Working Paper
Series

National Centre of Competence in Research
Financial Valuation and Risk Management

Working Paper No. 749

Affine Variance Swap Curve Models

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First version: March 2012
Current version: March 2012

This research has been carried out within the NCCR FINRISK project on
“Dynamic Asset Pricing”
Affine Variance Swap Curve Models*

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20 March 2012

Abstract
This paper provides a brief overview of the stochastic modeling of variance swap curves. Focus is on affine factor models. We propose a novel drift parametrization which assures that the components of the state process can be matched with any pre-specified points on the variance swap curve. This should facilitate the empirical estimation for such stochastic models. Moreover, sufficient and yet flexible conditions that guarantee positivity of the rates are readily available. We finally discuss the relation and differences to affine yield-factor models introduced by Duffie and Kan [7]. It turns out that, in contrast to variance swap models, their yield factor representation requires imposing constraints on systems of nonlinear equations that are often not solvable in closed form.

1 Variance Swaps

Variance swap rates are becoming increasingly available over-the-counter at many different maturities. It becomes vital to design and estimate stochastic term structure models for the variance swap rates, see e.g. Carr and Wu [3]. We first give a brief overview of the stochastic modeling of variance swap curves.

Let $S$ denote the price process of an underlying stock index modeled on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$. We assume that $\mathbb{Q}$ is a risk-neutral pricing measure, and that $S$ is a semimartingale of the form

$$\frac{dS_t}{S_{t-}} = r_t \, dt + \sigma_t \, dW_t + \int_{\mathbb{R}} (e^x - 1) \left( \mu(dt, dx) - \nu_t(dx) \, dt \right).$$

Here $W$ is a standard Brownian motion, and $\mu(dt, dx)$ denotes the jump measure associated to $\log S$. That is, $\Delta \log S_t = \int_{\mathbb{R}} x \mu(dt, dx)$. The $\mathbb{Q}$-compensator $\nu_t(dx) \, dt$ of $\mu(dt, dx)$ is assumed absolutely continuous. Finally, we have some

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*I thank Tom Hurd and Wolfgang Runggaldier for helpful discussions. Research support from NCCR FINRISK of the Swiss National Science Foundation is gratefully acknowledged.
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nonnegative predictable risk-free short rate \( r \) and volatility process \( \sigma \), and we assume that

\[
\int_0^T r_t \, dt < \infty, \quad \mathbb{E}_Q \left[ \int_0^T \sigma_t^2 \, dt + \int_0^T \int_{\{|x| \geq 1\}} e^{x/\nu_t} (dx) \, dt \right] < \infty
\]

for all finite \( T \). See Jacod and Shiryaev [13] for the relevant background on semimartingales.

The annualized realized variance of \( S \) over a given time horizon \([t, t+\tau]\) is defined as the annualized quadratic variation of the log returns, which equals

\[
RV(t, \tau) = \frac{1}{\tau} (\log S_{t+\tau} - \log S_t) = \frac{1}{\tau} \left( \int_t^{t+\tau} \sigma_s^2 \, ds + \int_t^{t+\tau} \int_{\mathbb{R}} x^2 \mu(ds, dx) \right).
\]

A variance swap on \( S \), initiated at \( t \) and with time to maturity \( \tau \), pays the difference

\[
RV(t, \tau) - VS(t, \tau)
\]

between the annualized realized variance and the variance swap rate, \( VS(t, \tau) \), fixed at \( t \). By convention the variance swap rate is such that the variance swap contract has value zero at inception. Assuming further that \( r \) and \( \log S \) are \( \mathcal{F}_t \)-conditionally independent under \( Q \), risk-neutral pricing then implies that

\[
VS(t, \tau) = \mathbb{E}_Q \left[ RV(t, \tau) \mid \mathcal{F}_t \right] = \frac{1}{\tau} \int_t^{t+\tau} f(t, s) \, ds
\]

where we define the \( T \)-forward variance \( f(t, T) \) prevailing at \( t \) as

\[
f(t, T) = \mathbb{E}_Q \left[ \sigma_T^2 + \int_{\mathbb{R}} x^2 \nu_T(dx) \mid \mathcal{F}_t \right].
\]

For \( t = T \) we obtain the spot variance

\[
v_t = f(t, t) = \sigma_t^2 + \int_{\mathbb{R}} x^2 \nu_t(dx).
\]

Note the analogy—and difference—to the forward curve of interest rates, see e.g. [11]. The \( T \)-forward variance \( f(t, T) \) is a martingale for \( t \in [0, T] \) under the risk-neutral measure \( Q \). In contrast, the \( T \)-forward interest rate is a martingale under the respective \( T \)-forward measure \( Q^T \).

The aim is now to model the stochastic evolution of the forward variance curve \( f(t, T) \), or equivalently, the spot variance \( v_t \). Any semimartingale price process \( S \) whose characteristics satisfy the consistency condition (1) is then compatible to this variance swap model. Related literature where this program has been carried out includes Büchler [2], Egloff et al. [8], Cont and Kokholm [6], and Filipović et al. [9].
2 The VIX Formula

Before we proceed with modeling the variance swap curve, we derive in this auxiliary section an alternative valuation formula for variance swap rates based on a replication argument. This formula underlies the computation of the VIX (Chicago Board of Options Exchange Volatility Index), which is defined as variance swap on the S&P 500 index with a time to maturity of 30 days.

We start with an elementary identity.

**Lemma 2.1.** For reals \( a, b > 0 \) we have

\[
\int_0^a \frac{1}{K^2} (K-b)^+ \, dK + \int_a^\infty \frac{1}{K^2} (b-K)^+ \, dK = \log(a) - \log(b) + \frac{1}{a} (b-a).
\]

(2)

We now assume that the futures contract on \( S \) with maturity \( T \) is traded and continuously marked to market. Moreover, in this section we assume that interest rates \( r_t \) are deterministic. (3)

We denote by \( P(t,T) = e^{-\int_t^T r_s \, ds} \) the discount bonds, and the futures price process by \( F_t = S_t/P(t,T) \). Then

\[
d \log F_t = \frac{dF_t}{F_t} - \frac{\sigma_t^2}{2} dt - \int_R (e^x - 1 - x)^\mu(dt,dx).
\]

Hence the realized variance over the time interval \([t,T]\), with time to maturity \( \tau = T-t \), can be written as

\[
\tau RV(t,\tau) = \int_t^T \sigma_s^2 \, ds + \int_t^T \int_R x^2 \mu(ds,dx)
\]

\[
= 2(\log(F_t) - \log(F_T)) + 2 \int_t^T \frac{dF_s}{F_s} - 2 \int_t^T \int_R \left(e^x - 1 - x - \frac{x^2}{2}\right) \mu(ds,dx)
\]

\[
= 2 \left( \int_0^{F_t} \frac{1}{K^2} (K - s_T)^+ \, dK + \int_{F_t}^{\infty} \frac{1}{K^2} (s_T - K)^+ \, dK \right)
\]

\[
+ 2 \int_t^T \left( \frac{1}{F'_s} - \frac{1}{F_t} \right) dF_s - 2 \int_t^T \int_R \left(e^x - 1 - x - \frac{x^2}{2}\right) \mu(ds,dx).
\]

In the last equality we used (2) and the fact that \( F_T = S_T \). We thus obtained a replication strategy for the realized variance modulo some error term due to the jumps of \( S \): holding static a basket of European options and trading dynamically in the futures contract by holding \( 2 \left( \frac{1}{F'_s} - \frac{1}{F_t} \right) \) futures at time \( s \). Taking \( \mathbb{Q} \)-expectation gives for the variance swap rate

\[
VS(t,\tau) = VIX(t) + \epsilon
\]

with the VIX rate

\[
VIX(t) = \frac{2}{\tau} \int_0^\infty \frac{\Theta_t(K,t+\tau)}{P(t,t+\tau)K^2} dK
\]
and error term

\[ \epsilon = -2 \frac{\tau}{T} \mathbb{E}_Q \left[ \int_t^{t+\tau} \int_{\mathbb{R}} \left( e^x - 1 - x - \frac{x^2}{2} \right) \nu(dx) ds \mid \mathcal{F}_t \right] \]

where \( \Theta_t(K, t + \tau) \) denotes the price an out-of-the-money European option with strike price \( K \) and maturity \( t + \tau \) (a call option when \( K > F_t \) and a put option when \( K \leq F_t \)). The error term \( \epsilon \) only appears when \( S \) has jumps. It is typically non-negative, since log-returns are negatively skewed. Hence the VIX is lower biased since it neglects price risk premium. This model-free valuation has been derived in increasing order of generality by Britten-Jones and Neuberger [1], Jiang and Tian [14], Carr and Wu [3].

### 3 Variance Swap Rate Factor Models

We now introduce a—possibly non-Markovian—multi-factor model for the variance swap term structure. We let \( X \) be a semimartingale state process with values in \( \mathbb{R}^m \), solving the stochastic differential equation

\[ dX_t = (b + \beta X_t) dt + dM_t \]

for some parameters \( b \in \mathbb{R}^m, \beta \in \mathbb{R}^{m \times m} \), and where \( M \) is a martingale, which can possibly also depend on other “unspanned stochastic volatility” factors.\(^1\)

We further assume that the spot variance is an affine function of the state variable:

\[ v_t = \phi_0 + \psi_0^T X_t \]

for some parameters \( \phi_0 \in \mathbb{R} \) and \( \psi_0 \in \mathbb{R}^m \). Under these assumptions, it follows that the term structure of annualized variance swap rates becomes affine in \( X_t \):

\[ VS(t, \tau) = \frac{1}{\tau} \int_0^\tau \mathbb{E}_Q[v_{t+s} \mid \mathcal{F}_t] ds = \frac{1}{\tau} \int_0^\tau (\phi(s) + \psi(s)^T X_t) ds \]

\[ = \frac{\Phi(\tau)}{\tau} + \frac{\Psi(\tau)}{\tau}^T X_t \]

where \( \phi \) and \( \psi \) are given by

\[ \phi(\tau) = \phi_0 + b^\top \int_0^\tau \psi(s) ds, \quad \psi(\tau) = e^{b^\top \tau} \psi_0, \]

and we denote their integrals by

\[ \Phi(\tau) = \int_0^\tau \phi(s) ds, \quad \Psi(\tau) = \int_0^\tau \psi(s) ds. \]

Indeed, for any \( T > 0 \) it is easily seen that \( N_t = \phi(T - t) + \psi(T - t)^T X_t \) is a martingale with \( N_T = v_T \). Hence \( \phi(T - t) + \psi(T - t)^T X_t = \mathbb{E}_Q[v_{T} \mid \mathcal{F}_t] \), and (6) follows for \( T = t + s \), as desired.

\[^1\]The reader is referred to Collin-Dufresne and Goldstein [5] for the definition of the concept of unspanned stochastic volatility.
Note that the constituents of the term structure, i.e. the functions \(\phi\) and \(\psi\), only depend on the drift parameters \(b\) and \(\beta\), and the spot variance parameters \(\phi_0\) and \(\psi_0\). The aim is to find a specification so that the factors \(X_t\) match pre-specified points on the variance swap curve:

\[
X_{it} = VS(t, \tau_i) \quad i = 1, \ldots, m
\]  

for some fixed maturities \(0 < \tau_1 < \cdots < \tau_m\). In view of (6) it follows that property (7) holds if and only if

\[
\frac{\Phi(\tau_i)}{\tau_i} = 0 \quad \text{and} \quad \frac{\Psi(\tau_i)}{\tau_i} = e_i, \quad i = 1, \ldots, m,
\]

where \(e_i\) denotes the \(i\)th standard basis vector in \(\mathbb{R}^m\). Following Duffie and Kan’s [7] terminology for yield curve models, we call a factor model satisfying (6) and (7) an affine variance swap rate factor model.

We shall now derive conditions on \(b\), \(\beta\), \(\phi_0\), and \(\psi_0\) for (8) to hold under the standing assumption that

\[
\beta \text{ has } m \text{ distinct eigenvalues } \lambda_1 < \cdots < \lambda_m.
\]

Assumption (9) implies that \(\beta\) is diagonalizable, see e.g. [12, Theorem 1.3.9]. That is, there exists an invertible real \(m \times m\)-matrix \(S\) whose columns are linearly independent eigenvectors of \(\beta\), and such that

\[
\beta = SLS^{-1}
\]

for the diagonal matrix \(L = \text{diag}(\lambda_1, \ldots, \lambda_m)\). We denote by \(1 = (1, \ldots, 1)^\top\) the vector with all entries equal to 1, and we shall follow the convention

\[
\frac{1}{s} (e^s - 1) = 1 \quad \text{if } s = 0.
\]

Here is our main result.

**Theorem 3.1.** For any choice of real parameters \(\ell\) and \(\lambda_1 < \cdots < \lambda_m\) there exists a unique set of parameters \(\phi_0 \in \mathbb{R}\), \(\psi_0 \in \mathbb{R}^m\), \(b \in \mathbb{R}^m\), and \(\beta \in \mathbb{R}^{m \times m}\) in the class of (9) such that the matching condition (7) holds. More specifically, they are explicitly given by

\[
\phi_0 = \ell (1 - \psi_0^\top 1)
\]

\[
\psi_0 = (S^{-1})^\top 1
\]

\[
b = -\ell \beta 1
\]

\[
\beta = SLS^{-1}
\]

\[\text{Here we make the mild assumption that the support of the random variable } X_t \text{ contains an open set in } \mathbb{R}^m.\]
for the diagonal matrix of eigenvalues \( L = \text{diag}(\lambda_1, \ldots, \lambda_m) \), and where

\[
S = (w_1 | \cdots | w_m)^\top
\]

with \( w_i \) defined as

\[
w_i = (L \tau_i)^{-1} (e^{L \tau_i} - I) \mathbf{1}, \quad i = 1, \ldots, m.
\]

In particular, the theorem also holds in the boundary case \( \tau_1 = 0 \), where \( X_{1t} = V S(t, 0) = v_t \) is the spot variance, with \( \phi_0 = 0 \) and \( \psi_0 = e_1 \).

It is reasonable to restrict to non-positive eigenvalues since then, and only then, the variance swap curve (6) is bounded as a function of \( \tau \in [0, \infty) \).

Proof. To simplify notation, we shall write \( B = \beta^\top = S L S^{-1} \) with \( S = (S^{-1})^\top \) in what follows. We shall first assume that \( B \) is invertible, that is, \( \lambda_i \neq 0 \) for all \( i \), and that \( \tau_1 > 0 \). We then obtain the explicit expressions

\[
\psi(\tau) = e^{B \tau} \psi_0 = S e^{L \tau} S^{-1} \psi_0
\]

\[
\phi(\tau) = \phi_0 + b^\top \Psi(\tau)
\]

\[
\Psi(\tau) = B^{-1} (e^{B \tau} - I) \psi_0 = S L^{-1} (e^{L \tau} - I) S^{-1} \psi_0
\]

\[
\Phi(\tau) = \tau \phi_0 + b^\top B^{-1} (\Psi(\tau) - \tau \psi_0).
\]

Property (8) is thus equivalent to the following two conditions:

\[
\phi_0 + b^\top B^{-1} (e_i - \psi_0) = 0 \quad (17)
\]

\[
S (L \tau_i)^{-1} (e^{L \tau_i} - I) S^{-1} \psi_0 = e_i \quad (18)
\]

for all \( i = 1, \ldots, m \). We first discuss (17). Thereto define \( z = \beta^{-1} b \). Then (17) reads

\[
(\psi_0 - e_i)^\top z = \phi_0, \quad i = 1, \ldots, m.
\]

This can hold if and only if all components \( z_i = e_i^\top z = \psi_0^\top z - \phi_0 \) of \( z \) are identical. Hence (17) holds if and only if \( z = -\ell \mathbf{1} \) for some \( \ell \in \mathbb{R} \) such that \( \ell (1 - \psi_0^\top \mathbf{1}) = \phi_0 \). We conclude that condition (17) holds if and only if \( b = -\ell \beta \mathbf{1} \) and \( \phi_0 = \ell (1 - \psi_0^\top \mathbf{1}) \) for some \( \ell \in \mathbb{R} \), which proves (11) and (13).

Next we denote the linearly independent column vectors of \( S^{-1} \) by \( w_i = S^{-1} e_i \). Then (18) can be rewritten as

\[
(e^{L \tau_i} - I)^{-1} (L \tau_i) w_i \equiv S^{-1} \psi_0 = \sum_{j=1}^m \psi_{0j} w_j, \quad i = 1, \ldots, m.
\]

We claim that all components of the vector \( S^{-1} \psi_0 \) must be nonzero. Indeed, suppose the \( k \)-th component of \( S^{-1} \psi_0 \) were zero. Then (19) implies that \( w_{ik} = 0 \) for all \( i \). But then \( S^{-1} \) cannot be invertible, which is absurd. Now suppose there are some linearly independent vectors \( w_i \) which satisfy (19), and let \( D \) be any invertible diagonal matrix. It then follows by inspection that (19) also holds for
Consequently, after some appropriate transformation if necessary, we can assume that $S^{-1}\psi_0 = 1$ without loss of generality. We thus have shown that (18) holds if and only if $w_i$ is of the form (16) and $\psi_0 = S \mathbf{1}$, where $S^{-1} = (w_1 \mid \cdots \mid w_m)$, which is (12) and (15). Notice that, in view of assumption (9), the vectors $w_i$ defined by (16) are indeed linearly independent.

Moreover, we have shown that, for any choice of eigenvalues $\lambda_1 < \cdots < \lambda_m$, the parameters $B = SLS^{-1}$ and $\psi_0 = S \mathbf{1}$ are uniquely determined by (18). Any choice of $\ell$ in turn then uniquely determines $\phi_0$ and $b$ by (17).

Thus the theorem is proved under the premise of nonzero eigenvalues $\lambda_i$ and $\tau_1 > 0$. However, with the convention (10) it becomes obvious that $\phi_0$, $\psi_0$, $b$, and $\beta$, and thus $\Phi(\tau_1)/\tau_1$ and $\Psi(\tau_1)/\tau_1$, are jointly continuous in $\lambda_1 < \cdots < \lambda_m$ and $\tau_1 < \cdots < \tau_m$. Thus (8) is also satisfied if $\lambda_i = 0$ for some $i$ or $\tau_1 = 0$.

Note that $\tau_1 = 0$ implies $w_1 = 1$ and thus $\psi_0 = e_1$ by linear independence of $w_i$, see also (19).

4 Nonnegative Variance Swap Rates

Theorem 3.1 does not assert nonnegativity of the implied variance swap curve (6). Negative variance swap rates are clearly non-desirable. In this section we give sufficient conditions on the specification (11)–(16) to produce nonnegative swap rates. We first observe that the variance swap rates $VS(t, \tau)$ remain non-negative for all $t$ and $\tau$ if and only if the spot variance $v_t$ is nonnegative for all $t$. Moreover, from (11) and (12) it follows that

$$v_t = \phi_0 + \psi_0^\top X_t = \mathbf{1}^\top S^{-1} (X_t - \ell \mathbf{1}) + \ell.$$  \hspace{1cm} (20)

On the other hand, in view of (13) and (14), the stochastic differential equation for $X$ reads

$$dX_t = (b + \beta X_t) dt + dM_t = SLS^{-1} (X_t - \ell \mathbf{1}) dt + dM_t.$$  \hspace{1cm} (21)

This suggests to look at the transformed state process

$$Z_t = S^{-1} (X_t - \ell \mathbf{1}) + \ell \theta,$$

for some fixed $\theta \in [0, 1]^m$ such that $\mathbf{1}^\top \theta = 1$. It satisfies the stochastic differential equation

$$dZ_t = S^{-1} dX_t = -L (\ell \theta - Z_t) dt + S^{-1} dM_t.$$  \hspace{1cm} (21)

Note that the drift of $Z_t$ is fully decoupled. It is relatively easy to provide sufficient conditions on $\ell$ and the martingale part $M_t$ such that $Z_t$ takes values in the positive orthant $\mathbb{R}_+^m$. This in turn implies nonnegative spot variance

$$v_t = \mathbf{1}^\top Z_t.$$  \hspace{1cm} (21)

The recipe for constructing observable nonnegative variance swap term structure models now reads as follows:

\footnote{We note however that the eigenvalues $\lambda_i$ and the maturities $\tau_i$ must be mutually distinct, respectively.}
(i) Fix some $\ell \geq 0$ and $m$ distinct eigenvalues $\lambda_1 < \cdots < \lambda_m \leq 0$. Define $L$ and $S$ as in Theorem 3.1.

(ii) Let $Z_t$ be a jump-diffusion process with state space $\mathbb{R}_+^m$ of the form

$$dZ_t = -L(\ell \theta - Z_t)\,dt + dN_t$$

for some appropriate martingale part $N_t$. For example

$$dN_t = \Sigma(Z_t)\,dW_t$$

where $W_t$ is a $m$-dimensional Brownian motion, and $\Sigma(z) \in \mathbb{R}^{m \times m}$ is an appropriate dispersion function such that for any boundary point $z \in \partial \mathbb{R}_+^m$ the orthogonal diffusion components vanish:

$$\Sigma(z)^T \mathbf{e}_i = 0 \quad \text{if} \quad z_i = 0. \quad (22)$$

(iii) Then $v_t = \mathbf{1}^T Z_t$ defines a nonnegative variance swap process with matched points on the curve

$$X_{it} = \text{VS}(t, \tau_i)$$

for the transformed state process $X_t = S(Z_t - \ell \theta) + \ell \mathbf{1}$. The jump-diffusion $X_t$ a fortiori takes values in $\mathbb{R}_+^m$ and is characterized by its stochastic differential equation (21) with martingale part $dM_t = SdN_t$. The spot variance can be expressed in terms of $X_t$ by (20). For the above diffusion example we would have

$$dM_t = S\Sigma(S^{-1}(X_t - \ell \mathbf{1}) + \ell \theta)\,dW_t$$

where $\Sigma$ is any dispersion function satisfying the invariance condition (22). One can introduce additional parameters $\pi$ by letting $\Sigma(\cdot) = \Sigma(\pi, \cdot)$.

A more detailed analysis and empirical study of matching variance swap rate factor models is given in [10].

5 Comparison to Affine Yield-Factor Models

In their seminal paper, Duffie and Kan [7] introduce the class of affine factor models for the term structure of interest rates. Generically, such an affine term structure model is first written in terms of some latent diffusion state vector $X_t$.

That is, the yield curve is an affine function of $X_t$,

$$y(t, \tau) = \frac{A(\tau)}{\tau} + \frac{B(\tau)^T}{\tau} X_t,$$

for some deterministic functions $A(\tau)$ and $B(\tau)$, which are given as solutions to some system of non-linear (Riccati) ODEs determined by the characteristics of $X$. The diffusion $X$ in turn is necessarily an affine process, meaning that its
drift and diffusion matrix are affine functions in the state $X_t$. Duffie and Kan [7] then also impose matching conditions similar to (7), such that

$$y(t, \tau_i) = X_{it}, \quad i = 1, \ldots, m$$

(24)

for some fixed maturities $0 \leq \tau_1 < \cdots < \tau_m$. As above, this is equivalent\(^4\) to

$$\frac{A(\tau_i)}{\tau_i} = 0 \quad \text{and} \quad \frac{B(\tau_i)}{\tau_i} = e_i, \quad i = 1, \ldots, m.$$  

Since here $A$ and $B$ solve non-linear ODEs, it is much more difficult to find an a priori parametrization, such as in Theorem 3.1 for the variance swap curves, that are consistent with the matching condition (24). Indeed, Duffie and Kan [7] state that they “do not have theoretical results describing how certain coefficients can be fixed in advance so as to achieve consistency with [the matching condition (24)].” Instead, they propose the practical but indirect solution of first specifying an arbitrary affine factor model (23) with latent state vector $X_t$. In a second step they change the variables via the affine transformation

$$Y_{it} = \frac{A(\tau_i)}{\tau_i} + \frac{B(\tau_i)^T}{\tau_i} X_t.$$  

Provided the $m \times m$-matrix $K$ with $i$th row vector given by $B(\tau_i)^T/\tau_i$ is non-singular, this change of variables is possible. The process $Y$ is an affine diffusion and the yield curve becomes affine in $Y_t$ of the form

$$y(t, \tau) = \frac{A^*(\tau)}{\tau} + \frac{B^*(\tau)^T}{\tau} Y_t,$$

for $A^*(\tau) = A(\tau) + B(\tau)^T K^{-1} k$ and $B^*(\tau)^T = B(\tau)^T K^{-1}$, where $k_i = A(\tau_i)/\tau_i$. By construction, the matching condition (24) holds in these new coordinates: $y(t, \tau_i) = Y_{it}$. However, the characteristics of $Y$ are now given in terms of $K$, which again depends via a solution of the non-linear Riccati equations on the original parameters of $X$. Apart from simple two-factor models, this approach is often difficult to implement and therefore has not been widely used, see also Collin-Dufresne et al. [4].

References


\(^4\)Under the mild assumption that the support of the random variable $X_t$ contains an open set in $\mathbb{R}^m$. 

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