Information Percolation in Centralized Markets

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First version: January 2011
Current version: July 2011

This research has been carried out within the NCCR FINRISK project on “Dynamic Asset Pricing”
Information Percolation in Centralized Markets*  

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July 14, 2011  

PRELIMINARY AND INCOMPLETE. COMMENTS WELCOME  

Abstract  

We explore the effects of information propagation in a centralized financial market. Specifically, we embed search frictions within the Grossman and Stiglitz (1980) framework, relying on information percolation as modeled in Duffie, Malamud, and Manso (2009). First, we show that information percolation produces a positive autocorrelation in stock returns. Second, we introduce a rumor among the population of investors which we let to propagate during two periods. As trading approaches its final round and that the fundamental value is close to be revealed, we let the rumor die out and show that it produces a strong price reversal toward the fundamental value. Importantly, information percolation introduces heterogeneity in individual precision among agents. This leads to drastically different investment strategies: agents who have been more efficient at gathering signals will tend to act as market makers. Whereas, agents who collected a lesser amount of signals will tend to be trend followers.

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*The authors are grateful for comments received from Rui Albuquerque, Philippe Bacchetta, Snehel Banerjee, Nittai Bergman, Hui Chen, Darrell Duffie, Bernard Dumas, Andrei Jirnyi, Harrison Hong, Ron Kaniel, Leonid Kogan, Arvind Krishnamurthy, Semyon Malamud, Gustavo Manso, Jianjun Miao, Dimitris Papanikolaou, Rémy Praz, Sergio Rebelo, Costis Skiadas, Jules van Binsbergen, and seminar participants at Boston University, Kellogg School of Management and MIT Sloan. Financial support from the Swiss Finance Institute and from NCCR FINRISK of the Swiss National Science Foundation is gratefully acknowledged.  
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1 Introduction

The concept of rational-expectations equilibrium has been proposed to illustrate how markets aggregate information that is initially dispersed across individual investors\(^1\). While centralized markets contribute to centralize information that is dispersedly held by market participants, they only do imperfectly so. In particular, the presence of noise trading and idiosyncratic shocks slows down information aggregation through prices. As noted by Grossman and Stiglitz (1980), “this is perhaps lucky, for were it (the price) to do it perfectly, an equilibrium would not exist”. As a result, the Efficient-Market Hypothesis fails and prices slowly drift toward the fundamental value. This fact has alternatively been viewed as an illustration of stock markets being beauty contests. Pursuing Keynes’ idea, Allen, Morris, and Shin (2006) argue that investors try to forecast the forecasts of others instead of assets’ fundamental value. Doing so, investors end up chasing the crowd. More generally, depending on the amount of noise in their private signals, investors tend to either follow the trend or to speculate. Whether or not this phenomenon is susceptible to produce price drift has been carefully analyzed in Cespa and Vives (2009) and Banerjee, Kaniel, and Kremer (2009).

In the present paper, we attempt to introduce social interactions within the Grossman and Stiglitz (1980) framework. In particular, we assume that markets are centralized, but that information is not. Accordingly, agents have to search for each other’s private information. Search dynamics are modeled based on the recent theoretical developments in the Information Percolation literature\(^2\). We show that, although markets are centralized, social interactions still have a strong effect on stock prices. On the one hand, social networks are prone to generate momentum in stock returns. On the other hand, they may help spreading a rumor through financial markets, producing significant reversal once the uncertainty pertaining to the rumor is resolved.

The rational-expectations literature focuses on the market as a whole and vastly ignores private interactions among investors. This is unfortunate as prices being imperfect aggregators, social networks have an important role to play, in particular as an alternative channel conveying information through private discussions. For instance, Shiller (2000) writes: “Word-of-mouth transmission of ideas appears to be an important contributor to day-to-day or hour-to-hour stock market fluctuations...”. In the context of decentralized markets, Duffie, Malamud, and Manso (2009) stud-

ies the percolation of information through a population of investors. Beyond the particular scope of financial markets, examples of the relevance of social networks in Economics abound: Acemoglu, Bimpikis, and Ozdaglar (2010) investigate the effectiveness of dynamics of communication as an aggregator of dispersed information. Burnside, Eichenbaum, and Rebelo (2010) show how social dynamics may explain the moves on the housing market. Stein (2008) describes conversations as being a central part of economic life. Shiller and Pound (1989) provide evidence that direct interpersonal communications are an important determinant of investors decisions.

Empirically, few works (to the best of our knowledge) have documented the relevance of word-of-mouth communications for stock return autocorrelation. The empirical investigation suffers from an identification problem: Due to the so-called Manski critique, a given strategy could be implemented by two individuals because they either met and shared their private information or because they observed the same public news. Two important contributions are still to be mentioned: Hong, Lim, and Stein (2000) show that stocks with greater analyst coverage exhibit more momentum. More to the point, Hong, Hong, and Ungureanu (2010) investigate, both theoretically and empirically, the relation between the serial correlation of returns and information diffusion based on word-of-mouth communication.

We show that, depending on the intensity at which agents meet with each other, information percolation produces momentum in stock returns. More precisely, when the frequency of meetings is low, noise trading overwhelms the impact of the diffusion of private information and agents’ portfolio positions tend to be mostly driven by noise trading. As a consequence, stock returns exhibit reversal. As information percolation intensifies, the average private information precision goes up and stock returns exhibit momentum. Drawing on this insight, we introduce a rumor which we let to propagate among the population of investors for several periods. As trading approaches its final round and that the fundamental value is close to be revealed, we let the rumor die out and show that it produces a strong price reversal toward the fundamental value.

In another context, Hong and Stein (1999) also use gradual diffusion of information in order to explain momentum and reversal. However, importantly, unlike the latter contribution, we do not need to postulate different types of agents in order to produce this effect. Neither do we prevent investors to extract information from

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prices. On the contrary, we assume that agents start off being identical indi-
viduals, and behave exactly as in the Grossman and Stiglitz (1980) setup. As they
start meeting with each other, they progressively become informationally heteroge-
neous. This ultimately leads to drastically different investment strategies: Agents
who have been more efficient at gathering signals tend to act as market makers.
Whereas, agents who collected a lesser amount of signals tend to be trend followers.

The paper is structured as follows: In Section 2, we derive a noisy rational
equilibrium in which agents meet and share their private information. In Section
3, we introduce a rumor. Section 4 concludes. For convenience, we report all
computations in the Appendix.

2 The Benchmark Model

2.1 Setup

There is a risky asset with payoff realized at date 1. The payoff is \( \tilde{U} \), a random
normal variable with mean 0 and precision \( H \). There is a continuum of investors
\( i \in [0,1] \). Each investor \( i \) is endowed at time 0 with quantities of the risky as-
sets represented by \( X^i \). Investors have exponential utility function with common
coefficient of absolute risk aversion \( 1/\gamma \). The aggregate per capita supply of the
risky asset, \( \tilde{X}_0 = \int_0^1 X^i di \), is normally and independently distributed with mean 0
and precision \( \Phi \). Trading takes place in 4 trading sessions which are held at times
\( \tau_t = t/4, t = 0, ..., 3 \). The asset payoff is realized and consumption takes place at
time 1 after the last trading session.

Immediately prior to each trading session, each investor \( i \) obtains a private signal
about the asset payoff, \( \tilde{Z}^i_t \), with

\[
\tilde{Z}^i_t = \tilde{U} + \tilde{\varepsilon}^i_t,
\]

where \( \tilde{\varepsilon}^i_t \) is distributed normally and independently of \( \tilde{U} \), has mean zero, precision
\( S \), and is independent of \( \tilde{\varepsilon}^k_t \), if \( k \neq i \) and \( j \neq t \).

New liquidity traders are assumed to enter the market in each trading session.
The incremental net supply of these traders is \( \tilde{X}_t \), which have zero mean and preci-

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5For other explanations of momentum and reversal with rational agents, see Albuquerque and
Vayanos and Woolley (2010), Makarov and Rytchkov (2009), Biais, Bossaerts, and Spatt (2010) or

6We could assume, as in He and Wang (1995) for instance, an AR(1) noise trading process. Still,
we prefer to assume that noise trading follows a random walk in order to isolate the momentum
effect. Introducing an AR(1) process for noise trading would only reinforce our momentum results.
sion Φ. All the information concerning the present setup is grouped for convenience in Table 1.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Precision</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fundamental</td>
<td>U</td>
<td>0</td>
<td>H</td>
</tr>
<tr>
<td>Per capita supply</td>
<td>Xᵢ</td>
<td>0</td>
<td>Φ</td>
</tr>
<tr>
<td>Liquidity traders</td>
<td>Xᵢ</td>
<td>0</td>
<td>Φ</td>
</tr>
<tr>
<td>Private signals</td>
<td>Zᵢ</td>
<td>(for ̂εᵢ)0</td>
<td>S</td>
</tr>
</tbody>
</table>

Table 1: Random variables in the benchmark model

The precision S, is assumed to be the same for all investors and uniformly bounded. It follows that the population average of the precision of private signals is equal to S.

Let ̂Pₜ denote the equilibrium risky asset price and ̂Dᵢₜ the risky asset demand for investor i. We denote by ̂Fᵢ the common information available to all investors at date t, and by ̂Fᵢ the total information available to investor i at date t. The following theorem describes the risky asset prices and investor asset demands at each date in a noisy rational expectations equilibrium.

**Theorem 1.** There exists a partially revealing rational expectations equilibrium in the 4 trading session economy in which the risky asset price, ̂Pₜ, and individual asset demands, ̂Dᵢₜ, for t = 0, .., 3, are given by:

\[
  ̂Pₜ = \frac{1}{Kₜ} \left( (Kₜ - S (t + 1)) \bar{μ}ₜ + S \bar{U} (t + 1) - \frac{1}{γ} \sum_{j=0}^{t} ̂Xⱼ \right),
\]

\[
  ̂Dᵢₜ = γ \left( \sum_{j=0}^{t} \left[ S ̂Zⱼ + \frac{̂Xⱼ}{γ} \right] - S \bar{U} (t + 1) \right),
\]

where

\[
  \bar{μ}ₜ \equiv \mathbb{E} \left[ U | ̂Fᵢ \right] = \frac{1}{Kₜ} S \sum_{j=0}^{t} ̂Zⱼ + γ²SΦ ̂Qⱼ,
\]

\[
  ̂Qⱼ \equiv ̂U - \frac{1}{γS} ̂Xⱼ,
\]

\[
  Kₜ \equiv \text{Var}^{-1} \left[ U | ̂Fᵢ \right] = H + S (1 + γ²SΦ) (t + 1)
\]

5
The optimal trading strategy of the individual investor \( i \) at time \( t = 1, 2, 3 \) is given by

\[
\Delta \tilde{D}_t^i \equiv \tilde{D}_t^i - \tilde{D}_{t-1}^i = \gamma \left( S \tilde{Z}_t^i - S \tilde{U} + \frac{\tilde{X}_t}{\gamma} \right) = \gamma S \left( \tilde{Z}_t^i - \tilde{Q}_t \right)
\]

(5)

**Proof.** See Appendix A. Q.E.D.

The trading strategy of investor \( i \) in periods \( t = 1, 2, 3 \) depends on the difference between his private signal in period 1 and the normalized price signal, weighted by his private signal precision and his risk aversion. If the precision is high, he will take a larger position. If the risk aversion is high (\( \gamma \) is low), he will take a smaller position. The sign of each investor’s position depends on how his private information compares with public information. In such a setting, stock returns exhibit positive autocorrelation provided that the quality of information shows sufficient improvement across periods. Hence, in order to obtain a significant amount of momentum, one needs to assume exogenously large improvement in precision over time. To see this, let us consider the covariance between the first and second period price differences, \( \text{cov} \left( \tilde{P}_1 - \tilde{P}_0, \tilde{P}_2 - \tilde{P}_1 \right) \):

\[
- \frac{HS \left( \gamma^2 S \Phi + 1 \right)}{\gamma^2 \Phi (H + \gamma^2 S^2 \Phi + S) (H + 2S \left( \gamma^2 S \Phi + 1 \right))^2 (H + 3S \left( \gamma^2 S \Phi + 1 \right))}
\]

This covariance is always negative, irrelevant of the calibration. As a consequence, in a standard rational-expectations setting, price returns tend to generally exhibit reversal. The same argument applies for the next period covariance, \( \text{cov} \left( \tilde{P}_2 - \tilde{P}_1, \tilde{P}_3 - \tilde{P}_2 \right) \). We borrow the intuition from Makarov and Rytchkov (2009): In a standard full information setting, investors being risk averse require a higher compensation in the form of a risk premium for holding a higher share of the risky asset. This results into a positive relation between expected returns and noisy supply and a negative relation between prices and noisy supply. If the noisy supply is assumed to be i.i.d., as in the present model, realized returns are negatively autocorrelated. In a partially revealing equilibrium, the same mechanism is at play, although slightly obscured by private information\(^7\).

If we consider different precisions at each date \( S \) for the private signals, it then turns out that a very simple condition needs to be satisfied in order for stock returns to exhibit momentum:

\[
(S_1 - S_0) \left( 1 + \gamma^2 S_0 \Phi \right) > H
\]

\(^7\)See Makarov and Rytchkov (2009) for a detailed derivation of the conditions for negative serial correlation to obtain.
Accordingly, in order to obtain momentum in a standard rational-expectations framework, we need to increase the average precision of private information, at the expense of losing in realism. This fact has been carefully analyzed in Cespa and Vives (2009).

While significant surge in precision over time appears difficult to sustain exogeneously, this arises as a natural phenomenon if we let investors chat with each other. In particular, information percolation delivers a convenient and intuitive mechanism to produce that effect: As agents meet with each other and gather others’ signals, they become more and more precise as time goes by.

Grossman (1981) proposed the concept of rational expectations equilibrium to capture the idea that stock prices aggregate information that is initially dispersed across investors. However, because of noise trading and idiosyncratic shocks, it only does imperfectly so. Hence, even though centralized markets contribute to centralize information, social networks thus still have an important role to play, in particular as an alternative channel conveying information through private discussions. For instance, Stein (2008) or Acemoglu, Bimpikis, and Ozdaglar (2010) have emphasized the importance of communication in social networks. In the next subsections, we introduce social dynamics among investors and show how it allows to naturally produce an improvement in investors’ precision over time, thus generating momentum.

2.2 Introducing Information Percolation

For the remainder of this section, we shall assume that each investor $i$ only receives a single private signal $\tilde{Z}_{i0}$ prior to the first trading session. Still, we let investors receive additional signals at $t = 1, 2, 3$, yet in a very specific way that we shall explain in this subsection. From date $t = 0$ onward, we let agents meet with each other and share the private signal $\tilde{Z}_{i0}$ that they received at date $t = 0$. It is important to emphasize that the signals they give to each other are not exactly the signals they were initially endowed with at date $t = 0$, for these signals have already been accounted for in the price at date $t = 0$. Rather, initial signals should be seen as being reshuffled among the population of agents.

Although trading is centralized, we assume that information is not. Agents have to search for each other’s information. There are at least four ways to motivate
this assumption: As suggested by Hong, Kubik, and Stein (2004), two possibilities
include word-of-mouth learning and enjoyment from talking with friends about the
market\textsuperscript{9}. Although we pursue a rational explanation, two more behavioral motiva-
tions involve limited attention and news being only understandable when explained
by friends, as suggested by Hong, Hong, and Ungureanu (2010). Related to the
limited attention story, The Economist notes\textsuperscript{10}: “In the face of this torrent of in-
formation, it is perhaps unsurprising that investors rely on whom they know, rather
than what they know.” Given a friend network or degree of sociability, some agents
may be more successful in matching other agents and may, thus, end up gathering
more signals than others. This produces a natural source of heterogeneity among
agents: In our model, while they all start off being identical agents, they become
informationally heterogeneous as soon as they start meeting with each other.

We now formalize this idea using the setup described in Duffie, Malamud, and
Manso (2009) and the references therein. We fix a probability space \((\Omega, \mathcal{F}, P)\) and
a nonatomic space \((A, \mathcal{A}, \alpha)\) of agents which represents the continuum described
above. As in the previous subsection, all agents benefit from information about the
random variable \(\hat{U}\). Each agent initially gets a signal. For almost every pair \((i, j)\)
of agents, their signal sets are disjoint.

Any particular agent is matched to other agents at each of a sequence of Pois-
son arrival times with a mean arrival rate \(\lambda\), which is common across agents. At
each meeting time, the matched agent is randomly selected from the population of
agents. We assume that, for almost every pair of agents, this matching procedure is
independent. There is a zero probability that the set of agents that \(i\) has met before
time \(t\) overlaps with the set of agents that \(j\) has met before time \(t\).

By the joint Gaussian assumption and by induction in the number of prior meet-
ings of each of a pair of currently matched agents, it is enough for the purpose of
updating the agents’ conditional expectation of \(\hat{U}\) that agent \(i\) tells his counterparty
at any meeting at time \(t\) his or her current conditional mean \(\hat{\mu}_t\) of \(\hat{U}\) and the total
number \(N_{it}\) of signals that played a role in calculating the agent’s current conditional
distribution of \(\hat{U}\). This number of signals is initially 1, and is then incremented at
each meeting by the number \(N_{jt}\) of signals that similarly influenced the information
about \(\hat{U}\) that had been gathered by his counterparty \(j\) by time \(t\).

The cross-sectional distribution \(\mu_t\) of information precision (i.e. \(N\)) at time \(t\) is
defined, at any set \(B\) of positive integers, as the fraction \(\mu_t(B) = \alpha \{i : N_{it} \in B\} / q\)
of agents whose precisions are currently in the set \(B\) and where \(q\) is the total number

\textsuperscript{9}The authors specify that there might be great distinctions to be made between these two social
interaction mechanisms.

\textsuperscript{10}We refer to Too much information - Buttonwood, The Economist, 14 July 2007.
of agents (which is fixed because there are no entries nor exits). The cross-sectional precision distribution satisfies the following Boltzmann equation

\[
\frac{d}{dt} \mu_t = \lambda \mu_t * \mu_t - \lambda \mu_t \\
= \lambda \sum_{m=1}^{n-1} \mu_t(n-m) \mu_t(m) - \lambda \mu_t
\]  

(6)

where * denotes the discrete convolution product.

Since agents are assumed to be initially endowed with a single signal, the initial distribution of signals is a Dirac mass at 1, i.e. \( \mu_0 = \delta_1 \). This feature has the major advantage of leading to a closed-form solution for the cross-sectional distribution of types. The latter is simply given by

\[
\mu_t(n) = e^{-n\lambda} (e^{\lambda} - 1)^{n-1}.
\]  

(7)

The social dynamics in (7) admit a strikingly simple solution. The simplicity of (7) is particularly compelling when considering the rich heterogeneity among agents that it is prone to generate. Producing information heterogeneity with tractable social dynamics is a delicate task. For instance, Hong and Stein (1999) build a simple social network which does not produce any heterogeneity in the amount of information across agents. Similarly, Hong, Hong, and Ungureanu (2010) make use of a social network borrowed from the Epidemics literature and which brings about two classes of agents while staying admittedly difficult to manipulate. Rather, the information percolation mechanism delivers both tractable cross-sectional dynamics and heterogeneity among agents. Moreover, while we only restrict our attention to a particular type of initial distributions (a Dirac mass), this mechanism is is flexible and remains tractable for a vast class of initial distributions. Figure 1 illustrates the evolution of the cross-sectional distribution \( \mu_t(n) \) of signals through time.

In order to solve for the equilibrium, we shall work with the incremental signals \( y \) that have accrued between time \( t \) and time \( T \) rather than the total amount of signals \( n \) at a given date \( t \). Notice that both may be equivalently used; we choose to use distributions of increments because of its convenience. Hence, it will prove useful to determine the cross-sectional distribution \( \pi_t(y) \) of the incremental signals \( y \) that kicked in between \( t \) and \( T \). The following purpose is implemented in the proposition below.

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\[11\] The authors have to restrict their analysis to a network comprised of a restricted number of agents.
**Figure 1**: Evolution of the probability density function of the number of signals through time when agents are initially endowed with one signal. Each graph depicts $\mu$ at time $t = 1$ and $t = 2$, respectively, with $\lambda = 1$.

**Proposition 2.** The cross-sectional distribution $\pi$ of incremental signals that accrued during time $t$ and $T$ satisfies

$$
\pi_T = e^{\lambda \int_t^T \mu_s ds -(T-t)}
$$

or equivalently solves the differential equation

$$
d\pi_t = -\lambda \pi_t + \lambda \pi_t * \mu_t
$$

with initial condition given by the Dirac measure $\pi_0 = \delta_0$ at zero.

**Proof.** See Appendix B. Q.E.D.

This distribution satisfies the same dynamics as the probability density of the posterior type at time $t$ of a given agent’s type who was randomly drawn with density $\pi(\cdot,0)$ at time zero in Duffie and Manso (2007) except that the distribution always starts at a Dirac mass $\delta_0$ in order to reflect that the counter has been reset to zero at the beginning of the relevant period. Therefore, initially, at time $t = 0$, $\pi$ and $\mu$ are the same distributions expect that $\pi$ has a Dirac mass at 0 while $\mu$ has a Dirac mass at one. The distribution $\pi_t$ also admits a closed-form solution:

$$
\pi_t(y) = e^{-\mu_t(y)(\lambda^{T-1})-\mu_T(y)(\lambda^{T-1})} \frac{\lambda(T-t)}{\pi(y)}. \tag{8}
$$

We now have all the necessary elements at hand to compute the equilibrium when the social dynamics described above are involved. Some additional notation is required. An important statistic for the present economy is the cross-sectional average of additional signals at time $t$, $\Omega_t = \sum_{n \in \mathbb{N}} \pi_t(n) n$. Furthermore, let $k_t = \{k_j\}_{j=0}^t$ denote an investor type, i.e. the whole history of signals he has gathered up
to and including \( t \). By construction, we have \( k_0 = k_3 = 1 \). At times \( t = 2 \) and \( t = 3 \), it follows from Gaussian theory that the average of the \( k_t \) private signals received by agent \( i \), denoted hereafter \( \tilde{Z}_{i,t,k_t} \), is sufficient in the estimation of \( \tilde{U} \). The demand at time \( t \) of an individual investor of type \( k_t \) is denoted by \( \tilde{D}_{i,t,k} \). Proceeding with similar notations, let us denote by \( \tilde{F}_{i,t,k} \) the information that agent \( i \) of type \( k_t \) has at his disposal at date \( t \). The following theorem is the analogue of Theorem 1 when agents meet and share their private information.

**Theorem 3.** There exists a partially revealing rational expectations equilibrium in the 4 trading session economy in which the risky asset price, \( \tilde{P}_t \), and individual asset demands, \( \tilde{D}_{i,t,k} \), for \( t = 0, \ldots, 3 \), are given by:

\[
\tilde{P}_t = \frac{K_t - H}{K_t} \tilde{U} - \frac{1}{\gamma K_t} \sum_{j=0}^{t} \frac{1 + \gamma^2 S \Phi \Omega_j}{\gamma K_t} \tilde{X}_j, \\
\tilde{D}_{i,t,k} = \gamma \left[ \sum_{j=0}^{t} \left( k_j S \tilde{Z}_{i,j,k_j} - S \Omega_j \tilde{U} + \frac{\tilde{X}_j}{\gamma} \right) - \Delta K_{i,k} \tilde{P}_t \right]
\]

where

\[
\tilde{\mu}_{i,t,k} \equiv \mathbb{E} \left[ \tilde{U} \mid \tilde{F}_{i,t,k} \right] = \frac{1}{K_{i,t,k}} \sum_{j=0}^{t} \left( k_j S \tilde{Z}_{i,j,k_j} + \gamma^2 S^2 \Phi \Omega_j \tilde{Q}_j \right)
\]

(10)

and

\[
K_{i,t,k} \equiv \text{Var}^{-1} \left[ \tilde{U} \mid \tilde{F}_{i,t,k} \right] = H + \sum_{j=0}^{t} \left[ k_j S + \gamma^2 S^2 \Phi \Omega_j \tilde{Q}_j \right]
\]

(11)

The difference \( \Delta K_{i,k} \equiv K_{i,k} - K_j \) represents the precision (dis)advantage of agent \( i \) with respect to the average precision of all investors. The optimal trading strategy of the individual investor \( i \) at time \( t = 1, 2, 3 \) is given by

\[
\Delta \tilde{D}_{i,t,k} \equiv \tilde{D}_{i,t,k} - \tilde{D}_{i,t-1,k} = \gamma \left[ k_t S \left( \tilde{Z}_{i,t,k_t} - \tilde{P}_t \right) - S \Omega_t \left( \tilde{U} - \tilde{P}_t \right) + \frac{\tilde{X}_t}{\gamma} - \Delta K_{i-1,k} \Delta \tilde{P}_t \right]
\]

(12)

**Proof.** See Appendix C. Q.E.D.

Theorem 3 contains two important results, one of which is summarized in the following lemma.

**Lemma 4.** Investors are myopic with respect to their future amount of signals.
Intuitively, investors’ demand in (12) does not depend on the future number of signals. That is, investors simply do not take into account the uncertainty with respect to the signals they will gather, or equivalently the number of other investors they will meet in the future.

Second, the last term of the optimal strategy as per equation (12) of Theorem 3 carries a signification that is central to the implication of noisy rational expectations equilibria with information percolation. In particular, introducing information percolation produces heterogeneous precisions among agents through the distribution $\pi$ of signals amount across the population of agents. Analogous to Brennan and Cao (1996) in the general case of heterogeneous precisions, optimal demands contain the following term

$$-\Delta K_{t-1,k}^i \left( \hat{P}_t - \hat{P}_{t-1} \right).$$

In the case of the latter reference, this simply means that agents who are more precise than the average agent tend to be contrarians while agents who are less precise tend to be trend followers. When information percolation prevails, it means that investors who have been more efficient at gathering signals, liquidate their positions. On the other hand, investors who have gathered a lesser amount of signals tend to buy, thus being trend followers.

This result bear some similarity with the setup of Hong and Stein (1999). Better informed investors act as the “newswatchers” from their model, while poorly informed investors act as the “momentum traders”. This is because poorly informed investors adjust more heavily on the public price signal, and thus they become trend-chasers. Note however that, unlike Hong and Stein (1999), we do not artificially create these two different investor types. It arises naturally from the information percolation mechanism. Neither do we restrict “newswatchers” to learn from prices, or “momentum traders” to have access to private information.

Importantly, this reveals a crucial implication of information percolation: Starting within an homogenous agents setting, information percolation first produces informationally heterogeneous agents which ultimately leads to drastically different investment strategies. In the present case, because $\Delta K_{0,k}^i = 0$, investors only trade on price variations from period $t = 1$ and thereafter. Our model may alternatively be viewed as one with a time-varying representative agent\(^\text{12}\).

Two papers verify empirically this result. First, in Brennan and Cao (1996), similar trading behavior on price variations comes from an accumulated information (dis)advantage. The main difference is that, in our case, the accumulation comes from the quantity of signals and not from their relative precision. Brennan and Cao\(^\text{12}\)

\(^{12}\)We thank Sergio Rebelo for pointing out this interpretation.
(1996) assume initial heterogeneity in the information precision, whereas investors in the present model start off being all identical agents and become heterogeneous as they search for each other. This has a strong economic appeal: For instance, if the present setup were to be applied to the home bias puzzle with which the latter reference is concerned, we would not need to assume that home investors have initial higher precision. In contrast to Brennan and Cao (1996), we could simply assume that agents are initially identical but, due to home networks being more dense, agents have more chances to meet home investors than foreign investors, thus focusing their precision on local concerns.

Second, Feng and Seasholes (2004) obtain the same implication by using a model similar to Brennan and Cao (1996), on a sample of individual brokerage accounts from the Chinese stock market. One of their results is that investors who live near the headquarters of a firm, and thus are able to obtain better private information, have net trades that are negatively correlated with stock returns. While “far” investors have net trades that are positively correlated with stock returns.

2.3 Price Drift

The fact that information percolation increases precision over time produces a significant amount of price drift, an implication to which we now turn. As in Cespa and Vives (2009), we concentrate on the covariance between the first and second period returns and the second and third period returns. This allows to capture the sign of the serial comovements in returns. We then normalize these covariances in order to obtain autocorrelations. The latter covariances are exposed in the proposition below.

**Proposition 5.** The covariances between the first and second period returns and the second and third period returns are respectively given by

\[
\text{cov} \left( \tilde{P}_1 - \tilde{P}_0, \tilde{P}_2 - \tilde{P}_1 \right) = \left( \frac{1}{K_1} \sum_{j=0}^{1} \Omega_j \left( \gamma^2 S \Phi \Omega_j + 1 \right) - \frac{1}{K_0} \left( \gamma^2 S \Phi + 1 \right) \right) \frac{S^2}{H} \\
\times \left( \frac{1}{K_2} \sum_{j=0}^{2} \Omega_j \left( \gamma^2 S \Phi \Omega_j + 1 \right) - \frac{1}{K_1} \sum_{j=0}^{1} \Omega_j \left( \gamma^2 S \Phi \Omega_j + 1 \right) \right) + \left( \frac{1}{K_2} - \frac{1}{K_1} \right) \left( \gamma S \Phi + \frac{1}{7} \right)^2 \left( \frac{1}{\gamma_1} - \frac{1}{\gamma_2} \right) \frac{1}{\Phi}.
\]

\[(13)\]
and

\[
\text{cov} \left( \tilde{P}_3 - \tilde{P}_2, \tilde{P}_2 - \tilde{P}_1 \right) = \left( \frac{1}{K_2} \sum_{j=0}^{2} \Omega_j \left( \gamma^2 S \Phi \Omega_j + 1 \right) - \frac{1}{K_1} \sum_{j=0}^{1} \Omega_j \left( \gamma^2 S \Phi \Omega_j + 1 \right) \right) \\
\times \left( \frac{1}{K_3} - \frac{1}{K_2} \right) \sum_{j=0}^{2} \Omega_j \left( \gamma^2 S \Phi \Omega_j + 1 \right) \right) S^2 \frac{H}{H} \right) + \left( \frac{1}{K_3} - \frac{1}{K_2} \right) \left( \left( \frac{1}{K_1} - \frac{1}{K_2} \right) \sum_{j=0}^{1} \left( \gamma S \Phi \Omega_j + \frac{1}{\gamma} \right)^2 \right) \frac{1}{\Phi}.
\]

(14)

**Proof.** See Appendix D. Q.E.D.

By substituting the average precisions from (11) into Proposition 5, we obtain a simple condition for momentum to arise in the first period:

\[
(\Omega_1 - 1) \left( 1 + \gamma^2 S \Phi \right) > H
\]

Since the cross-sectional average of signals at time 1, \( \Omega_1 \) is strictly increasing in the meeting intensity \( \lambda \), there must be a threshold \( \lambda^* \) such that for \( \lambda > \lambda^* \) we obtain momentum, and for \( \lambda < \lambda^* \) we obtain reversal. Figure 2 shows that, for our calibration, this threshold is approximately \( \lambda^* = 1 \).

Consistent with the impact of information percolation exposed in Subsection 2.2, gathering a large amount of signals significantly improves the precision over time. Since this phenomenon is best captured by the intensity \( \lambda \) at which agents meet with each other, we plot serial autocorrelations in return between period \( t = 0, 1 \), \( t = 1, 2 \) and \( t = 2, 3 \) against the average meeting in Figure 2 below.

![Figure 2](image-url)

**Figure 2:** The solid thick line represents the first period autocorrelation while the dashed line represents the second period autocorrelation. The calibration is \( H = S = \Phi = 1 \) and \( \gamma = \frac{1}{3} \).
As is apparent from Figure 2, we observe that for a low meeting intensity, returns exhibit reversal. This case has been referred to the Keynesian equilibrium in Cespa and Vives (2009). This happens when the improvement in precision across time is too low. Specifically, when agents do not meet at all, prices are martingales and exhibit no drift. This result is reminiscent of Kyle (1985) and extensions thereof, where informed traders only receive a private signal at date $t = 0$. It is however not strictly comparable to the usual Kyle (1985) setup because, in such setting, the market maker, whose role is explicitly determined, is widely assumed to be risk neutral.

For low meeting frequencies, the situation is akin to the setup of Theorem 1. Because this setup compares to standard rational-expectations equilibria, following the discussion of Subsection 2.1, returns exhibit reversal. As information percolation intensifies, meetings accelerate and the average precision goes up. This eventually offsets and dominates the former effect, thus producing momentum. That is, information percolation eventually produces momentum, a phenomenon that would hardly arise within a classical rational-expectations equilibrium except maybe for some far-fetched calibration.

Crucially, notice that this figure is strikingly in line with empirical evidences, particularly so with Figure 1 in Hong, Lim, and Stein (2000). The latter reference documents a pronounced nonlinear and non-monotonic relation between firm size and momentum. The study tend to cluster the analysis on the part of the sample for which the relation between momentum and size is decreasing. In Figure 2, this corresponds to $\lambda > 2$. Importantly, unlike Hong and Stein (1999) and because we allow for supply shocks, our model is able to reproduce this pattern over their whole sample range. As a result, consistent with Hong, Lim, and Stein (2000), we find that firms for which information slowly diffuses exhibit reversal. Finally, not only are we able to reproduce the whole empirical pattern, but we do this within a rational expectations framework and with a single parameter $\lambda$.

### 2.4 Volume and Price Volatility

Information percolation carries other implications for asset pricing, particularly regarding trading volume and price volatility. The information flow generated by word-of-mouth communications strongly impacts the way individuals build their portfolios: Information percolation increases the speed at which individuals tilt their portfolios toward speculation or market making. Thus volume is closely related to

---

13 The authors use a measure of analyst coverage corrected for firm size in order to proxy for the extent of information diffusion.
the information flow. That information percolation accelerates portfolio adjustments should be particularly compelling when looking at the trading volume. In order to see how the diffusion of information affects trading volume, we first consider the setting of Theorem 1. In this setting, agents rebalance their portfolios according to (5). In particular, it is apparent that \( \Delta \tilde{D}_i \) is driven by the extent of the noise \( \tilde{\epsilon}_i \) in agent \( i \)'s private signal and noise trading \( \tilde{X}_t \). It reflects both the speculative and the market-making part of an investor’s portfolio. Following He and Wang (1995), we define the volume in the benchmark setting as

\[
V_B(t) = \int_0^1 |\Delta \tilde{D}_i| \, di.
\]

The expected volume \( \bar{V}_B(t) \) in the Benchmark model is thus given by

\[
\bar{V}_B(t) = E\left[\int_0^1 |\Delta \tilde{D}_i| \, di\right] = \sqrt{\frac{2}{\pi}} \left( \gamma^2 S + \frac{1}{\Phi} \right). \tag{15}
\]

The expected trading volume in the Benchmark setup is decreasing in the risk aversion and the precision of noise trading and increasing the average precision of private information. Importantly, as in He and Wang (1995), although (15) contains the integral over the continuum of agents, it is actually never used in the computations. That is, aggregation does not affect the determinants of trading volume. This is due to information precision \( S \) being identical across individuals. Since He and Wang (1995) also assume, as in the Benchmark setup, that the precision in private information is homogenous across the population of agents, aggregation does not bite either when computing expected trading volume. However, when information percolation prevails, this turns out to be no longer true. To see this, first notice that the expression (12) for \( \Delta \tilde{D}_{t,k} \) in Theorem 3 can be rewritten as

\[
\Delta \tilde{D}_{t,k} = \frac{K_{i,t,k}^i}{K_t} \tilde{X}_t + \gamma Sk_{i,t,k}^i + \gamma^2 S \Phi \Omega \left( \frac{K_{i,t,k}^i}{K_t} - 1 \right) \tilde{X}_t \tag{16}
\]

\[
+ \gamma \left( \frac{K_{i,t,k}^i}{K_t} - \frac{K_{i,t-1,k}^i}{K_{t-1}} \right) H \tilde{U} + \left( \frac{K_{i,t,k}^i}{K_t} - \frac{K_{i,t-1,k}^i}{K_{t-1}} \right) \sum_{j=0}^{t-1} \left( 1 + \gamma^2 S \Phi \Omega_j \right) \tilde{X}_j
\]

where \( \tilde{\epsilon}_{i,t,k} \) is the average noise in investor \( i \)'s private signal. From (16), it is apparent that the ratio \( \frac{K_{i,t,k}^i}{K_t} \) between investor \( i \)'s individual precision and the precision of the average agent plays an important role. In particular, if we shut down the information percolation channel and let each individual receive a single signal, then \( k_t = \Omega_t = \frac{K_{t,n+k}^i}{K_{t,n}} = \frac{K_{t-1,k}^i}{K_{t-1}} = 1 \) and \( \tilde{\epsilon}_{i,t,k} = \tilde{\epsilon}_i \). As a consequence, only the two first terms in (16) remain and we recover the asset demand that prevails in the Benchmark setup.

Second, it will prove convenient for aggregation purposes to notice that agent \( i \)'s
individual precision $K_{t,k}^i$ may be written as

$$K_{t,n+k}^i = K_{t-1,n}^i + kS + \gamma^2 S^2 \Phi \Omega_t^2$$

where $k \sim \pi_t$ is the increment of new signals that accrued between time $t - 1$ and time $t$ while $n \sim \mu_{t-1}$ is the total number of signals gathered up to time $t - 1$. Disentangling an agent’s total current amount $n + k$ of signals in that manner allows to separate the aggregation of signals that have been gathered up to time $t - 1$ from the aggregation of those which just accrued at date $t$.

Now, we can use the fact that the random variables $\tilde{U}, \tilde{X}_j$ appearing in (16) are independent and that for $X \sim \mathcal{N}(0, \sigma^2)$, $E \|X\| = \sqrt{\frac{2}{\pi}} \sigma$, and directly obtain the expression for the expected trading volume when information percolation is involved

$$\bar{V}_t = \sqrt{\frac{2}{\pi}} \times \sum_{n \in \mathbb{N}} \sum_{k \in \mathbb{N}} \pi_t(k) \mu_{t-1}(n)$$

$$\times \left[ \gamma^2 k^2 S + \left( \frac{K_{t,n+k}^i}{K_t^i} - \frac{K_{t-1,n}^i}{K_{t-1}^i} \right)^2 \left( \gamma^2 H + \frac{1}{\Phi} \sum_{j=0}^{t-1} (1 + \gamma^2 S \Phi \Omega_j)^2 \right) \right].$$

As mentioned above, the striking feature of the expected trading volume in (17) is that aggregation matters. This yields a novel implication regarding the behavior of the trading volume in a rational expectations equilibrium with respect to the work of He and Wang (1995). Because the aggregation of agents is driven by the diffusion of information which is itself parametrized by $\lambda$, we plot the expected trading volume as a function of $\lambda$ for times $t = 1, 2$ in the figure below. More precisely, we plot the Benchmark-adjusted expected trading volume $\bar{V}_t - \bar{V}_t^B$. In Figure 3, the blue line represents the time $t = 1$ expected volume which is monotonically increasing in $\lambda$. This confirms that the diffusion of information contributes to significantly increase the volume due to the acceleration in trading strategies. In particular, from (12) one observes that price variations do not yet play a role in agents’ portfolio rebalancing during the first period. The reason is that they only initially receive a single signal and thus $\Delta K_{0,k}^i = 0$. The next period, as information spreads out through word-of-mouth communications, agents start to trade on price variations and the volume exhibits a non-monotonic pattern as seen from the red line. Specifically, for sufficiently low or large information diffusion intensity (say $\lambda < 1$ and $\lambda > 4$), the time $t = 1$ and time $t = 2$ volumes are very close. This is intuitive: For low search intensities, information percolation takes more time to take off and, thus, the amounts of signal gathered respectively at time $t = 1$ and $t = 2$ are not tremen-
Figure 3: The solid thick line represents the first period expected trading volume while the dashed line represents the second period expected trading volume. The calibration is $H = S = \Phi = 1$ and $\gamma = \frac{1}{3}$.

dissimilarly different. For high search intensities, a large fraction of the population has already gathered a significant amount of signals at time $t = 1$ and the incremental change in portfolio position is accordingly of small magnitude. The most interesting pattern in the trading volume appears for intermediate information diffusion intensities. In the latter case, the information is sufficiently spread over the whole population to allow individuals who are better informed to take advantage of their superior knowledge in order to speculate on price variations. This result is in line with findings of Holden and Subrahmanyam (2002).

Turning to price volatility, in order to keep the same structure, we consider the volatility $\sigma_t \equiv \sqrt{\text{var} \left( \Delta \tilde{P}_t \right)}$ of price differences. The latter is easily shown to be

\[
\sigma_t = \left( \frac{1 + \gamma^2 S \Phi \Omega_t}{\gamma K_t} \right)^2 \frac{1}{\Phi} + \left( \frac{1}{K_t} - \frac{1}{K_{t-1}} \right)^2 \left( H + \frac{1}{\gamma^2 \Phi} \sum_{j=0}^{t-1} \left( 1 + \gamma^2 S \Phi \Omega_j \right)^2 \right) \tag{18}
\]

We plot the volatility of the price differences for time $t = 1, 2$ in the Figure 4.

In the first period, the volatility of the price difference is increasing in the information diffusion intensity. The reason is that the price at time $t = 0$ reflects the trading of homogenous investors who only receive a single signal. The price at time $t = 1$ reflects a correction induced by information percolation. As the intensity of communications among investors increases, a larger fraction of investors gathered a sizable amount of signals and, as a result, the price correction is stronger. Thus, the volatility of the price difference increases. During the second period, volatility exhibits the opposite pattern. As information further diffuses between time $t = 2$
and $t = 3$, the price further corrects. Still, if the intensity was already high at time $t = 2$, the price adjustment at time $t = 3$ is of small magnitude. This temporal pattern in price difference volatility also matches Holden and Subrahmanyam (2002)’s results.

3 Residual Uncertainty, or Rumors

We now proceed to introduce a rumor\footnote{As defined by Peterson and Gist (1951), a rumor is viewed as “an unverified account or explanation of events circulating from person to person and pertaining to an object, event, or issue in public concern.”} about the final payoff $\tilde{U}$. The way we model the rumor strongly relates to the residual uncertainty component in He and Wang (1995) and Cespa and Vives (2009). In particular, we introduce additional uncertainty in the private signals that agents are initially endowed with and that they communicate to each other. That is, their signals are not centered on the final payoff $\tilde{U}$ but on some random variable $\tilde{U} + \tilde{V}$ where $\tilde{V}$ denotes the rumor. This error is common across agents and, although agents are aware of it, they do not learn about it, at least for some time. The rumor, thus, brings some market opaqueness into the benchmark setting of Section 2. However, unlike He and Wang (1995) and Cespa and Vives (2009), the rumor is not part of the final payoff, but only pertains to the private signals agents gather. It can be clearly interpreted as some common noise in the private signals of investors. Then, by means of the information percolation mechanism, the rumor propagates and circulates through the markets, up to some point when it vanishes.
This timing in the resolution of uncertainty makes another important difference between our setting and the latter references: In the present model, we let the rumor die out prior to the terminal date. As a result, the additional uncertainty brought about by the rumor is \textit{partially revealed} before $\tilde{U}$ is resolved. More precisely, the uncertainty resolution is modeled as follows: we let the initial signals $\tilde{Z}_0^i$ to be reshuffled and shared across the population of agents through information percolation until time $t = 2$. At time $t = 3$, we let agents receive a private signal $\tilde{Z}_3^i = \tilde{U} + \epsilon_3^i$ centered on the fundamental value $\tilde{U}$. As a result, investors realize on average that the information they have been sharing so far was biased and, thus, correct their beliefs. On the one hand, this mechanism shares some common features with the informational cascades literature initiated by Bikhchandani, Hirshleifer, and Welch (1998), namely i) erroneous mass behavior and ii) fragility. Unlike the informational cascade mechanism, the first aspect is not strategic and is only produced by the fact that signals contain an error. The second aspect stems from the occurrence of an unusual signal which, in the present case, overturns the effect of information percolation. It is worth mentioning that this effect could also be caused by a public signal. Hence, we could alternatively use a public signal $\tilde{Y}_3$ in order to reverse i).

On the other hand, the timing of the rumor and the way it produces reversal both relate to the advance information mechanism in Albuquerque and Miao (2010). In the latter contribution, advance information produces reversal when it materializes or, equivalently, when it becomes stale information. That is, reversal occurs in the setting of Albuquerque and Miao (2010) when advance information loses of its value (because it becomes public). Yet, while advance information carries useful information about future fundamental values, our notion of rumor eventually turns out to be unrelated to the value of the fundamental.

He and Wang (1995) have shown that this type of uncertainty offers a natural mean to enhance trading volume, yet under certain conditions. If the residual uncertainty is too large and, as a result, markets too opaque, $\tilde{V}$ may overwhelm potential gains from speculation. We now want to investigate how additional uncertainty in the form of the rumor described feedbacks with information percolation. In particular, we want to gauge whether both rumor and information propagation are amenable to a phase of momentum followed by a second phase of reversal.

### 3.1 Equilibrium Computation

We now consider the Benchmark setup and proceed to introduce a rumor. As in the benchmark model, there is a risky asset with payoff realized at date 1. There are 4 trading sessions which are held at times $\tau_t = t/4$, $t = 0, ..., 3$. Immediately prior to
trading sessions 0, 1, and 2, each investor \(i\) obtains a private signal about the asset payoff, \(\tilde{Z}_t^i\), with

\[
\tilde{Z}_t^i = \tilde{U} + \tilde{V} + \tilde{\varepsilon}_t^i, \quad t = 0, 1, 2
\]  \hspace{1cm} (19)

The extra random variable \(\tilde{V}\) has zero mean, precision \(\nu\), and is independent of all other variables from the model. This random variable makes the total amount of private information sufficient to infer \(\tilde{U} + \tilde{V}\), but not the true asset payoff \(\tilde{U}\). Therefore, there exists an unobservable component of the private information that is never revealed and makes the setup similar to the general case exposed in He and Wang (1995), and Cespa and Vives (2009). Hence, two things are worth mentioning about this economy. First, even if there is an infinite number of private signals at times 0, 1 and 2, the uncertainty about the value of the stock remains until the end of the economy. Second, given that investors are aware of all of them having this component in their private signals, \(\tilde{V}\) can be interpreted as a rumor pertaining to the final payoff.

Still, we shall let this rumor exist only at times 0, 1 and 2. That is, at time 3, investors receive non-biased private signals about the asset payoff:

\[
\tilde{Z}_3^i = \tilde{U} + \tilde{\varepsilon}_3^i
\]  \hspace{1cm} (20)

As a consequence, at time 3 the rumor is partially resolved, by making the private signals centered on the true fundamental. By applying the Law of Large Numbers, at time 3 agents are right on average about the final payoff of the asset. Notice that this does not imply that some agents know more than others, but simply that agents know that at time 3 the rumor will die out. Intuitively, as an asset approaches some payoff date, some rumors pertaining to its fundamental value simply disappear.

All the other elements of the model are as in the benchmark case. Finding a closed form solution, although very complicated, is possible for 2 or 3 periods, but the problem becomes intractable for more than 3 periods. We resort therefore to numerical methods. However, even if the model is solved numerically, we prefer to keep some analytical structure for the coefficients. This will prove to be beneficial when interpreting the equilibrium solution.

We first begin by assuming as usual that prices are linear functions of the random variables. Then, we normalize the price signals to find the \(\tilde{Q}_t\)s, which are informationally equivalent to the prices. We assume the following structure for the normalized price signals:

\[
\tilde{Q}_t = \tilde{U} + \nu_t \tilde{V} - \frac{1}{\gamma S \Omega_t}
\]  \hspace{1cm} (21)

where \(\nu_t = 1\) for \(t = 0, 1, 2\) and \(\nu_3 = 0\). Since there is no rumor at time 3, we fix
\( \Omega_3 = 1 \). The other 3 coefficients, \( \Omega_{0,1,2} \), are to be found numerically. In fact, it turns out that it is sufficient to solve for them in order to obtain a complete solution to the equilibrium. They solve 3 non-linear equations that are easily found. Denoting by \( \theta_t \) the coefficient of the private signal \( \tilde{Z}_i^t \) in the optimal demand of investor \( i, \tilde{D}_i^t \), for \( t = 0, 1, 2 \), it can be proven that the equations to be solved by the \( \Omega_t \)s are

\[
\begin{align*}
\theta_2 &= \frac{\gamma S (\Omega_0 + \Omega_1 + \Omega_2)}{3} \\
\theta_1 &= \frac{\gamma S (\Omega_0 + \Omega_1)}{2} \\
\theta_0 &= \gamma S \Omega_0
\end{align*}
\] (22)

If \( \nu \to \infty \), e.g., if the rumor has infinite precision and is therefore equal to zero, all the \( \Omega_t \)s are equal to one. Otherwise, they lie between 0 and 1. For most of the calibrations that we use, \( \Omega_t \) is decreasing in time, except at time 0, when it is jumping to 1.

The individual conditional precision at time \( t \) is the same for all the investors and is equal to

\[
K_t = H + \Lambda_t \left( (t + 1) S + \gamma^2 S^2 \Phi \sum_{j=0}^{t} \Omega_j^2 \right)
\] (23)

with

\[
\Lambda_t = \frac{\nu}{\nu + (t + 1) S + \gamma^2 S^2 \Phi \sum_{j=0}^{t} \Omega_j^2}
\] (24)

One can easily check that, when \( \nu \to \infty \), \( \Lambda_t \) is equal to 1, and that \( \Lambda_t \) is decreasing as \( t \) is increasing, except at time 4, when it is jumping to 1. Thus, \( \Lambda_t \) can be interpreted as a dampening factor which limits the increase in conditional precision as long as there is a rumor in the private signals. Once the rumor disappears, there is a massive increase in the conditional precision, driven by the sudden shift in \( \Lambda_t \). This sudden increase occurs because investors’ private information is now centered on the true fundamental.

It can be verified that, for \( t = 0, 1, 2 \), \( K_t = H + \nu (1 - \Lambda_t) \). Thus, during the rumor periods, the conditional precision of the investors is still increasing, but only at a slower rate. Every new period of trading is beneficial for learning, but at a lower marginal rate than what obtains in the case of no rumor.

The conditional expectation at each period is equal to

\[
\tilde{\mu}_i^t = \frac{\Lambda_t}{K_t} \sum_{j=0}^{t} \left( S \tilde{Z}_j^i + \gamma^2 S^2 \Phi \Omega_j^2 \tilde{Q}_j \right)
\] (25)
There are two differences as compared to the standard case. First, investors use less of the public information available at time $t$ (the history of prices, reflected in the $\tilde{Q}_j$s). The effect comes from the presence of $\Omega_j$ in the second term of the sum. Moreover, since $\Omega_j$ is decreasing in time during the rumor periods, the effect is stronger as the number of trading sessions increases. Observe that, at time $t = 3$, $\Omega_3$ jumps to 1, producing a substantial shift in expectations. Second, the term $\Lambda_t$, being smaller than one and decreasing during the rumor periods, is reinforcing the effect. It is again a dampening factor, making investors use less of all the available information that they have, when they form their conditional expectation.

During the rumor periods, investors know that on average there exists an error in their learning. As a result, they optimally modify their learning behavior to take this into account. Intuitively, they become more prudent regarding the available information at each period. Once the rumor is revealed on average (the private signals are centered on the true fundamental), the learning mechanism suddenly shifts to what is observed in a standard rational expectations equilibrium with no residual uncertainty.

In separate calculations, we added a periodic public signal in the investors information set. We supposed that the public signal is centered on the true fundamental, $\tilde{U}$. It turns out that the coefficient of this public signal has exactly the same form as that in the standard setup with no rumor. Since $K_t$ is smaller due to the $\Omega$s and the $\Lambda$s, the investors give a much higher weight to it as it is the case in a standard setup. This result might offer an alternative explanation for why investors may over-react to information such as public announcement, news, etc.

Let us analyze now the optimal demand of the investors. It is given by

$$\tilde{D}_t = \begin{cases} \gamma \sum_{j=0}^{t} \left( S \sum_{k=1}^{t \Omega_j} \tilde{Z}_j - S \tilde{Q}_j \right) & t = 0, 1, 2 \\ \tilde{D}_2 + S \tilde{Z}_3 - S \tilde{Q}_3 & t = 3 \end{cases}$$

(26)

According to this optimal demand, the optimal trading strategy of the individual investor at time $t = 1, 2, 3$, $\tilde{D}_t - \tilde{D}_{t-1}$, is given by

15More precisely, it is equal to $N/K_t$, where $N$ is the precision of the public signal.
16Allen, Morris and Shin (2006) stated that higher order beliefs might push agents to give a larger weight to public information. Even if the higher order belief mechanism is present in our paper as well, we have a somewhat different view, by arguing that it depends on which public information. If investors are aware of some rumor in their private information, they might rely less on prices as public signals and more on financial news, public announcements, etc.
\[
\Delta \tilde{D}_t^i = \begin{cases} 
\gamma \left[ \sum_{j=0}^{t-1} S \left( \sum_{k=0}^{j+1} \Omega_k \right) - \sum_{j=0}^{t+1} \Omega_k \right] \tilde{Z}_j^i + S \sum_{j=0}^{t+1} \Omega_k \tilde{Z}_j^i - S \Omega_t \tilde{Q}_t \\
S \tilde{Z}_3^i - S \tilde{Q}_3 
\end{cases} \quad t = 1, 2 \\
S \tilde{Z}_3^i - S \tilde{Q}_3 
\]

The first term in the square brackets would be equal to zero in a standard setup. However, here it supports that investors trade at each trading session on their previous private information, and in a meaningful way. During the rumor periods, the coefficients of \( \tilde{Z}_j^i \) is negative (because \( \Omega_j \) is decreasing in time, its average is also decreasing in time). That is, investors unwind to some extent the positions that they build following the past private information. With the arrival of the new information, investors have a better knowledge of what the rumor might be, and thus re-adjust their portfolios. It is somewhat like they were overshooting in the past and now they are adjusting for that. This feature is well documented in He and Wang (1995), where they show that the presence of the residual uncertainty increases the trading volume.

In period 3, once the rumor is revealed on average, investors switch back to a standard trading strategy. That is, they stop trading on previous signals.

Finally, the equilibrium price of the asset satisfies

\[
\tilde{P}_t = \frac{S \Omega_t + \gamma^2 S^2 \Phi \Omega_t^2 \Lambda_t \Gamma_t}{K_t} \left( \tilde{U} + \tilde{V} \right) + \\
\sum_{j=0}^{t-1} \left( \frac{S \Omega_j + \gamma^2 S^2 \Phi \Omega_j^2 \Lambda_j \Gamma_j}{K_j} \tilde{Q}_j \right) - \frac{1 + \gamma^2 S \Phi \Omega_t \Lambda_t \Gamma_t}{\gamma K_t} \tilde{X}_t 
\]

with \( \Gamma_t \) a coefficient that is found numerically once the \( \Omega_j \)'s are known. This coefficients takes the value 1 as \( \nu \to \infty \), and is again decreasing in time during the rumor periods.

### 3.2 Information Percolation, Momentum and Reversal

The agents’ optimal investment strategy described in the previous section was pointing to a feature that will become obvious now. Namely, we saw that the presence of a rumor pushes investors to re-adjust their past trades. Hence, since we are in a general equilibrium setting, these successive portfolio adjustments generate reversal in asset returns. It appears that this is truly the case: asset returns in this setting are strongly negatively autocorrelated. Moreover, at date 3, when aggregate private signals reveal the true value of the fundamental, the reversal is even stronger.

At this point, we emphasize once again that neither the rumor nor the funda-
mental are completely revealed at time 3. What happens is that investors’ private signals do not contain the common noise that we refer to as the rumor, and investors are aware of that fact. As a consequence, a significant adjustment toward the fundamental value of the asset price occurs at time 3.

The question that we address now is whether the information percolation is still able to produce significant momentum, despite the fact that, in the present setup, returns are strongly negatively autocorrelated. In order to do so, we embed once more the search dynamics exposed in Subsection 2.2 in the present model. However, given the presence of the rumor, computations are more tedious. We prefer to leave technical details in Appendix E, and show here some very preliminary results.

In Figure 5, we build a graph similar to Figure 2 in the benchmark case. The only additional parameter with respect to the benchmark case is the precision of the rumor $\nu$. We find it convenient for the exposition to build 3 graphs, for 3 different values of $\nu$. If $\nu$ is large, that is, if the rumor is negligible, we approach the benchmark case. This can be seen in the left panel. When $\nu$ is getting smaller, and thus the rumor getting “large”, stock returns tend to be more negatively serially correlated. However, once the information percolation mechanism is involved, the first period autocorrelation tends to be positive, while the second period autocorrelation remains negative. That is, stock returns clearly exhibit momentum and reversal: the momentum is generated by the diffusion of the information, while the reversal is generated by the extinction of the rumor. This can be seen in the middle panel. If the rumor gets too large, then the negative autocorrelation gets stronger and offsets the information diffusion effect. As a result, momentum almost disappears, as seen in the right panel.

We build impulse response functions to better investigate the effect of information percolation. A similar methodology has been used in Hong and Stein (1999) and Daniel, Hirshleifer, and Subrahmanyam (1998). At each trading session, we compute the average price, conditional on the fundamental values $\tilde{U}$ and $\tilde{V}$. We consider an initial date “-1” at which everybody is symmetrically informed only with priors regarding $\tilde{U}$, $\tilde{V}$. Since all agents are identical at this date and have a prior of zero for the liquidation value of the risky asset, $P_{-1} = 0$. The asset is liquidated at date 4, thus $P_4 = \tilde{U}$. Obviously, the parameter $\lambda$ will have an impact on the average path of the prices from $P_{-1}$ to $P_4$.

As can be seen from Figure 6, in the absence of information percolation, the price converges slowly toward the fundamental. In this numerical exercise we fixed the rumor at $\tilde{V} = 0.5$. Without information percolation the price does not overshoot the fundamental value. Once the information percolation intensifies, a hump-shapped
Figure 5: The solid thick line represents the first period autocorrelation while the dashed line represents the second period autocorrelation. The calibration is $H = S = \Phi = 1$ and $\gamma = \frac{3}{4}$. There are three panels, each with a different value of $\nu$. 
pattern appears in the average price path. The rumor propagates into the market and pushes prices to overshoot the fundamental value. A reversal takes place at date 3, when the rumor is on average revealed.

4 Conclusion

This paper attempts to show that the diffusion of information through financial markets has a significant impact on the dynamics of asset prices, even in a centralized market setting. Since, in such a setting, prices do not completely reveal the fundamental, there is certainly a benefit for investors to meet and communicate with their peers.

As a first step, we show that the percolation of information produces price drift. The mechanism is simple and intuitive: Investors’ precision in their estimate of the fundamental increases gradually through time, by gathering informations from their peers. This increase in the conditional precision tends to accelerate the price convergence to its path toward the fundamental, thus producing momentum in asset returns. Additionnally, information percolation appears to be a natural way of generating heterogeneity in investments strategies. Each investor starts with one private signal and randomly meets their peers. As time shifts to the the next trading round, the diffusion of information produces heterogeneity in the number of signals that each agent gathers. Some of them become better informed, and act as contrarians, whereas others, who collected less information, implement trend-following portfolio strategies.
Since the information percolation mechanism appears to be a natural way for news to propagate through financial markets, we address a second question. What if the informations that spread through the market are not completely accurate in the aggregate, e.g., what if they contain a rumor? We add such a feature to the benchmark model, and we show that returns actually continue to exhibit momentum, yet to a lesser extent. Crucially, the momentum phase is followed by a significant reversal, in line with the common under- and over-reaction patterns in stock returns widely documented.

At this point, many fascinating extensions are pending, some of which are work in progress.
A Proof of Theorem 1

We provide the proof for a two trading session economy, that is, we eliminate for simplicity the dates \( t = 2 \) and \( t = 3 \). Once the equilibrium quantities are written in a recursive form, as in Brennan and Cao (1996), or in He and Wang (1995), it is straightforward to derive the full recursive equilibrium solution. The model is solved backwards, starting from date 1 and then going back to date 0.

First, conjecture that prices in period 0 and period 1 are

\[
\tilde{P}_0 = \beta_0 \tilde{U} - \gamma_0 \tilde{X}_0 \tag{29}
\]

\[
\tilde{P}_1 = \varphi_1 \tilde{U} - \gamma_0 \tilde{X}_0 - \gamma_1 \tilde{X}_1. \tag{30}
\]

Consider the normalized price signal in period zero

\[
\tilde{Q}_0 = \frac{1}{\beta_0} \tilde{P}_0 = \tilde{U} - \frac{\gamma_0}{\beta_0} \tilde{X}_0. \tag{31}
\]

\( \tilde{Q}_0 \) is informationally equivalent to \( \tilde{P}_0 \). Replace \( \tilde{X}_0 \) from (31) into (30) to obtain

\[
\tilde{P}_1 = \beta_1 \tilde{U} + \xi_{01} \tilde{Q}_0 - \gamma_1 \tilde{X}_1. \tag{32}
\]

Notice that the coefficients of \( \tilde{U} \) and \( \tilde{V} \) have changed. We can now normalize the price signal in period one

\[
\tilde{Q}_1 = \frac{1}{\beta_1} \left( \tilde{P}_1 - \xi_{01} \tilde{Q}_0 \right) = \tilde{U} - \frac{\gamma_1}{\beta_1} \tilde{X}_1 \tag{33}
\]

We shall make the following assumptions (see Admati (1985)):

\[
\frac{\gamma_0}{\beta_0} = \frac{1}{\gamma S_0}, \quad \frac{\gamma_1}{\beta_1} = \frac{1}{\gamma S_1}. \tag{34}
\]

These assumptions will be verified once the solution is obtained.

A.1 Period 1

The total wealth of the investor \( i \) at date \( t = 1 \) is

\[
\tilde{W}_i = X^i \tilde{P}_0 + \tilde{D}_0^i \left( \tilde{P}_1 - \tilde{P}_0 \right) + \tilde{D}_1^i \left( \tilde{U} - \tilde{P}_1 \right) \tag{35}
\]

At date \( t = 1 \), the information of investor \( i \) is \( \tilde{Z}_i, \tilde{Z}_0^i, \tilde{Q}_1 \) and \( \tilde{Q}_0 \). With this information at hand, investor \( i \) will try to forecast \( \tilde{U} \). The state variables corresponding to investor \( i \) are therefore

<table>
<thead>
<tr>
<th>Precision</th>
<th>( \tilde{U} )</th>
<th>( \tilde{X}_1 )</th>
<th>( \tilde{X}_0 )</th>
<th>( \tilde{X}_1 )</th>
<th>( \tilde{X}_0 )</th>
<th>( \tilde{X}_0 )</th>
<th>( \tilde{X}_0 )</th>
</tr>
</thead>
<tbody>
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<td>( \tilde{U} )</td>
<td>( \tilde{Z}_1 )</td>
<td>( \tilde{Z}_0 )</td>
<td>( \tilde{X}_1 )</td>
<td>( \tilde{X}_0 )</td>
<td>( \tilde{X}_0 )</td>
<td>( \tilde{X}_0 )</td>
</tr>
<tr>
<td>( \tilde{U} )</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \tilde{Z}_1 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \tilde{Z}_0 )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \tilde{Q}_1 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-\frac{1}{\sqrt{3}}</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \tilde{Q}_0 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-\frac{1}{\sqrt{3}}</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

With this information at hand, it is straightforward to calculate

\[
K_1 = Var^{-1} \left[ \tilde{U} | \tilde{Z}_1, \tilde{Z}_0, \tilde{Q}_1, \tilde{Q}_0 \right] \tag{36}
\]

\[
\tilde{\mu}_1 = E \left[ \tilde{U} | \tilde{Z}_1, \tilde{Z}_0, \tilde{Q}_1, \tilde{Q}_0 \right] \tag{37}
\]

by using the projection theorem.
Theorem 6. Consider a n-dimensional normal random variable
\[(\theta, s) \sim N\left(\begin{bmatrix} \mu_{\theta} \\ \mu_s \end{bmatrix}, \begin{bmatrix} \Sigma_{\theta,\theta} & \Sigma_{\theta,s} \\ \Sigma_{s,\theta} & \Sigma_{s,s} \end{bmatrix}\right)\].

The conditional density of \(\theta\) given \(s\) is normal with conditional mean
\[\mu_{\theta} + \Sigma_{\theta,s}\Sigma_{s,s}^{-1}(s - \mu_s)\]
and variance-covariance matrix
\[\Sigma_{\theta,\theta} - \Sigma_{\theta,s}\Sigma_{s,s}^{-1}\Sigma_{s,\theta}\]
provided \(\Sigma_{s,s}\) is non-singular.

The optimal demand of trader \(i\) in period 1 has a simple expression (from the normality of distribution assumption in conjunction with the exponential utility function):
\[\tilde{D}_i^1 = \gamma K_1 \left(\tilde{\mu}_i - \tilde{P}_1\right)\].

Integrate the optimal demands to get the total demand. Follow the convention used by Admati (1985) that implies \(\int_{1}^{0} \tilde{Z}_{i,j} = \tilde{U}_{i,j}\), a.s., and replace \(\tilde{U}_{i,j} - \frac{1}{\gamma S_{i,j}} \tilde{X}_i\) to obtain
\[\tilde{D}_1 = \gamma \left(S \left(\tilde{Z}_{i,0} + \tilde{Z}_{i,1}\right) + \gamma^2 S^2 \Phi \left(\tilde{Q}_0 + \tilde{Q}_1\right) - K_1 \tilde{P}_1\right)\].

Integrate the optimal demands to get the total demand. Follow the convention used by Admati (1985) that implies \(\int_{1}^{0} \tilde{Z}_{i} = \tilde{U}\ a.s., and replace \(\tilde{U} - \frac{1}{\gamma S_{i,j}} \tilde{X}_i\) to obtain
\[\tilde{D}_1 = \gamma \left(S \left(2 + \gamma^2 S \Phi\right) \tilde{U} + \gamma^2 S^2 \Phi \tilde{Q}_0 - \gamma^2 S^2 \Phi \tilde{X}_1 - K_1 \tilde{P}_1\right)\].

In order to clear the market, we need to have \(\tilde{D}_1 = \tilde{X}_0 + \tilde{X}_1\). Once we impose market clearing, we can use the conjectured equation (32) to get the undetermined coefficients. The solution is shown below.

\[\beta_1 = \frac{S \left(1 + \gamma^2 S \Phi\right)}{K_1},\]
\[\xi_{01} = \frac{S \left(1 + \gamma^2 S \Phi\right)}{K_1}\]
\[\gamma_1 = \frac{1 + \gamma^2 S \Phi}{\gamma K_1}\]

From these solutions, we can verify one of the two equalities assumed in (34), namely \(\frac{\gamma_1}{\beta_1} = \frac{1}{\gamma S}\).

A.2 Period 0

The problem of the investor is now
\[\max_{\tilde{D}_0} E\left[-e^{-\frac{1}{\gamma} \tilde{W}_i^i} \mid \tilde{Z}_{i,0}^i, \tilde{Q}_0\right]\]
where, as stated before, the final wealth is given by
\[\tilde{W}_i^i = X^i \tilde{P}_0 + \tilde{D}_0^i \left(\tilde{P}_1 - \tilde{P}_0\right) + \tilde{D}_1^i \left(\tilde{U} - \tilde{P}_1\right)\].

Replace \(\tilde{D}_i^1\) from (38) into this equation and observe that the investor needs to estimate \(\tilde{U}, \tilde{P}_1\) and \(\tilde{\mu}_1\), after observing \(\tilde{Z}_{i,0}^i\) and \(\tilde{Q}_0\). This is a quadratic form of a multivariate normal variable. Accordingly, we use the following convenient theorem:
Theorem 7. Consider a random vector $z \sim N(0, \Sigma)$. Then,

$$E \left[ e^{\frac{1}{2} Fz + Gz + H} \right] = |I - 2SS^{-1}F|^{-\frac{1}{2}} e^{\frac{1}{2} G(I - 2SS^{-1}F)^{-1}G + H}$$

The matrices $F$, $G$ and $H$ have complicated forms and are not exposed here for simplicity. They are replaced in the objective function, which is then differentiated with respect to $\tilde{P}_0$ and equalized to zero in order to solve for the optimal demand. Once the optimal demand is found, it can be integrated to solve for $\tilde{P}_0$. This allows to finally find the undetermined coefficients of $\tilde{P}_0$:

$$\beta_0 = \frac{S(1 + \gamma^2 S\Phi)}{K_0},$$
$$\gamma_0 = \gamma S\Phi + \frac{1}{\gamma}. \quad (44)$$

At this point, the remaining equality conjectured in (34), namely $\frac{\beta_0}{\gamma_0} = \frac{1}{\gamma_2}$ is verified as well. Once we know the coefficients, we can write the solution in a recursive form as done in Theorem 1.

\hfill □

B Proof of Proposition 2

In order to prove Proposition 2, we compute the distribution of new signals that have been gathered between time 0 and time $T$, i.e. during the time period $T$. Denote by $X_{t_i}$ the number of new signals accruing if a meeting occurs at time $t_i$ and observe that it is distributed as

$$X_{t_i} \sim \mu(t_i, \cdot)$$

where the distribution $\mu(t, x)$ satisfies the differential equation in (6). It is reminded that, in the latter differential equation, $\lambda$ represents the intensity of the Poisson meeting process and $\ast$ the convolution product. That is, the number $N(T)$ of meetings that have taken place between time 0 and $T$ is a Poisson counter with intensity $\lambda$. The total number $Y_T$ of new signals that have been gathered between time 0 and $T$ is accordingly given by $\sum_{i=1}^{N(T)} X_{t_i}$. We now proceed to characterize its distribution. Notice that the $X_{t_i}$’s are independent but not identically distributed, which makes this task somewhat more difficult. Yet, this problem is very similar to the accumulated loss claim problem with time-dependent claim amount in the actuarial sciences field (e.g. see Buehmann (2005)). Our problem also somewhat compares to Duffie, Malamud, and Manso (2010).

A first step is to observe that $Y_T$, conditional on the set of times $\{0 \leq t_1 \leq t_2 \leq \ldots \leq t_{N(T)} \leq T\}$ at which a meeting has occurred during $T$ and the total number of meetings $N(T)$ (that is, conditioning on the whole trajectory $A^N_T$ of the Poisson process), is distributed as

$$Y_T | A^N_T \sim \int_{R^{N-1}} \mu \left( Y_{t_N} - Y_{t_{N-1}}, t_N \right) d\mu \left( Y_{t_{N-1}} - Y_{t_{N-2}}, t_{N-1} \right) \ldots d\mu \left( Y_{t_1} - 0, t_1 \right)$$
$$\equiv \int_{R^{N-1}} \mu \left( X_{t_N}, t_N \right) d\mu \left( X_{t_{N-1}}, t_{N-1} \right) \ldots d\mu \left( X_{t_1}, t_1 \right).$$

In order to bridge this with Duffie, Malamud, and Manso (2010), we observe that the distribution $\mu(X_{t_i}, t_i)$ of the increment can be expressed as a translation $T$ of the type measure $\mu$ and the increment $x$. Hence, as in the latter reference, the distribution above may be written as

$$Y_T | A^N_T \sim \Gamma_{i=1}^{N(T)} \mu_{t_i}$$

where, for any probability measures $\alpha_1, \ldots, \alpha_k$, we write $\Gamma_{i=1}^{k} = \alpha_1 \ast \alpha_2 \ast \ldots \ast \alpha_k$.

Now, as a second step, observe that each $t_i$ in the sequence of meetings $\{0 \leq t_1 \leq t_2 \leq \ldots \leq$
\( t_{N(T)} \leq T \}\) conditional on \( N(T) \) is uniformly distributed over \( T \). Accordingly, we have that

\[
Y_T | N(T) \sim \Gamma_{i=1}^N(1) \frac{1}{T} \int_0^T \mu_t dt = \left( \frac{1}{T^{N(T)}} \left( \int_0^T \mu_s ds \right)^{*N(T)} \right)
\]

where \(*n\) denotes the \( n\)-fold convolution.

A third and final step is to notice that \( N(T) \) is a Poisson\((\lambda)\) counter and thus

\[
Y_T \sim \sum_{k=0}^\infty e^{-\lambda T} \frac{(\lambda T)^k}{k!} \left( \int_0^T \mu_s ds \right)^{*k} = \sum_{k=0}^\infty e^{-\lambda T} \frac{\lambda^k}{k!} \left( \int_0^T \mu_s ds \right)^{*k}.
\]

Using the fact that, by Taylor expansion, \( e^x \) is equivalently written as \( e^x = \sum_{k=0}^\infty \frac{x^k}{k!} \), we thus can write

\[
\pi_T = e^\lambda \left( \int_0^T \mu_s ds - T \right).
\]

Equivalently, \( \pi_t \) satisfies the differential equation

\[
d\pi_t = -\lambda \pi_t + \lambda \pi_t * \mu_t
\]

with initial condition given by the Dirac measure \( \delta_0 \) at zero.

\[\square\]

### C Proof of Theorem 3

As for Theorem 1, we proceed to prove Theorem 3 by means of a simpler example. The proof then follows by induction. We consider two dates \( t = 0, 1 \) and two types of agents \( i = 1, 2 \). Agents of type 1 receive a private signal at time \( t = 0 \) while agents of type 2 receive two private signals at \( t = 0 \). There is a proportion \( \lambda \) of agents of type 2. Both types receive a single signal at date \( t = 1 \). The remaining elements of the economy are identical to the Benchmark economy of Section 2. Because we use the same theorems and the same steps as in Appendix A, we do not repeat them. Rather, we directly expose the key elements of the solutions of the particular case treated here.

The conditional precisions for each agent type are

\[
K_{0,1} = H + S + \gamma^2 S^2 \Phi \Omega_0^2 \\
K_{1,1} = H + 2S + \gamma^2 S^2 \Phi \Omega_0^2 + \gamma^2 S^2 \Phi
\]

and

\[
K_{0,2} = H + 2S + \gamma^2 S^2 \Phi \Omega_0^2 \\
K_{1,2} = H + 3S + \gamma^2 S^2 \Phi \Omega_0^2 + \gamma^2 S^2 \Phi
\]

with \( \Omega_0 = (1 - \lambda) + 2\lambda \), that is the cross-sectional average \( 1 + \lambda \). Given these two investor categories, there exists an average \( K \):

\[
K_0 = H + S \Omega_0 + \gamma^2 S^2 \Omega_0^2 \\
K_1 = H + S (1 + \Omega_0) + \gamma^2 S^2 \Omega_0^2 + \gamma^2 S^2 \Phi.
\]

The conditional expectation for each agent type are

\[
\tilde{\mu}_{0,1} = \frac{1}{K_{0,1}} \left( S \tilde{Z}_{0,1} + \gamma^2 S^2 \Phi \tilde{Q}_0 \right) \\
\tilde{\mu}_{1,1} = \frac{1}{K_{1,1}} \left( S \tilde{Z}_{0,1} + S \tilde{Z}_{1} + \gamma^2 S^2 \Phi \tilde{Q}_0 + \gamma^2 S^2 \Phi \tilde{Q}_1 \right)
\]

32
The optimal strategy for investors of type 1 satisfies

\[ \hat{\mu}_0 = \frac{1}{K_{0,2}} \left( 2S \frac{\tilde{Z}_{0,1} + \tilde{Z}_{0,2}}{2} + \gamma^2 S^2 \Phi \Omega_0 \tilde{Q}_0 \right) \]

\[ \hat{\mu}_1 = \frac{1}{K_{1,2}} \left( 2S \frac{\tilde{Z}_{1,1} + \tilde{Z}_{1,2}}{2} + S \tilde{Z}_1 + \gamma^2 S^2 \Phi \Omega_1 \tilde{Q}_0 + \gamma^2 S^2 \Phi \tilde{Q}_1 \right) \]

with

\[ \tilde{Q}_0 = \tilde{U} - \frac{1}{\gamma S \Omega_0} \tilde{X}_0 \]
\[ \tilde{Q}_1 = \tilde{U} - \frac{1}{\gamma S \tilde{X}_1} \]

There also exists an average conditional average given by

\[ \bar{\mu}_0 = \frac{1}{K_0 - S \Omega_0} \left( \gamma^2 S^2 \Phi \Omega_0 \tilde{Q}_0 \right) \]
\[ \bar{\mu}_1 = \frac{1}{K_1 - S (1 + \Omega_1)} \left( \gamma^2 S^2 \Phi \Omega_1 \tilde{Q}_0 + \gamma^2 S^2 \Phi \tilde{Q}_1 \right) . \]

The optimal demands are given by

\[ \tilde{D}_{0,1} = \gamma \left( S \tilde{Z}_{0,1} - S \Omega_0 \tilde{U} + \frac{1}{\gamma} \tilde{X}_0 \right) - \Delta K_{0,1} \tilde{P}_0 \]
\[ \tilde{D}_{1,1} = \gamma \left( S \tilde{Z}_{1,1} + S \tilde{Z}_1 - S \Omega_0 \tilde{U} - S \tilde{U} + \frac{1}{\gamma} \left( \tilde{X}_0 + \tilde{X}_1 \right) \right) - \Delta K_{1,1} \tilde{P}_1 \]

and

\[ \tilde{D}_{0,2} = \gamma \left( 2S \frac{\tilde{Z}_{0,1} + \tilde{Z}_{0,2}}{2} - S \Omega_0 \tilde{U} + \frac{1}{\gamma} \tilde{X}_0 \right) - \Delta K_{0,2} \tilde{P}_0 \]
\[ \tilde{D}_{1,2} = \gamma \left( 2S \frac{\tilde{Z}_{1,1} + \tilde{Z}_{1,2}}{2} + S \tilde{Z}_1 - S \Omega_0 \tilde{U} - S \tilde{U} + \frac{1}{\gamma} \left( \tilde{X}_0 + \tilde{X}_1 \right) \right) - \Delta K_{1,2} \tilde{P}_1 \]

where

\[ \Delta K_{0,1} \equiv K_{0,1} - K_0 = S (1 - \Omega_0) \]
\[ \Delta K_{1,1} \equiv K_{1,1} - K_1 = \Delta K_{0,1} \]
\[ \Delta K_{0,2} \equiv K_{0,2} - K_0 = S (2 - \Omega_0) \]
\[ \Delta K_{1,2} \equiv K_{1,2} - K_1 = \Delta K_{0,2} \]

The equilibrium prices are given by

\[ \tilde{P}_0 = \frac{1}{K_0} \left[ (K_0 - S \Omega_0) \tilde{\mu}_0 + S \Omega_0 \tilde{U} - \frac{1}{\gamma} \tilde{X}_0 \right] \]
\[ \tilde{P}_1 = \frac{1}{K_1} \left[ (K_1 - S (1 + \Omega_0)) \tilde{\mu}_1 + S (1 + \Omega_0) \tilde{U} - \frac{1}{\gamma} \left( \tilde{X}_0 + \tilde{X}_1 \right) \right] \]

The optimal strategy for investors of type 1 satisfies

\[ \Delta \tilde{D}_{1,1} \equiv D_{1,1} - D_{0,1} = \gamma \left( S \left( \tilde{Z}_1 - \tilde{P}_1 \right) - S \left( \tilde{U} - \tilde{P}_1 \right) + \frac{\tilde{X}_1}{\gamma} - \Delta K_{0,1} \Delta \tilde{P}_1 \right) \]
while the optimal strategy for investors of type 2 satisfies

$$\Delta \tilde{D}_{1,2}^i \equiv D_{1,2}^i - D_{0,2}^i = \gamma \left( S \left( \tilde{Z}_i^1 - \tilde{P}_1 \right) - S \left( \tilde{U} - \tilde{P}_t \right) + \frac{\tilde{X}_1}{\gamma} - \Delta K_{0,2}^i \Delta \tilde{P}_t \right).$$

Now, proceeding by induction and denoting by $k$ the investor type, i.e. the number of private signals that the investor has, we can write

$$K_{t,k}^i \equiv \text{Var}^{-1} \left[ \tilde{U} \mid \tilde{F}_{t,k}^i \right] = H + \sum_{j=0}^{t} \left[kS + \gamma^2 S^2 \Phi \Omega_j^2 \tilde{Q}_j\right].$$

with $\Omega_0 = \sum_{k \in \mathbb{N}} \pi_0 (k) k$ and $\Omega_1 = 1$ and where $\pi$ is defined in Proposition 2. The average precision satisfies

$$K_t^i \equiv \text{Var}^{-1} \left[ \tilde{U} \mid \tilde{F}_t \right] = H + \sum_{j=0}^{t} \left[S + \gamma^2 S^2 \Phi \Omega_j^2 \tilde{Q}_j\right].$$

The individual conditional expectation of $\tilde{U}$ is

$$\mu_{t,k}^i \equiv E \left[ \tilde{U} \mid \tilde{F}_{t,k}^i \right] = \frac{1}{K_{t,k}^i} \sum_{j=0}^{t} \left(kS \tilde{Z}_{j,k}^i + \gamma^2 S^2 \Phi \Omega_j^2 \tilde{Q}_j\right)$$

where $\tilde{Z}_{j,k}^i$ denotes the average of the private signals received by agents of type $k$ and

$$\tilde{Q}_j \equiv \tilde{U} - \frac{1}{\gamma S \Omega_j} \tilde{X}_j$$

and

$$\hat{\mu}_t = \frac{1}{K_j - S \Omega_j} \sum_{j=0}^{t} \gamma^2 S^2 \Phi \Omega_j^2 \tilde{Q}_j.$$

The individual optimal demands are of the form

$$\tilde{D}_{t,k}^i = \gamma \left( \sum_{j=0}^{t} \left[kS \tilde{Z}_{j,k}^i - S \Omega_j \tilde{U} + \frac{1}{\gamma} \tilde{X}_j\right] - \Delta K_{j,k}^i \tilde{P}_j \right).$$

The price obeys

$$\tilde{P}_t = \frac{1}{K_t} \left( (K_t - S \Omega_t) \hat{\mu}_t + \sum_{j=0}^{t} \left[S \Omega_j \tilde{U} - \frac{1}{\gamma} \tilde{X}_j\right]\right)$$

and the optimal strategy for investors of type $k$ is

$$\Delta \tilde{D}_{t,k}^i \equiv \tilde{D}_{1,k}^i - \tilde{D}_{0,k}^i = \gamma \left( S \left( \tilde{Z}_1^i - \tilde{P} - 1 \right) - S \left( \tilde{U} - \tilde{P} - 1 \right) + \frac{1}{\gamma} \tilde{X}_1 - \Delta K_{0,k}^i \Delta \tilde{P}_1 \right).$$

By induction on the number of trading sessions, we obtain Theorem 3. □
D Proof of Proposition 5

Using the recursive form of the price in Theorem 3, we can write the prices difference between $t = 0$ and $t = 1$ as

$$
\tilde{P}_1 - \tilde{P}_0 = (...) + \frac{1}{\kappa_1} \left( \left( \gamma^2S^2\Phi\Omega_1^2 \left( \tilde{U} - \frac{1}{\gamma S\Phi_1} \tilde{X}_1 \right) \right) + \left( S\Omega_1 \tilde{U} - \frac{1}{\gamma} \tilde{X}_1 \right) \right)
$$

and the prices difference between $t = 1$ and $t = 2$ as

$$
\tilde{P}_2 - \tilde{P}_1 = (...) + \frac{1}{\kappa_2} \left( \left( \gamma^2S^2\Phi\Omega_2^2 \left( \tilde{U} - \frac{1}{\gamma S\Phi_2} \tilde{X}_2 \right) \right) + \left( S\Omega_2 \tilde{U} - \frac{1}{\gamma} \tilde{X}_2 \right) \right)
$$

and the prices difference between $t = 2$ and $t = 3$, we have that

$$
\tilde{P}_3 - \tilde{P}_2 = (...) + \frac{1}{\kappa_3} \left( \left( \gamma^2S^2\Phi\Omega_3^2 \left( \tilde{U} - \frac{1}{\gamma S\Phi_3} \tilde{X}_3 \right) \right) + \left( S\Omega_3 \tilde{U} - \frac{1}{\gamma} \tilde{X}_3 \right) \right)
$$

Accordingly, we obtain the following covariance

$$
cov \left( \tilde{P}_1 - \tilde{P}_0, \tilde{P}_2 - \tilde{P}_1 \right) = \left( \frac{1}{K_1} \sum_{j=0}^{1} \Omega_j (\gamma^2S^2\Phi\Omega_j + 1) - \frac{1}{K_0} \Omega_0 (\gamma^2S^2\Phi\Omega_0 + 1) \right) \frac{S^2}{H}
$$

$$
\times \left( \frac{1}{K_2} \sum_{j=0}^{2} \Omega_j (\gamma^2S^2\Phi\Omega_j + 1) - \frac{1}{K_1} \sum_{j=0}^{1} \Omega_j (\gamma^2S^2\Phi\Omega_j + 1) \right)
$$

$$
+ \frac{1}{K_2} - \frac{1}{K_1} \frac{1}{\Phi} \left( \left( \gamma S\Phi\Omega_0 + \frac{1}{\gamma} \right)^2 \left( \frac{1}{\kappa_1} - \frac{1}{\kappa_2} \right) + \frac{1}{\kappa_1} \left( \gamma S\Phi\Omega_1 + \frac{1}{\gamma} \right)^2 \right).
$$

Now, considering the price difference between $t = 2$ and $t = 3$, we have that

$$
\tilde{P}_3 - \tilde{P}_2 = (...) + \frac{1}{K_3} \left( \left( \gamma^2S^2\Phi\Omega_3^2 \left( \tilde{U} - \frac{1}{\gamma S\Phi_3} \tilde{X}_3 \right) \right) + \sum_{j=0}^{2} \Omega_j \left( \gamma^2S^2\Phi\Omega_j \left( \tilde{U} - \frac{1}{\gamma S\Phi_j} \tilde{X}_j \right) \right) \right)
$$

$$
+ \frac{1}{K_3} - \frac{1}{K_2} \left( \left( \gamma^2S^2\Phi\Omega_2^2 \left( \tilde{U} - \frac{1}{\gamma S\Phi_2} \tilde{X}_2 \right) \right) + \sum_{j=0}^{2} \Omega_j \left( \gamma^2S^2\Phi\Omega_j \left( \tilde{U} - \frac{1}{\gamma S\Phi_j} \tilde{X}_j \right) \right) \right)
$$

Accordingly, we obtain the following covariance

$$
cov \left( \tilde{P}_3 - \tilde{P}_2, \tilde{P}_2 - \tilde{P}_1 \right) = \left( \frac{1}{K_2} \sum_{j=0}^{2} \Omega_j (\gamma^2S^2\Phi\Omega_j + 1) - \frac{1}{K_1} \sum_{j=0}^{1} \Omega_j (\gamma^2S^2\Phi\Omega_j + 1) \right)
$$

$$
\times \left( \left( \frac{1}{\kappa_2} - \frac{1}{\kappa_1} \right) \sum_{j=0}^{1} \Omega_j \left( \gamma^2S^2\Phi\Omega_j + \frac{1}{\gamma} \right) \right) \frac{S^2}{H}
$$

$$
+ \frac{1}{K_3} - \frac{1}{K_2} \left( \left( \frac{1}{\kappa_3} - \frac{1}{\kappa_2} \right) \sum_{j=0}^{1} \Omega_j \left( \gamma S\Phi\Omega_j + \frac{1}{\gamma} \right) \right) \frac{1}{\Phi}.
$$

□

35
E  Information Percolation in the Rumor Case

[TO BE COMPLETED]
References


37


