of initial option prices for all strikes in \( \mathbb{R}^+ \) and all maturities in the strip \( 0 < t \leq T \). We require that \( \Phi \) should be specified in such a way that once the data are provided, then \( \Phi \) is determined and the “master equation” provides an evolution of the conditional density. Once we have the conditional density process, we can work out the evolution of the option price system for the specified strip, and hence the evolution of the associated implied volatility surface. It should not be assumed that the data have to be presented exactly in the way specified in the previous paragraph—there may be situations where more data are available (e.g., in the form of barrier option prices or other derivative prices) or where less data are available (less well-developed markets). One should think of the parametric form of \( \Phi \) as being adapted in a flexible way to the nature of a specific problem. The philosophy is that there are many different markets for options, and one needs a methodology that can accommodate these with reasonable generality.

4 Models with deterministic volatility structures

In what follows, we present a methodology for the development of models for conditional density processes. We begin with the most general class of models we explicitly construct in this paper. Later we will treat explicit examples which perhaps are more apt to applications in financial mathematics, and in particular to option pricing.

We consider a probability space \((\Omega, \mathcal{F}, Q)\), and construct density models characterized by a deterministic volatility structure given by a function of time and a random variable \( X \).

Proposition 4.1. Let \( T \in (0, \infty) \) and let \( \tilde{f}_0(x) : \mathbb{R} \to \mathbb{R}^+ \) be a density function. Let \( \nu(t, x) \) be a function on \([0, T) \times \mathbb{R} \), and let \( \gamma(t) \) be a function on \([0, T)\), such that

\[
\lim_{t \to T} \sqrt{t} \gamma(t) = 0, \quad \lim_{t \to T} \int_0^t \nu(s, x) \, ds = g(x),
\]

for some invertible function \( g(x) \) on \( \mathbb{R} \). For finite \( T \) we have \( \lim_{t \to T} \gamma(t) = 0 \). Let \( \{B_t\} \) be a Brownian motion and let \( X \) be an independent random variable with density \( \tilde{f}_0(x) \). Let \( \{F_t\} \) denote the filtration generated by the process \( \{I_t\} \) defined by

\[
I_t = B_t + \int_0^t \nu(s, X) \, ds.
\]

Let the \( \{F_t\} \)-adapted density process \( \{f_t(x)\} \) be defined by

\[
f_t(x) = \frac{\tilde{f}_0(x) \exp \left[ \int_0^t \nu(s, x) \, dI_s - \frac{1}{2} \int_0^t \nu^2(s, x) \, ds \right]}{\int_{-\infty}^\infty \tilde{f}_0(y) \exp \left[ \int_0^t \nu(s, y) \, dI_s - \frac{1}{2} \int_0^t \nu^2(s, y) \, ds \right] \, dy}.
\]
Then (a) the random variable $X$ is $\mathcal{F}_T$-measurable, (b) the process $\{W_t\}$ defined by

$$W_t = I_t - \int_0^t \mathbb{E}^Q [v(s, X) \mid \mathcal{F}_s] \, ds$$

(4.4)
is an $\{\mathcal{F}_t\}$-adapted Brownian motion, and (c) the equations

$$\mathbb{Q} [X \in dx \mid \mathcal{F}_t] = f_t(x) \, dx$$

(4.5)

and

$$f_t(x) = \tilde{f}_0(x) + \int_0^t f_s(x) \left[ v(s, x) - \int_{-\infty}^{\infty} v(s, y) f_s(y) \, dy \right] \, dW_s$$

(4.6)

are satisfied for all $t \in [0, T)$.

**Proof.** (4.1) implies that $g(X) = \lim_{t \to T} \gamma(t) I_t$ is $\mathcal{F}_T$-measurable. Since $g(x)$ is invertible, we conclude that $X$ is $\mathcal{F}_T$-measurable. Now let the filtration $\{G_t\}$ be given by

$$G_t = \sigma \left( \{B_s\}_{0 \leq s \leq t}, X \right).$$

(4.7)

Clearly $G_t \supset \mathcal{F}_t$. The random variable $X$ is $G_t$-measurable, and $\{B_t\}$ is a ($\{G_t\}, \mathbb{Q}$)-Brownian motion. We introduce a ($\{G_t\}, \mathbb{B}$)-martingale $\{M_t\}$ defined by

$$M_t = \exp \left[ - \int_0^t v(s, X) dB_s - \frac{1}{2} \int_0^t v^2(s, X) \, ds \right], \quad t \in [0, T),$$

(4.8)
as well as a probability measure $\mathbb{B}$ by setting

$$\frac{d\mathbb{B}}{d\mathbb{Q}} \mid_{G_t} = M_t.$$  

(4.9)

Now consider the process $\{I_t\}$. We observe that $dI_t = dB_t + v(t, X) \, dt$. By Girsanov’s theorem, $\{I_t\}$ is a ($\{G_t\}, \mathbb{B}$)-Brownian motion. We also note that $\{M_t^{-1}\}$ is a ($\{G_t\}, \mathbb{B}$)-martingale. Let $H$ be a bounded measurable function on $\mathbb{R}$. Then, since $H(X)$ is $G_t$-measurable, we have the generalised Bayes formula

$$\mathbb{E}^Q [H(X) \mid \mathcal{F}_t] = \frac{\mathbb{E}^B [M_t^{-1} H(X) \mid \mathcal{F}_t]}{\mathbb{E}^B [M_t^{-1} \mid \mathcal{F}_t]}.$$  

(4.10)

In the following steps, we calculate the right side of this equation. To this end we show that $X$ and $I_t$ are $\mathbb{B}$-independent for all $t$. In particular, we show that the expectation

$$\mathbb{E}^B [\exp (yI_t + zX)]$$

(4.11)
factorises. We have:

$$\mathbb{E}^B [\exp (yI_t + zX)] = \mathbb{E}^Q [M_t \exp (yI_t + zX)],$$

(4.12)

$$= \mathbb{E}^Q \left[ \exp \left( - \int_0^t v(s, X) dB_s - \frac{1}{2} \int_0^t v^2(s, X) \, ds \right.ight.$$  

$$+ y \left[ B_t + \int_0^t v(s, X) \, ds \right] \exp (zX) \biggr),$$

(4.13)

$$= \mathbb{E}^Q \left[ m_t \exp \left( \frac{1}{2} y^2 t \right) \exp (zX) \right],$$

(4.14)
where
\[ m_t = \exp \left( \int_0^t [-v(s, X) + y] dB_s - \frac{1}{2} \int_0^t [-v(s, X) + y]^2 ds \right). \] (4.15)

By use of the tower property and the independence of \( \{B_t\} \) and \( X \) under \( \mathbb{Q} \), we have
\[ \mathbb{E}^\mathbb{B}[\exp (yI_t + zX)] = \mathbb{E}^\mathbb{Q}[m_t \exp (\frac{1}{2} y^2 t) \exp (zX)], \] (4.16)
\[ = \exp (\frac{1}{2} y^2 t) \mathbb{E}^\mathbb{Q}[\mathbb{E}^\mathbb{Q}[m_t | X] \mathbb{E}^\mathbb{Q}[\exp (zX)]] \] (4.17)
\[ = \exp (\frac{1}{2} y^2 t) \mathbb{E}^\mathbb{Q}[\mathbb{E}^\mathbb{Q}[m_t | X] \mathbb{E}^\mathbb{Q}[\exp (zX)]] \] (4.18)

It is straightforward to see that \( \mathbb{E}^\mathbb{Q}[m_t | X] = 1 \). Thus we obtain
\[ \mathbb{E}^\mathbb{B}[\exp (yI_t + zX)] = \exp (\frac{1}{2} y^2 t) \mathbb{E}^\mathbb{Q}[\exp (zX)], \] (4.19)
which establishes the desired factorization. Now that we have shown that \( X \) is independent of \( I_t \) for all \( t \in [0, T] \) (and thus of \( \mathcal{F}_t \)) under \( \mathbb{B} \), we can work out the right-hand-side of (4.10). We have:
\[ \mathbb{E}^\mathbb{Q}[H(X) | \mathcal{F}_t] = \frac{\mathbb{E}^\mathbb{B}[M_t^{-1} H(X) | \mathcal{F}_t]}{\mathbb{E}^\mathbb{B}[M_t^{-1} | \mathcal{F}_t]],} \] (4.20)
\[ = \frac{\mathbb{E}^\mathbb{B}[H(X) \exp \left( \int_0^t v(s, X) dI_s - \frac{1}{2} \int_0^t v^2(s, X) ds \right) | \mathcal{F}_t]}{\mathbb{E}^\mathbb{B}[\exp \left( \int_0^t v(s, X) dI_s - \frac{1}{2} \int_0^t v^2(s, X) ds \right) | \mathcal{F}_t]}, \] (4.21)
\[ = \frac{\int_{-\infty}^\infty \tilde{f}_0(x) H(x) \exp \left( \int_0^t v(s, x) dI_s - \frac{1}{2} \int_0^t v^2(s, x) ds \right) dx}{\int_{-\infty}^\infty \tilde{f}_0(y) \exp \left( \int_0^t v(s, y) dI_s - \frac{1}{2} \int_0^t v^2(s, y) ds \right) dy}. \] (4.22)

In particular, by setting \( H(X) = 1(X \leq x) \), we deduce that (4.5) is satisfied, as required.

Next we show that the process \( \{W_t\} \) defined by (4.4) is an \((\{\mathcal{F}_t\}, \mathbb{Q})\)-Brownian motion. We need to show (1) that \( dW_t dW_t = dt \) and (2) that \( \mathbb{E}^\mathbb{Q}[W_u | \mathcal{F}_t] = W_t \) for \( 0 \leq t \leq u \). The first condition is evidently satisfied. The second condition can be shown to be satisfied as follows (for simplicity, we suppress the superscript \( \mathbb{Q} \)). We have
\[ \mathbb{E}[W_u | \mathcal{F}_t] = \mathbb{E}[I_u | \mathcal{F}_t] - \mathbb{E} \left[ \int_0^u \mathbb{E}[v(s, X) | \mathcal{F}_s] ds | \mathcal{F}_t \right]. \] (4.23)

First we work out \( \mathbb{E}[I_u | \mathcal{F}_t] \): since \( \{B_t\} \) is a \((\{\mathcal{G}_t\}, \mathbb{Q})\)-Brownian motion, we have
\[ \mathbb{E}[I_u | \mathcal{F}_t] = \mathbb{E} \left[ B_u + \int_0^u v(s, X) ds | \mathcal{F}_t \right] \] (4.24)
\[ = \mathbb{E}[B_u | \mathcal{F}_t] + \mathbb{E} \left[ \int_0^u v(s, X) ds | \mathcal{F}_t \right] \] (4.25)
\[ = \mathbb{E} \left[ \mathbb{E}[B_u | \mathcal{G}_t] | \mathcal{F}_t \right] + \mathbb{E} \left[ \int_0^u v(s, X) ds | \mathcal{F}_t \right] \] (4.26)
\[ = \mathbb{E}[B_t | \mathcal{F}_t] + \mathbb{E} \left[ \int_0^u v(s, X) ds | \mathcal{F}_t \right]. \] (4.27)
We insert this intermediate result in (4.23) to obtain

\[
\mathbb{E}[W_u | \mathcal{F}_t] = \mathbb{E}[I_u | \mathcal{F}_t] - \mathbb{E} \left[ \int_0^u \mathbb{E}[v(s, X) | \mathcal{F}_s] \, ds \mid \mathcal{F}_t \right],
\]

(4.28)

\[
= \mathbb{E}[B_t | \mathcal{F}_t] + \mathbb{E} \left[ \int_0^u v(s, X) \, ds \mid \mathcal{F}_t \right] - \mathbb{E} \left[ \int_0^u \mathbb{E}[v(s, X) | \mathcal{F}_s] \, ds \mid \mathcal{F}_t \right].
\]

(4.29)

Next we split the integrals in the last two expectations by writing,

\[
\mathbb{E}[W_u | \mathcal{F}_t] = \mathbb{E}[B_t | \mathcal{F}_t] + \mathbb{E} \left[ \int_0^t v(s, X) \, ds \mid \mathcal{F}_t \right] + \mathbb{E} \left[ \int_t^u v(s, X) \, ds \mid \mathcal{F}_t \right] - \mathbb{E} \left[ \int_0^t \mathbb{E}[v(s, X) | \mathcal{F}_s] \, ds \mid \mathcal{F}_t \right] - \mathbb{E} \left[ \int_t^u \mathbb{E}[v(s, X) | \mathcal{F}_s] \, ds \mid \mathcal{F}_t \right].
\]

(4.30)

Observing that

\[
\mathbb{E}[B_t | \mathcal{F}_t] + \mathbb{E} \left[ \int_0^t v(s, X) \, ds \mid \mathcal{F}_t \right] = I_t,
\]

(4.31)

and that

\[
\mathbb{E} \left[ \int_t^u v(s, X) \, ds \mid \mathcal{F}_t \right] = \mathbb{E} \left[ \int_t^u \mathbb{E}[v(s, X) | \mathcal{F}_s] \, ds \mid \mathcal{F}_t \right],
\]

(4.32)

we see that the expectation \( \mathbb{E}[W_u | \mathcal{F}_t] \) reduces to

\[
\mathbb{E}[W_u | \mathcal{F}_t] = I_t - \mathbb{E} \left[ \int_0^t \mathbb{E}[v(s, X) | \mathcal{F}_s] \, ds \mid \mathcal{F}_t \right]
\]

(4.33)

\[
= I_t - \int_0^t \mathbb{E}[v(s, X) | \mathcal{F}_s] \, ds
\]

(4.34)

\[
= W_t,
\]

(4.35)

by (4.4). Lemma 2.2 concludes the proof of Proposition 4.1.

\[\square\]

**Remark 4.1.** For the sake of simplicity we presented Proposition 4.1 for the one-dimensional state space only. However, it should become obvious from the proof how the results carry over to conditional densities on higher dimensional state spaces.

**Remark 4.2.** We note that for simulations of the dynamics of the conditional density process, the following alternative representation to (4.3) may prove useful:

\[
f_t(x) = \frac{\tilde{f}_0(x) \exp \left( \int_0^t [v(s, x) - v(s, X)] \, dB_s - \frac{1}{2} \int_0^t [v(s, x) - v(s, X)]^2 \, ds \right)}{\int_\mathbb{R} \tilde{f}_0(y) \exp \left( \int_0^t [v(s, y) - v(s, X)] \, dB_s - \frac{1}{2} \int_0^t [v(s, y) - v(s, X)]^2 \, ds \right) \, dy}
\]

(4.36)

The un-normalised density—the numerator in (4.36)—is conditionally log-normal given \( X \) for all \( x \in \mathbb{R} \). The simulation of the density requires only the numerical implementation of the standard Brownian motion and of \( X \).
Remark 4.3. The density models with deterministic volatility structure presented in Proposition 4.1 form a particular class of models which satisfy the following system:

Let \( \bar{f}_0(x) : \mathbb{R} \to \mathbb{R}^+ \) be a density function. A filtered probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q) \) can be constructed along with

(i) an \( \mathcal{F}_\infty \)-measurable random variable \( X \) with density \( \bar{f}_0(x) \),
(ii) an \( \{\mathcal{F}_t\}\)-adapted density process \( \{f_t(x)\} \),
(iii) an \( \{\mathcal{F}_t\}\)-adapted Brownian motion \( \{W_t\} \)

such that (a) for some function \( \gamma(t) \) on \([0, \infty)\), (b) for some function \( g(x) \) that is invertible onto \( \mathbb{R} \), and (c) for some suitably integrable function \( v(t, x) \) on \([0, \infty) \times \mathbb{R}\) with the properties

\[
\lim_{t \to \infty} \sqrt{t} \gamma(t) = 0, \quad \lim_{t \to \infty} \gamma(t) \int_0^t v(s, x) \, ds = g(x),
\]

(4.37)

the following hold for all \( t \in [0, \infty) \):

\[
Q[ X \in dx | \mathcal{F}_t ] = f_t(x) \, dx,
\]

(4.38)

and

\[
f_t(x) = \bar{f}_0(x) + \int_0^t f_s(x) \left[ v(s, x) - \int_{-\infty}^{\infty} v(s, y) f_s(y) \, dy \right] \, dW_s.
\]

(4.39)

Remark 4.4. We note that the construction of conditional density models admits rather surprisingly an interpretation as a kind of a filtering problem. The process (4.2) plays the role of what, in filtering theory, is called an “observation process”, and (4.4) has the interpretation of being what is usually referred to as an “innovation process” (see [1, 14, 15]).

Remark 4.5. It is reasonable, at least on a heuristic basis, to expect that the general deterministic volatility structure model can be calibrated to the specification of an essentially arbitrary volatility surface. In particular, the parameter freedom implicit in a deterministic volatility structure coincides with that of a volatility surface. The situation is rather similar to that of the relation arising in the Dupire (1994) model between the local volatility (which is determined by a deterministic function of two variables, one with dimensions of price and the other with dimensions of time) and the initial volatility surface (which represents a two-parameter family of option prices, labelled by strike and maturity). The precise characterisation of such relations constitutes a non-trivial and important inverse problem.

5 Semilinear volatility structure: Brownian-bridge approach

We consider the case where the random variable \( A_T \) has a prescribed unconditional density \( \bar{f}_0T(x) \) defined by

\[
\bar{f}_0T(x) \, dx = Q[ A_T \in dx ].
\]

(5.1)
We construct a family of conditional density processes \( \{ f_{tT}(x) \} \) that solve the master equation (2.12) on the time interval \([0, T)\). Here \( f_{tT}(x) \) is defined by

\[
f_{tT}(x)dx = \mathbb{Q}[A_T \in dx \mid \mathcal{F}_t]. \tag{5.2}
\]

In order for this definition to make sense we need to specify the filtration \( \{ \mathcal{F}_t \} \). We construct a process \( \{ \xi_{tT} \}_{0 \leq t \leq T} \) given by

\[
\xi_{tT} = \sigma A_T t + \beta_{tT}, \tag{5.3}
\]

where \( \sigma \) is a constant and \( \{ \beta_{tT} \}_{0 \leq t \leq T} \) is a standard Brownian bridge, taken to be independent from \( A_T \). Next we assume that \( \{ \mathcal{F}_t \} \) is given by

\[
F_t = \sigma (\{ \xi_{sT} \}_{0 \leq s \leq t}). \tag{5.4}
\]

Clearly \( \{ \xi_{tT} \} \) is \( \{ \mathcal{F}_t \} \)-adapted, and \( A_T \) is \( \mathcal{F}_T \)-measurable since \( A_T = \frac{\xi_{TT}}{\sigma} \). It is shown in Brody et al. (2007, 2008) that \( \{ \xi_{tT} \} \) is an \( \{ \mathcal{F}_t \} \)-Markov process.

**Proposition 5.1.** Let the initial density \( \bar{f}_{0T}(x) \) be prescribed, and let the volatility structure be of the semi-linear form

\[
\sigma_{tT}(x) = \sigma \frac{T}{T-t} x, \tag{5.5}
\]

for \( 0 \leq t < T \). Let \( \{ \mathcal{F}_t \} \) be defined by (5.4). Then the process \( \{ W_{t} \}_{0 \leq t < T} \) defined by

\[
W_t = \xi_{tT} - \int_0^t \frac{1}{T-s} (\sigma T \mathbb{E}[A_T \mid \xi_{sT}] - \xi_{sT}) \, ds \tag{5.6}
\]

is an \( \{ \mathcal{F}_t \} \)-Brownian motion and the process \( \{ f_{tT}(x) \} \), given by

\[
f_{tT}(x) = \frac{\bar{f}_{0T}(x) \exp \left[ \frac{T}{T-t} \left( \sigma_{tT} x - \frac{1}{2} \sigma^2 x^2 t \right) \right]}{\int_{-\infty}^{\infty} \bar{f}_{0T}(y) \exp \left[ \frac{T}{T-t} \left( \sigma_{tT} y - \frac{1}{2} \sigma^2 y^2 t \right) \right] \, dy}, \tag{5.7}
\]

satisfies the master equation (2.12) with the given initial condition.

**Proof.** The fact that \( \{ W_t \}_{0 \leq t < T} \) is an \( \{ \mathcal{F}_t \} \)-Brownian motion, is shown in Brody et al. (2007, 2008). We calculate \( \mathbb{E}[A_T \mid \xi_{tT}] \) by use of the Bayes formula,

\[
\mathbb{E}[A_T \mid \xi_{tT}] = \int_{\mathbb{R}} x f_{tT}(x) \, dx, \tag{5.8}
\]

where \( f_{tT}(x) \) is given by

\[
f_{tT}(x) = \frac{\bar{f}_{0T}(x) \rho(\xi_{tT} \mid A_T = x)}{\int_{\mathbb{R}} \bar{f}_{0T}(y) \rho(\xi_{tT} \mid A_T = y) \, dy}. \tag{5.9}
\]

Here \( \rho(\xi_{tT} \mid A_T = x) \) is the conditional density of \( \xi_{tT} \) given the value of \( A_T \). Conditional on the value of \( A_T \), the random variable \( \xi_{tT} \) is Gaussian:

\[
\rho(\xi_{tT} \mid A_T = x) = \sqrt{\frac{T}{2\pi t(T-t)}} \exp \left[ -\frac{1}{2} \frac{T}{t(T-t)} (\xi_{tT} - \sigma t x)^2 \right]. \tag{5.10}
\]
Thus the conditional density process is given by

$$f_{tT}(x) = \frac{\tilde{f}_{0T}(x) \exp \left[ -\frac{1}{2} \frac{T}{t(T-t)} (\xi_{tT} - \sigma t x)^2 \right]}{\int_{-\infty}^{\infty} \tilde{f}_{0T}(y) \exp \left[ -\frac{1}{2} \frac{T}{t(T-t)} (\xi_{tT} - \sigma t y)^2 \right] dy}. \quad (5.11)$$

The last expression can be simplified after some rearrangement to

$$f_{tT}(x) = \frac{\tilde{f}_{0T}(x) \exp \left[ \frac{T}{T-t} (\sigma \xi_{tT} x - \frac{1}{2} \sigma^2 x^2 t) \right]}{\int_{-\infty}^{\infty} \tilde{f}_{0T}(y) \exp \left[ \frac{T}{T-t} (\sigma \xi_{tT} y - \frac{1}{2} \sigma^2 y^2 t) \right] dy}. \quad (5.12)$$

With this intermediate result at hand, we can write the process \{W_t\}_{0 \leq t < T} as follows:

$$W_t = \xi_{tT} - \int_0^t \frac{1}{T-s} \left( \sigma T \int_x f_{sT}(x) dx - \xi_{sT} \right) ds. \quad (5.13)$$

We recall that the master equation (2.12) can be written in the form (2.13). We now prove that (5.7) together with (5.6) satisfy (2.12) by showing that (2.13) reduces to (5.7) if we insert (5.6) in (2.19) and choose the volatility structure to be given by (5.5). For the process \{Z_t\} in (2.19) we obtain

$$Z_t = \xi_{tT} + \int_0^t \frac{\xi_{sT}}{T-s} ds. \quad (5.14)$$

The next step is to insert \{Z_t\}—as given in the above—in the exponent

$$\int_0^t \sigma_{sT}(x) dZ_s - \frac{1}{2} \int_0^t \sigma_{sT}(x)^2 ds$$

appearing in equation (2.13). Expression (5.15) can be simplified by use of (5.5) to give

$$\sigma T x \int_0^t \frac{1}{T-s} \left( d\xi_{sT} + \frac{\xi_{sT}}{T-s} ds \right) - \frac{1}{2} \frac{1}{(T-s)^2} ds = \frac{T}{T-t} \left( \sigma x \xi_{tT} - \frac{1}{2} \sigma^2 x^2 t \right). \quad (5.16)$$

To derive this result, we make use of the relation

$$\int_0^t \frac{d\xi_{sT}}{T-s} = \frac{\xi_{tT}}{T-t} - \int_0^t \frac{\xi_{sT}}{(T-s)^2} ds. \quad (5.17)$$

With equation (5.16) at hand, we see that indeed (2.13) reduces to equation (5.7) if (5.5) and \(f_{0T}(x) = \tilde{f}_{0T}(x)\) hold.

6 Semilinear volatility structure: Brownian motion approach

We shall show how the models constructed in Section 5 are related to the conditional density models with deterministic volatility structure treated in Section 4. To obtain the models considered in Section 5, we consider a volatility function of the form

$$v(t, x) = \sigma \frac{T}{T-t} x. \quad (6.1)$$
We observe that for this volatility function the associated process \( \{I_t\} \) has the dynamics

\[
dI_t = \sigma \frac{T}{T-t} X \, dt + dB_t. \tag{6.2}
\]

We thus calculate the exponent

\[
\int_0^t v(s, x) \, dI_s - \frac{1}{2} \int_0^t v^2(s, x) \, ds \tag{6.3}
\]

in equation (4.3) making use of (6.1) and (6.2). We have:

\[
\int_0^t v(s, x) \, dI_s - \frac{1}{2} \int_0^t v^2(s, x) \, ds = T \left[ (T-t) \int_0^t dB_s + T(T-t) \sigma X \int_0^t \frac{ds}{(T-s)^2} \right] - \frac{1}{2} T^2 \sigma^2 x^2 \int_0^t \frac{ds}{(T-s)^2}. \tag{6.4}
\]

The first integral gives rise to a \((\{G_t, Q\})\)-Brownian bridge \( \{\beta_{tT}\} \) over the interval \([0, T]\). More specifically, we have

\[
\beta_{tT} = (T-t) \int_0^t \frac{dB_s}{T-s}. \tag{6.5}
\]

The deterministic integral in (6.4) gives

\[
\int_0^t \frac{ds}{(T-s)^2} = \frac{T}{T(T-t)}. \tag{6.6}
\]

Armed with these results, one can write (6.3) as follows:

\[
\int_0^t v(s, x) \, dI_s - \frac{1}{2} \int_0^t v^2(s, x) \, ds = T \left[ \sigma x (\sigma x t + \beta_{tT}) - \frac{1}{2} T \frac{\sigma^2 x^2 t}{T-t} \right]. \tag{6.7}
\]

Let the process \( \{\xi_{tT}\} \) be defined for \( t \in [0, T] \) by

\[
\xi_{tT} = \sigma X t + \beta_{tT}. \tag{6.8}
\]

Then for (6.7) we obtain

\[
\int_0^t v(s, x) \, dI_s - \frac{1}{2} \int_0^t v^2(s, x) \, ds = T \frac{\sigma x \xi_{tT} - \frac{1}{2} \sigma^2 x^2 t}{T-t}. \tag{6.9}
\]

We conclude that the conditional density process \( \{f_t(x)\} \) in (4.3) reduces to the following expression in the case for which the volatility structure is chosen to be given by (6.1):

\[
f_t(x) = \frac{f_0(x) \exp \left[ \frac{T}{T-t} \left( \sigma x \xi_{tT} - \frac{1}{2} \sigma^2 x^2 t \right) \right]}{\int_0^\infty f_0(y) \exp \left[ \frac{T}{T-t} \left( \sigma y \xi_{tT} - \frac{1}{2} \sigma^2 y^2 t \right) \right] \, dy}. \tag{6.10}
\]

From equation (6.9) we see that the process \( \{\xi_{tT}\} \) takes the role of the information process that generates \( \{\mathcal{F}_t\} \). Since \( \{\beta_{tT}\} \) vanishes for \( t = T \), the random variable \( X \) is “revealed”
at $T$. Thus the random variable $X$ is $\mathcal{F}_T$-measurable, and the process \{\xi_t\} defined in (6.8) is essentially identical to the one generating the information-based models in Brody et al. (2007, 2008). This conclusion is supported by the following construction.

We consider the measure $\mathbb{B}$ defined in (4.9). Under $\mathbb{B}$ the process \{\xi_t\} is a Brownian motion over the interval $t \in [0, T)$. We can construct a $\mathbb{B}$-Brownian bridge by use of the $\mathbb{B}$-Brownian motion \{\xi_t\} as follows. On $[0, T)$ we set

$$\xi_{tT} = \left( \frac{T}{T-t} \right)^{T-t} \int_0^t \frac{1}{T-s} dI_s.$$  \hfill (6.11)

Next we recall definition (4.2) and insert this in the expression for the Brownian bridge above. The result is

$$\xi_{tT} = (T-t) \int_0^t \frac{1}{T-s} dB_s + (T-t) \int_0^t v(s, X) \, ds.$$  \hfill (6.12)

The first integral defines a ($\mathcal{G}_t$, $\mathbb{Q}$)-Brownian bridge process over the interval $[0, T)$ which we denote \{\beta_{tT}\}. For the volatility function we set

$$v(t,x) = \sigma T \frac{x}{T-t}.$$  \hfill (6.13)

This leads to

$$\xi_{tT} = \sigma X (T-t) T \int_0^t \frac{1}{(T-s)^2} ds + \beta_{tT},$$  \hfill (6.14)

and therefore to

$$\xi_{tT} = \sigma X t + \beta_{tT}.$$  \hfill (6.15)

Since \{\mathcal{F}_t\} is generated by \{I_t\}, so it is equivalently by \{\xi_{tT}\}. That is what makes \{\xi_{tT}\} an “information process”. Due to the special form of the volatility function (6.13), the random variable $X$ is $\mathcal{F}_T$-measurable. Hence we see that the role of the $X$-factor inside the information process \{\xi_{tT}\} in Brody, Hughston & Macrina (2007, 2008), is taken over by $A_T$, in such a way that for (6.15) we can write

$$\xi_{tT} = \sigma A_T t + \beta_{tT}$$  \hfill (6.16)

instead, in accordance with the notation used in Section 5. The expression for the conditional density \{f_t(x)\} appearing in (6.10) is identical to the one used in the information-based models of Brody et al. (2007, 2008).

7 Bachelier model

The Bachelier model is $A_t = \gamma W_t$ where $\gamma$ is a constant. We shall show that the class of models defined by Proposition 5.1 also contains the Bachelier model. We consider a random variable $A_T$ associated with a fixed date $T$. We assume that $A_T \sim N[0, 1/T\sigma^2]$, where $N[m,v]$ denotes the class of normally-distributed random variables with mean $m$ and variance $v$. In the notation of Section 5, we have

$$\tilde{f}_{0T}(x) = \frac{\sigma \sqrt{T}}{\sqrt{2\pi}} \exp \left(-\frac{1}{2} \sigma^2 T x^2 \right).$$  \hfill (7.1)
we recall the process \( \{\xi_T\}_{0 \leq t \leq T} \) defined by (5.3). We observe that if \( A_T \sim N(0, 1/T \sigma^2) \) then \( \{\xi_T\} \) is an \( \{\mathcal{F}_t\} \)-Brownian motion over \([0, T]\). This can be shown by noticing that \( \{\xi_T\} \) is a continuous Gaussian process where \( \xi_{0T} = 0 \) and \( \text{Cov}[\xi_{sT}, \xi_{tT}] = s \) for \( 0 \leq s \leq t \leq T \). Next we recall the definition of the Brownian motion \( \{W_t\} \) associated with \( \{\xi_T\} \), given by (5.6). Since for \( A_T \sim N(0, 1/T \sigma^2) \) the process \( \{\xi_T\} \) is a Brownian motion, it follows that

\[
\mathbb{E}[A_T | \xi_{sT}] = \frac{1}{\sigma T} \mathbb{E}[\xi_{tT} | \xi_{sT}] = \frac{1}{\sigma T} \xi_{sT}. \tag{7.2}
\]

Thus we see that \( W_t = \xi_{tT} \). As a consequence we see that

\[
A_t = \mathbb{E}[A_T | \xi_{tT}] = \frac{1}{\sigma T} \mathbb{E}[W_T | W_t] = \frac{1}{\sigma T} W_t. \tag{7.3}
\]

Hence to match the Bachelier model with an element in the class of models constructed in Section 5, it is necessary to set \( \sigma = 1/(\gamma T) \).

**Proposition 7.1.** The conditional density process \( \{f_{tT}^B(x)\} \) of the Bachelier price process, defined over the interval \([0, T]\), given by

\[
f_{tT}^B(x) = \frac{\exp\left[-\frac{1}{2} \sigma^2 (T-t)(x - \gamma W_t)^2\right]}{\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2} \sigma^2 (T-t)(y - \gamma W_t)^2\right] dy}, \tag{7.4}
\]

is a special case of the family of models in Proposition 5.1 obtained by setting

\[
\tilde{f}_{0T}(x) = \frac{\sigma \sqrt{T}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \sigma^2 T x^2\right), \quad \sigma_{tT}(x) = \sigma \frac{T}{T-t} x, \quad \text{and} \quad \sigma = 1/(\gamma T). \tag{7.5}
\]

**Proof.** We insert (7.5) in (5.7). Completion of squares gives

\[
f_{tT}(x) = \frac{\tilde{f}_{0T}(x) \exp\left[\frac{T}{T-t} \left(\sigma_{tT} x - \frac{1}{2} \sigma^2 x^2 t\right)\right]}{\int_{-\infty}^{\infty} f_{0T}(y) \exp\left[\frac{T}{T-t} \left(\sigma_{tT} y - \frac{1}{2} \sigma^2 y^2 t\right)\right] dy}
\]

\[
= \frac{\exp\left[-\frac{1}{2} \sigma^2 (T-t) \left(x - \frac{1}{\sigma T} \xi_{tT}\right)^2\right]}{\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2} \sigma^2 (T-t) \left(y - \frac{1}{\sigma T} \xi_{tT}\right)^2\right] dy}. \tag{7.6}
\]

Recalling that \( \xi_{tT} = W_t \), and setting \( \sigma = 1/(\gamma T) \), we obtain the desired result. \(\Box\)

**Corollary 7.1.** Let the initial density \( f_{0T}(x) \), the volatility structure \( \sigma_{tT}(x) \), and the parameter \( \sigma \) be given as in (7.5). Then the Bachelier conditional density \( \{f_{tT}^B(x)\} \) satisfies the master equation (2.12), and the Brownian motion \( \{W_t\} \) coincides with \( \{\xi_{tT}\} \).

**Remark 7.1.** Suppose we chose an asset price model with a certain law. Then we know that we can derive the corresponding conditional density process where the related volatility structure and initial density are specified. We then may wonder how the conditional density transforms, and what the new volatility structure looks like, if we were to consider a new
law for the asset price model. For instance, we may begin with the Bachelier model and ask what is the conditional density and volatility structure associated with a log-normal model.

We present a “transformation formula” for the conditional density. This result allows for the construction of a variety of conditional density processes from a given one. Let \( \{ f_t(x) \} \) solve the differential equation (2.12), and let \( \psi : \mathbb{R} \to \mathbb{R} \) be some \( C^1 \)-bijection. Then it is known that if the random variable \( X \) has conditional density \( f_t(x) \) then \( Z = \psi(X) \) has conditional density \( g_t(z) \) given by

\[
g_t(z) = f_t(\psi^{-1}(z)) \quad \text{where } \psi^{-1}(z) \text{ is the inverse function and } \psi'(x) \text{ is the derivative of } \psi(x), \text{ respectively. Furthermore the conditional density } \{ g_t(z) \} \text{ solves}
\]

\[
dg_t(z) = g_t(z) \left[ \nu(t, z) - \langle \nu_t \rangle \right] dW_t, \quad \text{(7.8)}
\]

where \( \nu(t, z) := \nu(t, \psi^{-1}(z)) \) and

\[
\langle \nu_t \rangle = \int_{\mathbb{R}} v(t, y) g_t(y) dy. \quad \text{(7.9)}
\]

We see thus, that the volatility structure \( \nu(t, x) \) associated with the density \( f_t(x) \) becomes \( \nu(t, z) \) associated with \( g_t(z) \) through the transformation.

For example, consider the Bachelier model where \( A_T \sim N(0, 1/(T\sigma^2)) \). The associated initial density and volatility structure are given in (7.5). Now suppose that \( Z = \exp(A_T) \), that is \( \psi(x) = \exp(x) \). The asset price process \( \{ A_t \} \) is then given by the log-normal model

\[
A_t = \exp(\gamma W_t), \quad \text{(7.10)}
\]

where \( W_t = \xi_{tT} \) as stated in Corollary (7.1). By the transformation formula (7.7), it follows that the conditional density process \( \{ g_{tT}(x) \} \) associated with the log-normal price process (7.10) is given by

\[
g_{tT}(z) = \frac{1}{z} \int_0^\infty \exp \left[ -\frac{1}{2} \frac{1}{\gamma^2(T-t)} (\ln(y) - \gamma W_t)^2 \right] \exp \left[ -\frac{1}{2} \frac{1}{\gamma^2(T-t)} (\ln(y) - \gamma W_t)^2 \right] dy, \quad \text{(7.11)}
\]

for \( z > 0 \). Indeed we see that \( g_{tT}(z) \) is the normalised log-normal conditional density. The associated volatility structure is given by

\[
\nu(t, z) = \nu(t, \ln(z)) = \sigma_{tT}(z) = \sigma \frac{T}{T-t} \ln(z). \quad \text{(7.12)}
\]

8 Call option prices

We consider a European-style call option with maturity \( t \), strike \( K \) and price process \( \{ C_{st} \}_{0 \leq s \leq t} \) defined by

\[
C_{st} = \mathbb{E}^Q \left[ (A_t - K)^+ \mid \mathcal{F}_s \right], \quad \text{(8.1)}
\]

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The price process \{A_t\}_{0 \leq t < \infty} of the underlying asset is given by
\[
A_t = \mathbb{E}^Q [X \mid \mathcal{F}_t]
\]
\[
= \frac{\int_{-\infty}^{\infty} x f_0(x) \exp \left( \int_0^t v(s,x) dI_s - \frac{1}{2} \int_0^t v^2(s,x) ds \right) dx}{\int_{-\infty}^{\infty} f_0(y) \exp \left( \int_0^t v(s,y) dI_s - \frac{1}{2} \int_0^t v^2(s,y) ds \right) dy},
\]
where the filtration \{\mathcal{F}_t\}_{0 \leq s \leq t < \infty} is generated by
\[
I_t = B_t + \int_0^t v(s,X) ds.
\]
We also recall that \{B_t\} is a (\{\mathcal{G}_t\}, Q)-Brownian motion. We also recall that \{I_t\} can be written in terms of an (\{\mathcal{F}_t\}, Q)-Brownian motion \{W_t\} by the relation
\[
dI_t = dW_t + \langle v_t \rangle dt,
\]
where we recall that the bracket notation stands for the scalar product
\[
\langle v_t \rangle = \int_{\mathbb{R}} v(t,x) f_t(x) dx.
\]
Next we introduce a positive (\{\mathcal{F}_t\}, Q)-martingale \{\Lambda_t\} defined by
\[
\Lambda_t = \exp \left( \int_0^t \langle v_s \rangle dW_s + \frac{1}{2} \int_0^t \langle v_s \rangle^2 ds \right),
\]
which induces a change of measure from Q to a new measure Q\*:
\[
\frac{dQ^*}{dQ} \bigg|_{\mathcal{F}_t} = \Lambda_t.
\]
The Q\*-measure is characterised by the fact that \{I_t\} is an (\{\mathcal{F}_t\}, Q\*)-Brownian motion. Furthermore we observe that, by using the relationship
\[
\int_{\mathbb{R}} f_0(x) \exp \left( \int_0^t v(s,x) dI_s - \frac{1}{2} \int_0^t v^2(s,x) ds \right) dx
\]
\[
= \exp \left( \int_0^t \langle v_s \rangle dI_s - \frac{1}{2} \int_0^t \langle v_s \rangle^2 ds \right)
\]
\[
= \Lambda_t,
\]
we may use the denominator in (8.2) for the price process \{A_t\} to write the option price \(C_{st}\) in terms of a conditional expectation taken with respect to the measure Q\* under which \{I_t\} is a standard Brownian motion. We note here that the equality (8.8) must be satisfied so that the conditional density \(f_t(x)\) integrates to one. Equation (8.9) is obtained by applying the relationship (8.4). We then have that
\[
C_{st} = \mathbb{E}^Q \left[ (A_t - K)^+ \mid \mathcal{F}_s \right],
\]
\[
= \mathbb{E}^Q \left[ (N_t \Lambda_t^{-1} - K)^+ \mid \mathcal{F}_s \right],
\]
\[
= \mathbb{E}^Q \left[ \Lambda_t^{-1} (N_t - K \Lambda_t)^+ \mid \mathcal{F}_s \right],
\]
\[
= \Lambda_s^{-1} \mathbb{E}^Q^* \left[ (N_t - K \Lambda_t)^+ \mid \mathcal{F}_s \right],
\]
where

\[ N_t = \int_{-\infty}^\infty x f_0(x) \exp \left( \int_0^t v(s, x) dI_s - \frac{1}{2} \int_0^t v^2(s, x) ds \right) dx, \]  

\[ (8.14) \]

\[ \Lambda_t = \int_{-\infty}^\infty f_0(x) \exp \left( \int_0^t v(s, x) dI_s - \frac{1}{2} \int_0^t v^2(s, x) ds \right) dx. \]  

\[ (8.15) \]

Since \( \{I_t\} \) is an \( \{F_t, Q^*\}\)-Brownian motion, the conditional expectation boils down to the calculation of a Gaussian integral provided the zero of the max-function is computed. In case the goal is a closed-form expression for the computed conditional expectation, then it is necessary to consider a special case. In the following we consider a “binary” initial density function given by

\[ f_0(x) = q_1 \delta(x - x_1) + q_2 \delta(x - x_2), \]  

\[ (8.16) \]

where \( \delta(x) \) is the Dirac distribution and \( q_i = Q[X = x_i] \) for \( i = 1, 2 \). It follows that

\[ N_t = \sum_{i=1}^2 x_i q_i \mathcal{E}_t(x_i), \]  

\[ (8.17) \]

\[ \Lambda_t = \sum_{i=1}^2 q_i \mathcal{E}_t(x_i), \]  

\[ (8.18) \]

where we introduce the process

\[ \mathcal{E}_t(x_i) = \exp \left[ \int_0^t v(s, x_i) dI_s - \frac{1}{2} \int_0^t v^2(s, x_i) ds \right]. \]  

\[ (8.19) \]

In the special case that the random variable \( X \) takes values in \( \{x_1, x_2\} \), the call option price is

\[ C_{st} = \Lambda_s^{-1} \mathbb{E}_{s}^{Q^*} \left[ \left( \sum_{i=1}^2 (x_i - K) q_i \mathcal{E}_t(x_i) \right)^+ \middle| \mathcal{F}_s \right]. \]  

\[ (8.20) \]

We observe that \( \{\mathcal{E}(x_i)\} \) is a positive process with the property \( \mathbb{E}_{s}^{Q^*} [\mathcal{E}_t(x_i)] = 1 \). In particular we use the martingale \( \{\mathcal{E}_t(x_i)\} \) to define a change-of-measure from \( Q^* \) to a new measure that we denote \( \hat{Q} \) by

\[ \left. \frac{d\hat{Q}}{dQ^*} \right|_{\mathcal{F}_t} = \mathcal{E}_t(x_1), \]  

\[ (8.21) \]

together with \( dI_t = d\tilde{I}_t + v(t, x_1) dt \). Next we pull out \( \mathcal{E}_t(x_1) \) to the front of the max-function in equation (8.20) to obtain

\[ C_{st} = \Lambda_s^{-1} \mathbb{E}_{s}^{Q^*} \left[ \mathcal{E}_t(x_1) \left( q_1(x_1 - K) + q_2(x_2 - K) \frac{\mathcal{E}_t(x_2)}{\mathcal{E}_t(x_1)} \right)^+ \middle| \mathcal{F}_s \right]. \]  

\[ (8.22) \]

By use of the Bayes formula we express the call option price in terms of the new measure \( \hat{Q} \):

\[ C_{st} = \Lambda_s^{-1} \mathbb{E}_{s}(x_1) \mathbb{E}_{s}^{\hat{Q}} \left[ \left( q_1(x_1 - K) + q_2(x_2 - K) \frac{\mathcal{E}_t(x_2)}{\mathcal{E}_t(x_1)} \right)^+ \middle| \mathcal{F}_s \right]. \]  

\[ (8.23) \]
For the sake of a simplified notation we define $R_{0t} = \mathcal{E}_t(x_2)/\mathcal{E}_t(x_1)$ and notice that the process $\{R_{0t}\}$ is an exponential ($\{\mathcal{F}_t, \tilde{Q}\}$)-martingale:

$$R_{0t} = \exp \left( \int_0^t [v(s, x_2) - v(s, x_1)] d\tilde{I}_s - \frac{1}{2} \int_0^t [v(s, x_2) - v(s, x_1)]^2 ds \right).$$  \quad (8.24)

We write $R_{0t} = R_{0s} R_{st}$ so that

$$C_{st} = \Lambda_s^{-1} \mathcal{E}_s(x_1) \mathbb{E}^{\tilde{Q}} \left[ (q_1(x_1 - K) + q_2(x_2 - K) R_{0s} R_{st}^+) \mid \mathcal{F}_s \right],$$  \quad (8.25)

and note that $R_{0s}$ is $\mathcal{F}_s$-measurable. We are thus left with finding the range of values of the random variable $R_{st}$ for which the max-function vanishes. Thereafter we calculate the Gaussian integral arising from the conditional expectation. Recalling that $\{\tilde{I}_t\}$ is an ($\{\mathcal{F}_t, \tilde{Q}\}$)-Brownian motion, we observe that the logarithm of $R_{st}$ is Gaussian. Let $Y$ be a standard normal random variable. Then it holds that

$$\ln (R_{st}) = \sqrt{\int_s^t [v(u, x_2) - v(u, x_1)]^2 du} Y \left. - \frac{1}{2} \int_s^t [v(u, x_2) - v(u, x_1)]^2 du \right|_{u=0}^{u=t}.$$  \quad (8.26)

By solving the argument of the max-function in (8.25) for the logarithm of $R_{st}$, we see that the max-function is zero for all values $y^*$ that $Y$ may take for which

$$y^* \leq \frac{\ln \left[ \frac{q_1(K-x_1)}{q_2(K-x_2)R_{0s}} \right] + \frac{1}{2} \int_s^t [v(u, x_2) - v(u, x_1)]^2 du}{\sqrt{\int_s^t [v(u, x_2) - v(u, x_1)]^2 du}}.$$  \quad (8.27)

Therefore the option price can be written in terms of two Gaussian integrals as follows:

$$C_{st} = \Lambda_s^{-1} \mathcal{E}_s(x_1) \left[ q_1(x_1 - K) \frac{1}{\sqrt{2\pi}} \int_{y^*}^\infty \exp \left( -\frac{1}{2} y^2 \right) dy \right.$$

$$\left. + q_2(x_2 - K) R_{0s} \frac{1}{\sqrt{2\pi}} \int_{y^*}^\infty \exp \left( -\frac{1}{2} \eta^2(y) \right) dy \right].$$  \quad (8.28)

where

$$\eta(y) = y - \sqrt{\int_s^t [v(u, x_2) - v(u, x_1)]^2 du}.$$  \quad (8.29)

The two integrals can be written in terms of the cumulative normal distribution denoted $\Phi(x)$. In order to highlight the similarity with the Black-Scholes option price formula, we define

$$d_{st}^- = -y^*, \quad d_{st}^+ = d_{st}^- + \sqrt{\int_s^t [v(u, x_2) - v(u, x_1)]^2 du},$$  \quad (8.30)

so that we can write

$$C_{st} = \Lambda_s^{-1} \mathcal{E}_s(x_1) \left[ q_1(x_1 - K) \Phi(d_{st}^-) + q_2(x_2 - K) R_{0s} \Phi(d_{st}^+) \right].$$  \quad (8.31)
We can simplify the above equation further by use of relation (8.18). Finally the option price can be expressed by the following compact formula:

\[
C_{st} = (x_1 - K) \frac{q_1}{q_1 + q_2 R_{Os}} \Phi(d^-_{st}) + (x_2 - K) \frac{q_2}{q_1 R_{Os}^{-1} + q_2} \Phi(d^+_{st}).
\]  

(8.32)

References.


