No-Arbitrage in a Numéraire Independent Modelling Framework

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Abstract

Using the numéraire independent modelling framework developed in Herdegen (2012), we define a numéraire independent notion of no-arbitrage via superreplication prices, called NGE, which is equivalent to the BK or NUPBR notions. Moreover, we provide an interpretation of this and the classical NA and NFLVR notions in terms of maximal strategies which are optimal investments in a suitable sense. We proceed to show that in NGE markets, there exist “sufficiently many” numéraire representatives, which satisfy NFLVR and therefore admit equivalent σ-martingale measures. The latter result is a numéraire independent extension of the celebrated fundamental theorem of asset pricing of Delbaen & Schachermayer (1994, 1998).


JEL classification: G10, C00, G15

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1 Introduction

When describing the evolution of a financial market, the usual approach consists in fixing one currency and denominating all traded quantities in terms of the latter. Trading is then described in terms of admissible strategies. But since the notion of admissible strategies depends on the chosen currency or – in terms of mathematical finance – the numéraire, this implies that standard notions of no-arbitrage, including the no free lunch with vanishing risk (NFLVR) condition, depend on the chosen numéraire as well. This insight is of course not new and has been studied in Delbaen & Schachermayer (1995) under the assumption that the market satisfies NFLVR in units of the original numéraire. What has not been studied so far, however, is the somehow more general question under which condition on the market there exists a numéraire in which units the market satisfies NFLVR, and if so, how one can find such a “good” numéraire. Intimately linked with the

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latter is the question of how to define a notion of “no-arbitrage” which is a property of the whole market and not of some arbitrarily chosen numéraire.

The following motivating example shows that even if a market fails to satisfy NFLVR in units of each asset, it may still be possible to find a “good” numéraire in which units the market does satisfy NFLVR.

**Example 1.1.** Let \( W = (W_t)_{t \in [0, 1]} \) be a Brownian motion on some filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq 1}, \mathbb{P})\). Define the process \( X = (X_t)_{t \in [0, 1]} \) by

\[
X_t := \begin{cases}
\exp \left( \int_0^t \frac{1}{2} \, dW_s - \frac{t}{2} \right) & \text{if } t < 1, \\
\frac{1}{2} & \text{if } t = 1,
\end{cases}
\]

and the stopping time \( \tau \) by

\[
\tau := \inf \left\{ t \in [0, 1] : X_t = \frac{1}{2} \right\}.
\]

Then it is a standard exercise in stochastic analysis to check that \( X \) is a nonnegative continuous local \( \mathbb{P} \)-martingale, \( \tau < 1 \) \( \mathbb{P} \)-a.s. and \( X_\tau = 1/2 \) \( \mathbb{P} \)-a.s.

Let the evolution of two assets \((S^0, S^1) = (S^0_t, S^1_t)_{t \in [0, 1]}\), in units of \( S^0 \), be given by

\[
S^0_t := 1 \quad \text{and} \quad S^1_t := \begin{cases}
\frac{1}{2} + \frac{1}{2} X & \text{on } [0, \tau], \\
1 + X & \text{on } (\tau, 1].
\end{cases}
\]

Then \( S^1 \) is a strictly positive continuous semimartingale with \( S^0_0 = 1, S^1_0 = 3/2 \) and \( S^1_1 = 1 \).

This “weird” behaviour of \( S^1 \) suggest that \((S^0, S^1)\) is not free of arbitrage. Indeed, the strategy \( \vartheta = (-1, 1) \mathbb{1}_{[0, \tau]} + (1/2, 0) \mathbb{1}_{(\tau, 1]} \) is admissible and satisfies

\[
V_0(\vartheta)(S) = \vartheta_0 \cdot S_0 = 0 \quad \text{and} \quad V_1(\vartheta)(S) = \vartheta_1 \cdot S_1 = \vartheta_0 \cdot S_0 + \vartheta \cdot S_1 = 1/2 > 0 \quad \mathbb{P} \text{-a.s.}
\]

Next, one could ask if, in units of \( S^1 \), the market is free of arbitrage. To this end set

\[
\tilde{S}^0 := \frac{S^0}{S^1} = \begin{cases}
\frac{2X}{1 + X} & \text{on } [0, \tau], \\
\frac{1}{1 + X} & \text{on } (\tau, 1].
\end{cases}
\]

Then \( \tilde{S}^0 \) is a strictly positive continuous semimartingale with \( \tilde{S}^0_0 = 1, \tilde{S}^0_\tau = 3/2 \) and \( \tilde{S}^0_1 = 1 \).

The “weird” behaviour of \( \tilde{S}^0 \) suggest that \((\tilde{S}^0, \tilde{S}^1)\) is not free of arbitrage. Indeed, the strategy \( \tilde{\vartheta} = (3/2, -1) \mathbb{1}_{(\tau, 1]} \) is admissible and satisfies

\[
V_0(\tilde{\vartheta})(\tilde{S}) = \tilde{\vartheta}_0 \cdot \tilde{S}_0 = 0 \quad \text{and} \quad V_1(\tilde{\vartheta})(\tilde{S}) = \tilde{\vartheta}_1 \cdot \tilde{S}_1 = \tilde{\vartheta}_0 \cdot \tilde{S}_0 + \tilde{\vartheta} \cdot \tilde{S}_1 = 1/2 > 0 \quad \mathbb{P} \text{-a.s.}
\]

Finally, consider the strategy \( \eta = (-1, 2) \mathbb{1}_{[0, \tau]} + (2, 0) \mathbb{1}_{(\tau, 1]} \). Its value process \( V(\eta)(S) \) satisfies

\[
V(\eta)(S) = \eta \cdot S = \eta_0 \cdot S_0 + \eta \cdot S_1 = \frac{1}{X} > 0 \quad \mathbb{P} \text{-a.s.}
\]

We show that, in units of \( V(\eta)(S) \), the market is free of arbitrage. To this end set

\[
\tilde{S}^0 := \frac{S^0}{V(\eta)(S)} = X^\tau \quad \mathbb{P} \text{-a.s.} \quad \text{and} \quad \tilde{S}^1 := \frac{S^1}{V(\eta)(S)} = \frac{1}{2} + \frac{1}{2} X \quad \mathbb{P} \text{-a.s.}
\]

Since both \( \tilde{S}^0 \) and \( \tilde{S}^1 \) are local \( \mathbb{P} \)-martingales, the (easy direction) of the fundamental theorem of asset pricing by Delbaen & Schachermayer (1994), shows that \((\tilde{S}^0, \tilde{S}^1)\) satisfies NFLVR.

\[2\]
To tackle the above questions systematically, we use the *numéraire independent modelling framework* developed in Herdegen (2012). First, we introduce a numéraire independent notion of no-arbitrage called *no gratis events* (NGE) defined via (limit quantile) superreplication prices and study its fundamental properties. Next, we consider the question of arbitrage from a different point of view by looking at different kinds of *maximal strategies*, which are optimal investments in a suitable sense. We show that a (numéraire) market satisfies NGE (for undefaultable strategies) if and only if the zero strategy is strongly maximal (for undefaultable strategies), which is in turn equivalent to the BK or NUPBR property (for some fixed numéraire representative) considered in Kabanov (1997) or Karatzas & Kardaras (2007), respectively. Moreover, we interpret the classical notions NA and NFLVR in terms of maximal strategies, showing thereby why and in which respect they are *not* numéraire independent.

The first key result of this paper (Theorem 6.3) shows that if the zero strategy is strongly maximal (for undefaultable strategies), then for every undefaultable strategy \( \vartheta \), there exists a strategy \( \vartheta^* \) which is strongly maximal (for undefaultable strategies) and dominates \( \vartheta \) in an appropriate sense. The other key result of this paper (Theorem 7.10) is a numéraire independent extension of the celebrated *fundamental theorem of asset pricing* by Delbaen and Schachermayer. More, precisely we show that if a (numéraire) market satisfies NGE (for undefaultable strategies), then there exist “sufficiently many” numéraire representatives which satisfy NFLVR and therefore admit equivalent \( \sigma \)-martingale measures. This last result has independently (and in a slightly weaker form) been obtained by Kardaras (2012) and Takaoka (2012).

The rest of the paper is organised as follows. The remainder of Section 1 fixes the probabilistic setup and notation. Section 2 recalls from Herdegen (2012) the main concepts and results of the numéraire independent modelling framework to make the paper as self-contained as possible. Section 3 introduces the no-arbitrage notion NGE and studies its fundamental properties. Next, we consider the question of arbitrage from a different point of view by looking at different kinds of no-arbitrage called NGE markets, proves a numéraire independent version of the fundamental theorem of asset pricing and studies dual characterisations of superreplication prices as well as some other topics in the setup of the numéraire independent modelling framework. Finally, Section 8 compares the numéraire independent approach of modelling financial markets and studying no-arbitrage to the standard and other recent approaches to these issues.

1.1 Probabilistic setup and notation

Throughout this paper, we work on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) satisfying the usual conditions of right-continuity and completeness, where \(T > 0\) denotes the finite time-horizon. We assume that \(\mathcal{F}_0\) is \(\mathbb{P}\)-trivial.

We denote the collection of all \(\mathbb{P}\)-null sets (in \(\mathcal{F}_T\)) by \(\mathcal{N}\) and the set of all stopping times taking values in \([0, T]\) by \(\mathcal{T}_{[0,T]}\). Note that we also consider stopping times with values in \([0, T] \cup \{+\infty\}\). For the definition of hitting times etc., we agree that \(\inf \emptyset = +\infty\).

For an \(\mathbb{R}^d\)-valued semimartingale \(X = (X^1, \ldots, X^d)\), we denote by \(L(X)\) the set of all \(\mathbb{R}^d\)-valued *predictable* processes \(\zeta = (\zeta^1, \ldots, \zeta^d)\) that are integrable with respect to \(X\) in the sense of \(d\)-dimensional stochastic integration (consult Jacod & Shiryaev (2003) for details). For \(\zeta \in L(X)\) and \(0 \leq t \leq T\), we write \((\zeta \bullet X)_t\) for the stochastic integral \(\int_{[0,t]} \zeta_u dX_u\) and \(\zeta_t \cdot X_t\) for the inner product \(\sum_{i=1}^d \zeta^i_t X^i_t\).

We set \(\mathbb{R}_+ := [0, \infty)\) and \(\mathbb{R}_{++} = (0, \infty)\). Similarly, for \(\tau \in \mathcal{T}_{[0,T]}\), denote by \(\mathbf{L}_0^\tau(\mathcal{F}_\tau)\) the set of all nonnegative \(\mathcal{F}_\tau\)-measurable random variables and by \(\mathbf{L}^\tau_{+\infty}(\mathcal{F}_\tau)\) the subset of all \(g \in \mathbf{L}_0^\tau(\mathcal{F}_\tau)\) that satisfy \(\mathbb{P}[g > 0] = 1\). We identify \(\mathbf{L}_0^\tau(\mathcal{F}_0)\) with \(\mathbb{R}_+\) and \(\mathbf{L}^\tau_{+\infty}(\mathcal{F}_0)\) with \(\mathbb{R}_{++}\).
2 Summary of numéraire independent modelling

In this section we briefly recall the main definitions and results of the numéraire independent modelling framework introduced in Herdegen (2012). For motivations, proofs, examples and further comments, we refer to the latter paper.

2.1 Exchange rate processes and markets

Definition 2.1. A real-valued semimartingale $D$ satisfying $\inf_{0 \leq t \leq T} D_t > 0$ $\mathbb{P}$-a.s. is called an exchange rate process.

The set of all exchange rate processes forms an abelian group with respect to pointwise multiplication.

Definition 2.2. A nonempty set $S$ of $\mathbb{R}^d$-valued semimartingales is called a $(d$-dimensional) market if for all $S \in S$ and all exchange rate processes $D$, we have $DS \in S$, and if some (and hence all) $S \in S$ satisfy

$$\mathbb{P} \left[ \inf_{0 \leq t \leq T} \sum_{i=1}^d |S^i_t| > 0 \right] = 1. \quad (2.1)$$

Each $S \in S$ is called a representative of $S$. Moreover, $S$ is called nonnegative if some (and hence all) $S \in S$ are $\mathbb{P}$-a.s. (componentwise) nonnegative.

Remark 2.3.

(a) If $S$ is any $\mathbb{R}^d$-valued semimartingale satisfying (2.1), then there exists a unique market $S$ such that $S$ is a representative of $S$. We call $S$ the market corresponding to $S$.

(b) If $S$ is a market and $S, \tilde{S} \in S$, then there exists a $\mathbb{P}$-a.s. unique exchange rate process $D$ satisfying

$$\tilde{S} = DS \ \mathbb{P}\text{-a.s.} \quad (2.2)$$

2.2 Positions, investments and self-financing strategies

Definition 2.4. Let $S$ be a $d$-dimensional market. Any $d$-dimensional progressive process $\varphi$ is called a position process for the market $S$, and for each $S \in S$, the progressive process $V(\varphi)(S)$ defined by

$$V_t(\varphi)(S) := \varphi_t \cdot S_t, \quad t \in [0,T],$$

is called the position value process of $\varphi$ in the currency unit determined by $S$.

Formally, $V(\varphi)$ is a map from $S$ to the space of progressive processes, which for all $S \in S$ and all exchange rate processes $D$ satisfies the exchange rate consistency condition

$$V(\varphi)(DS) = DV(\varphi)(S) \ \mathbb{P}\text{-a.s.} \quad (2.3)$$

Remark 2.5. For notational convenience, we often write $V(\varphi) \geq 0 \ \mathbb{P}\text{-a.s.}$, $\mathbb{P}[V_\tau(\varphi) > 0] > 0$, etc. as a shorthand for $V(\varphi)(S) \geq 0 \ \mathbb{P}\text{-a.s.}$ for all $S \in S$, or $\mathbb{P}[V_\tau(\varphi)(S) > 0] > 0$ for all $S \in S$, etc. We use this notation only if it is well defined in the sense that the validity of the expression for some $S \in S$ implies its validity for all $S \in S$. This is for instance not the case in expressions like $\mathbb{P}[V_\tau(\varphi)(S) > c]$ for $c \neq 0$.

Definition 2.6. Let $S$ be a market and $\zeta$ a predictable position process for $S$. For fixed $S \in S$, $\zeta$ is called an investment process for the representative $S$ if $\zeta \in L(S)$, and the adapted process $\tilde{V}(\zeta)(S)$ defined by

$$\tilde{V}_t(\zeta)(S) = \zeta_0 \cdot S_0 + \zeta \cdot S_t, \quad t \in [0,T],$$

is called the investment value process of $\zeta$ in the currency unit determined by $S$. 

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Unlike position value processes, the investment value process $\tilde{V}(\zeta)(S)$ is in general only defined for some and not for all $S \in S$.

**Definition 2.7.** Let $S$ be a market and $\vartheta$ a position process for $S$ which is an investment process for all $S \in S$. Then $\vartheta$ is called a self-financing strategy for $S$ if for all $S \in S$, $\vartheta \cdot S = V(\vartheta)(S) = \tilde{V}(\vartheta)(S) = \vartheta_0 \cdot S_0 + \vartheta \cdot S \ \mathbb{P}\text{-a.s.}$ (2.4)

We denote the vector space of all self-financing strategies for $S$ by $L^{sf}(S)$.

It can be shown that the self-financing condition (2.4) holds for all $S \in S$ if and only if it holds for some $S \in S$.

Since the position value process $V(\vartheta)$ and the investment value process $\tilde{V}(\vartheta)$ coincide for any self-financing strategy $\vartheta$ and we only work with these (unless otherwise stated) in the sequel, we can and do call both processes simply the value process of $\vartheta$ and denote this by $V(\vartheta)$.

### 2.3 Numéraire strategies and undefaultable strategies

**Definition 2.8.** Let $S$ be a market. A self-financing strategy $\eta \in L^{sf}(S)$ is called a numéraire strategy for $S$ if $V(\eta)(S)$ is an exchange rate process for some (and hence all) $S \in S$. If such an $\eta$ exists, $S$ is called a numéraire market.

**Proposition 2.9.** Let $S$ be numéraire market and $\eta \in L^{sf}(S)$ a numéraire strategy. Then there exists a $\mathbb{P}$-a.s. unique representative $S^{(\eta)} \in S$ satisfying 

$$V(\eta)(S^{(\eta)}) \equiv 1 \ \mathbb{P}\text{-a.s.}$$

Moreover, for all $S \in S$,

$$S^{(\eta)} = \frac{S}{V(\eta)(S)} \ \mathbb{P}\text{-a.s.}$$ (2.5)

We call $S^{(\eta)} \in S$ the numéraire representative of $\eta$.

**Definition 2.10.** A market $S$ is called a bounded numéraire market if there exists a bounded numéraire strategy with a bounded numéraire representative.

A nonnegative market $S$ is a numéraire market if and only if the bounded (buy-and-hold) strategy $\eta := (1, \ldots, 1)$ is a numéraire strategy, in which case $S$ is a bounded numéraire market.

**Definition 2.11.** A strategy cone for the market $S$ is a nonempty convex cone $\Gamma \subset L^{sf}(S)$ containing the zero strategy. If $\Gamma$ is a strategy cone, we denote by bf$\Gamma$ the strategy cone of all bounded strategies in $\Gamma$.

**Definition 2.12.** Let $S$ be a market. We denote by $\mathcal{U}(S)$ the strategy cone of all $\vartheta \in L^{sf}(S)$ satisfying

$$V(\vartheta) \geq 0 \ \mathbb{P}\text{-a.s.},$$

and call $\vartheta \in \mathcal{U}(S)$ an undefaultable strategy.

### 2.4 Admissible investment processes

**Definition 2.13.** Let $S$ be a numéraire market and $\eta$ a numéraire strategy. An investment process $\zeta$ for $S^{(\eta)}$ with $\zeta_0 = 0$ is called an admissible investment process given $S^{(\eta)}$ if there exists $a \geq 0$ such that

$$\zeta \bullet S^{(\eta)} \geq -a \ \mathbb{P}\text{-a.s.}$$ (2.6)

We denote the set of all admissible investment processes given $S^{(\eta)}$ by $L^{ad}(S^{(\eta)})$. Moreover, if $\zeta \in L^{ad}(S^{(\eta)})$ and $a \geq 0$ satisfy (2.6), we say that $\zeta$ is an $a$-admissible investment processes given $S^{(\eta)}$ and write $\zeta \in L^{ad}(S^{(\eta)}, a)$. 

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Remark 2.14. The notion of admissible investment processes is a good generalisation of the concept of admissible strategies in the standard framework. Indeed, let \( S = (S^1, \ldots, S^d) \) be a \( d \)-dimensional semimartingale, which is interpreted as the evolution of \( d \) “risky” discounted assets. A \( d \)-dimensional predictable process \( \vartheta \in L(S) \) with \( \vartheta_0 = 0 \) is called an admissible strategy in the standard literature if there exists \( a \geq 0 \) such that \( \vartheta \cdot S \geq -a \) \( \mathbb{P} \)-a.s. (confer for example Delbaen & Schachermayer (2006), Section 8.1). From a purely mathematical point of view, if \( S^{(0)} \) is a numéraire representative and \( \zeta \) an admissible investment process given \( S^{(0)} \), then \( \zeta \) is an admissible strategy in the standard framework with \( d \) discounted risky assets described by \( S^{(0)} \). Therefore all mathematical results for admissible strategies carry over without further ado to admissible investment processes given a numéraire representative \( S^{(0)} \). Nevertheless, from an economic point of view, there is a fundamental difference between the two concepts: In the standard framework, there is a “riskless” traded asset \( S^0 \equiv 1 \) in the background, which enables one to extend every admissible strategy \( \vartheta \) to a \( (d+1) \)-dimensional self-financing strategy \( (\vartheta_0, \vartheta) \). In the case of admissible investment processes given a numéraire representative \( S^{(0)} \), however, there does not exist a traded asset \( S^{(0),0} \equiv 1 \) in the background, and so admissible investment processes cannot be directly identified with self-financing strategies.

Admissible investment processes and undefaultable strategies are closely linked.

Proposition 2.15. Let \( S \) be a numéraire market and \( \eta \) a numéraire strategy. Then for each \( \zeta \in L^{ad}(S^{(0)}) \), there exists \( \vartheta \in L^f(S) \) such that
\[
V(\vartheta)(S^{(n)}) = \vartheta \cdot S^{(n)} = \zeta \cdot S^{(n)} \quad \mathbb{P} \text{-a.s.} \tag{2.7}
\]
Moreover, for all \( a \geq 0 \) such that \( \zeta \in L^{ad}(S^{(0)}, a) \), there exists \( \tilde{\vartheta} \in \mathcal{U}(S) \) such that \( \vartheta = \tilde{\vartheta} - a \eta \).

Corollary 2.16. Let \( S \) be a numéraire market, \( \eta \) a numéraire strategy, \( \zeta \in L^{ad}(S^{(n)}) \) and \( v_0 \geq 0 \). Suppose that \( v_0 + \zeta \cdot S^{(n)} \geq 0 \) \( \mathbb{P} \)-a.s. Then there exists \( \vartheta \in \mathcal{U}(S) \) satisfying
\[
V(\vartheta)(S^{(n)}) = \vartheta_0 \cdot S^{(n)} + \vartheta \cdot S^{(n)} = v_0 + \zeta \cdot S^{(n)} \quad \mathbb{P} \text{-a.s.}
\]

2.5 Contingent claims

Definition 2.17. A contingent claim at time \( \tau \in \mathcal{T}_{[0,T]} \) for the market \( S \) is a map \( F : S \to L^0_{+} (\mathcal{F}_\tau) \) satisfying the exchange rate consistency condition
\[
F(DS) = D_{\tau} F(S) \quad \mathbb{P} \text{-a.s.} \tag{2.8}
\]
for all \( S \in S \) and all exchange rate processes \( D \). It is called strictly positive if \( F(S) \in L^0_{+} (\mathcal{F}_\tau) \) for all \( S \in S \).

The set of contingent claims at time \( \tau \) forms a convex cone containing 0.

Remark 2.18. For a contingent claim \( F : S \to L^0_{+} (\mathcal{F}_\tau) \), we often write as in the case of value processes (see Remark 2.5) \( F \geq 0 \) \( \mathbb{P} \)-a.s. or \( \mathbb{P}[F > 0] > 0 \), etc. as a shorthand for \( F(S) \geq 0 \) \( \mathbb{P} \)-a.s. for all \( S \in S \), or \( \mathbb{P}[F(S) > 0] > 0 \) for all \( S \in S \), etc.

Proposition 2.19. Let \( S \) be a market and \( \tau \in \mathcal{T}_{[0,T]} \) a stopping time. Then for any pair \((S, g)\), where \( S \in S \) and \( g \in L^0_{+} (\mathcal{F}_\tau) \), there exists a \( \mathbb{P} \)-a.s. unique contingent claim \( F : S \to L^0_{+} (\mathcal{F}_\tau) \) satisfying \( F(S) = g \) \( \mathbb{P} \)-a.s.

2.6 Classical superreplication prices

Definition 2.20. Let \( S \) be a market, \( \Gamma \) a strategy cone, \( \tau \in \mathcal{T}_{[0,T]} \) and \( F : S \to L^0_{+} (\mathcal{F}_\tau) \) a contingent claim. The function \( \Pi(F | \Gamma) : S \to [0, \infty] \) defined by
\[
\Pi(F | \Gamma)(S) := \inf \{ v \geq 0 : \text{there exists } \vartheta \in \Gamma \text{ such that } V_0(\vartheta)(S) = v \text{ and } V_\tau(\vartheta)(S) \geq F(S) \mathbb{P} \text{-a.s.} \}
\]
is called the classical superreplication price of \( F \) for \( \Gamma \).
Proposition 2.21. Let $S$ be a market, $\Gamma$ a strategy cone, $\tau \in 7_{[0,T]}$ and $F : S \rightarrow L^0_+(\mathcal{F}_\tau)$ a contingent claim. Then either $\Pi(F | \Gamma) \leq +\infty$ or $\Pi(F | \Gamma)$ is a contingent claim at time 0.

Remark 2.22. The above result implies in particular that if $S, S' \in S$ with $S_0 = S'_0$, then $\Pi(F | \Gamma)(S) = \Pi(F | \Gamma)(S')$. This simple observation already suggests that when looking for a dual characterisation of $\Pi(F | \Gamma)(S)$ for some fixed $S \in S$, we should consider all representatives $S' \in S$ satisfying $S_0 = S'_0$ (see Section 7.5).

The following result provides an economic interpretation of classical superreplication prices.

Proposition 2.23. Let $S$ be a market, $\Gamma$ a strategy cone, $\tau \in 7_{[0,T]}$ and $F : S \rightarrow L^0_+(\mathcal{F}_\tau)$ a contingent claim with $\Pi(F | \Gamma) < \infty$. Then for all $\delta > 0$ and all strictly positive contingent claims $C : S \rightarrow \mathbb{R}^+$ (at time 0) there exists $\mu \in \Gamma$ satisfying

$$V_0(\mu) \leq \Pi(F | \Gamma) + \delta C \quad \text{and} \quad V_\tau(\mu) \geq F \quad \mathsf{P}\text{-a.s.}$$

Classical superreplication prices are monotonic, convex and positively homogeneous.

Proposition 2.24. Let $S$ be a market, $\Gamma$ a strategy cone, $\tau \in 7_{[0,T]}$, $F,F_1,F_2,G : S \rightarrow L^0_+(\mathcal{F}_\tau)$ contingent claims with $F \leq G \mathsf{P}\text{-a.s.}$ and $\lambda \geq 0$. Then:

$$\Pi(F | \Gamma) \leq \Pi(G | \Gamma) \quad \text{(monotonicity),}$$

$$\Pi(\lambda F | \Gamma) = \lambda \Pi(F | \Gamma) \quad \text{(positive homogeneity),}$$

$$\Pi(F_1 + F_2 | \Gamma) \leq \Pi(F_1 | \Gamma) + \Pi(F_2 | \Gamma) \quad \text{(subadditivity).}$$

2.7 Limit quantile superreplication prices

Definition 2.25. Let $S$ be a market, $\Gamma$ a strategy cone, $\tau \in 7_{[0,T]}$, $F : S \rightarrow L^0_+(\mathcal{F}_\tau)$ a contingent claim and $\epsilon \in (0,1)$. Then the function $\Pi^\epsilon(F | \Gamma) : S \rightarrow [0, \infty]$ defined by

$$\Pi^\epsilon(F | \Gamma)(S) := \inf\{v \geq 0 : \text{there exists } \mu \in \Gamma \text{ such that} \quad V_0(\mu)(S) = v \quad \text{and} \quad \mathsf{P}[V_\tau(\mu)(S) \geq F(S)] \geq 1 - \epsilon\}$$

is called the $\epsilon$-quantile superreplication price of $F$ for $\Gamma$.

Definition 2.26. Let $S$ be a market, $\Gamma$ a strategy cone, $\tau \in 7_{[0,T]}$ and $F : S \rightarrow L^0_+(\mathcal{F}_\tau)$ a contingent claim. Then the limit quantile superreplication price of $F$ for $\Gamma$ is defined by

$$\Pi^*(F | \Gamma) := \lim_{\epsilon \to 0} \Pi^\epsilon(F | \Gamma).$$

(2.9)

Remark 2.27.

(a) Of course, the classical superreplication price $\Pi(F | \Gamma)$ is formally obtained as $\Pi^0(F | \Gamma)$ for $\epsilon = 0$. Taking the limit $\epsilon \to 0$ is more subtle: indeed, if $S$ is a market, $\Gamma$ a strategy cone, $\tau \in 7_{[0,T]}$ and $F : S \rightarrow L^0_+(\mathcal{F}_\tau)$ a contingent claim, then

$$\Pi^*(F | \Gamma) \leq \Pi(F | \Gamma),$$

(2.10)

where in general the inequality may be strict.

(b) For technical reasons, we consider limit quantile superreplication prices only for strategy cones consisting of undefaultable strategies. For this reason a lot of results in this paper are formulated under the hypothesis $\Gamma \subset \mathcal{U}(S)$.

Proposition 2.28. Let $S$ be a market, $\Gamma$ a strategy cone, $\tau \in 7_{[0,T]}$ and $F : S \rightarrow L^0_+(\mathcal{F}_\tau)$ a contingent claim. Then either $\Pi^*(F | \Gamma) \equiv +\infty$ or $\Pi^*(F | \Gamma)$ is a contingent claim at time 0.
Proposition 2.29. Let $S$ be a market, $\Gamma \subset \mathcal{U}(S)$ a strategy cone, $\tau \in \mathcal{T}_{[0,T]}$ and $F : S \rightarrow L^0_+(\mathcal{F}_\tau)$ a contingent claim. Then for all $\epsilon \in (0, 1)$, there exists $A \in \mathcal{F}_\tau$ with $P[A] \geq 1 - \epsilon$ such that

$$\Pi(F \mathbb{1}_A | \Gamma) \leq \Pi^*(F | \Gamma).$$

The following corollary provides an economic interpretation of limit quantile superreplication prices.

Corollary 2.30. Let $S$ be a market, $\Gamma \subset \mathcal{U}(S)$ a strategy cone, $\tau \in \mathcal{T}_{[0,T]}$ and $F : S \rightarrow L^0_+(\mathcal{F}_\tau)$ a contingent claim with $\Pi^*(F | \Gamma) < \infty$. Then for all $\epsilon \in (0, 1)$, all $\delta > 0$ and all strictly positive contingent claims $C : S \rightarrow \mathbb{R}_{++}$ (at time 0), there exists $\vartheta \in \Gamma$ satisfying

$$V_0(\vartheta) \leq \Pi^*(F | \Gamma) + \delta C \text{ P-a.s. and } P[V_\tau(\vartheta) \geq F] \geq 1 - \epsilon.$$

Like classical superreplication prices, limit quantile superreplication prices are monotonic, convex and positively homogeneous.

Proposition 2.31. Let $S$ be a market, $\Gamma \subset \mathcal{U}(S)$ a strategy cone, $\tau \in \mathcal{T}_{[0,T]}$ a stopping time, $F, F_1, F_2, G : S \rightarrow L^0_+(\mathcal{F}_\tau)$ contingent claims with $F \leq G$ P-a.s. and $\lambda \geq 0$. Then

$$\Pi^*(F | \Gamma) \leq \Pi^*(G | \Gamma) \quad \text{(monotonicity)},$$
$$\Pi^*(\lambda F | \Gamma) = \lambda \Pi^*(F | \Gamma) \quad \text{(positive homogeneity)},$$
$$\Pi^*(F_1 + F_2 | \Gamma) \leq \Pi^*(F_1 | \Gamma) + \Pi^*(F_2 | \Gamma) \quad \text{(subadditivity)}.$$

Limit quantile superreplication prices also have a monotone convergence property. This makes them mathematically much nicer than classical superreplication prices, which lack this property in general.

Lemma 2.32. Let $S$ be a market, $\Gamma \subset \mathcal{U}(S)$ a strategy cone, $\tau \in \mathcal{T}_{[0,T]}$ and $(F_n)_{n \in \mathbb{N}} : S \rightarrow L^0_+(\mathcal{F}_\tau)$ an increasing sequence of contingent claims with $F := \lim_{n \rightarrow \infty} F_n < \infty$ P-a.s. Then

$$\lim_{n \rightarrow \infty} \Pi^*(F_n | \Gamma) = \Pi^*(F | \Gamma).$$

Finally, limit quantile superreplication prices are invariant under an equivalent change of measure. In the following result we indicate by a left superscript the measure under which the limit quantile superreplication price is taken.

Proposition 2.33. Let $S$ be a market, $\Gamma \subset \mathcal{U}(S)$ a strategy cone, $\tau \in \mathcal{T}_{[0,T]}$, $F : S \rightarrow L^0_+(\mathcal{F}_\tau)$ a contingent claim and $Q \approx \mathbb{P}$ on $\mathcal{F}_\tau$ an equivalent probability measure. Then

$$P^Q \Pi^*(F | \Gamma) = 0 \Pi^*(F | \Gamma).$$

3 Arbitrage and gratis events

In this section, we develop a numéraire independent notion of no-arbitrage called no gratis events (NGE) and study basic properties of markets lacking this property.

Loosely speaking, arbitrage always means “making money out of nothing without risk”. One basic question is how to translate this economic meta-statement into a sound and precise mathematical definition. Already a cursory glance into the mathematical literature on (no-)arbitrage and its historical development reveals that there is no straightforward answer; the good definition of (no-)arbitrage does not seem to exist (so far). Historically, classical notions like NFLVR have been developed out of topological concepts like NFL in order to get a better economical interpretation (which the NFL notion lacks). Therefore, we adopt the viewpoint that (no-)arbitrage should be considered as a meta-notion and that a precise mathematical definition for it should be given a different name – being aware of the fact that it is perhaps not the only reasonable one.
3.1 Gratis events: definition and basic properties

One possible interpretation of the meta-notion of (no-)arbitrage is that a market is arbitrage-free at a time $\tau \in T_{[0,T]}$ if and only if there does not exist a non-zero contingent claim at time $\tau$ having a limit quantile superreplication price of 0.

Before making this concept of (no-)arbitrage precise, we introduce the support of a contingent claim.

**Definition 3.1.** Let $S$ be a market, $\Gamma$ a strategy cone, $\tau \in T_{[0,T]}$ and $F : S \to L^0_+(\mathcal{F}_\tau)$ a contingent claim. Then any set $A \in \mathcal{F}_\tau$ satisfying $A = \{F > 0\}$ $\mathbb{P}$-a.s. is called a support of $F$, and we write $A = \supp F$ $\mathbb{P}$-a.s.

If $A = \supp F$ $\mathbb{P}$-a.s., we say that $F$ is supported on $A$. Recall that $\{F > 0\}$ $\mathbb{P}$-a.s. is a shorthand for $\{F(S) > 0\}$ $\mathbb{P}$-a.s. for each $S \in S$ (see Remark 2.18), and so the support of $F$ is $\mathbb{P}$-a.s. unique.

The next definition introduces the notion of gratis events in a market $S$, which is the cornerstone of our analysis on arbitrage. Recall that the collection of all $\mathbb{P}$-null sets (in $\mathcal{F}_\tau$) is denoted by $N$.

**Definition 3.2.** Let $S$ be a market, $\Gamma$ a strategy cone, $\tau \in T_{[0,T]}$ and $A \in \mathcal{F}_\tau \setminus N$. Then $A$ is called a gratis event of $S$ at time $\tau$ for $\Gamma$ if there exists a contingent claim $F : S \to L^0_+(\mathcal{F}_\tau)$ with

$$A = \supp F \quad \text{and} \quad \Pi^*(F \mid \Gamma) = 0. \quad (3.1)$$

We denote by $\mathcal{G}_\tau(\Gamma)$ the collection of all gratis events for $S$ at time $\tau$ for $\Gamma$. If $\mathcal{G}_\tau(\Gamma) = \emptyset$ for all $\tau \in T_{[0,T]}$, the market $S$ is said to satisfy no gratis events (NGE) for $\Gamma$.

First, we collect two basic properties of the set $\mathcal{G}_\tau(\Gamma)$.

**Proposition 3.3.** Let $S$ be a market, $\Gamma \subset \mathcal{U}(S)$ a strategy cone and $\tau \in T_{[0,T]}$:

(a) The set $\mathcal{G}_\tau(\Gamma)$ is closed under finite unions.

(b) $A \in \mathcal{G}_\tau(\Gamma)$ and $B \in \mathcal{F}_\tau \setminus N$ with $B \subset A$ $\mathbb{P}$-a.s. implies $B \in \mathcal{G}_\tau(\Gamma)$.

**Proof.** Part (a) follows from subadditivity and part (b) from monotonicity of limit quantile superreplication prices.

Next, we establish that the limit quantile superreplication price of every contingent claim supported on an gratis event is indeed gratis, i.e. 0.

**Proposition 3.4.** Let $S$ be a market, $\Gamma \subset \mathcal{U}(S)$ a strategy cone, $\tau \in T_{[0,T]}$ and $F : S \to L^0_+(\mathcal{F}_\tau)$ a contingent claim with $\supp F \in \mathcal{G}_\tau(\Gamma)$. Then

$$\Pi^*(F \mid \Gamma) = 0.$$

**Proof.** Since $\supp F \in \mathcal{G}_\tau(\Gamma)$, by the definition of gratis events, there exists a contingent claim $G : S \to L^0_+(\mathcal{F}_\tau)$ such that

$$\supp F = \{F > 0\} = \{G > 0\} \quad \text{and} \quad \Pi^*(G \mid \Gamma) = 0.$$

For $n \in \mathbb{N}$, pick $A_n \in \mathcal{F}_\tau$ with $A_n = \{F \leq nG\}$ $\mathbb{P}$-a.s. and set $F_n := F 1_{A_n}$. Then $F_n \leq F_{n+1}$ $\mathbb{P}$-a.s. for all $n \in \mathbb{N}$, and $F = \lim_{n \to \infty} F_n$ $\mathbb{P}$-a.s. Hence, by Lemma 2.32, it suffices to show that for all $n \in \mathbb{N}$, we have $\Pi^*(F_n \mid \Gamma) = 0$. Since $F_n \leq nG$ $\mathbb{P}$-a.s., monotonicity and positive homogeneity of limit quantile superreplication prices yield

$$\Pi^*(F_n \mid \Gamma) \leq \Pi^*(nG \mid \Gamma) = n\Pi^*(G \mid \Gamma) = 0.$$

**Remark 3.5.**
(a) The last result shows in particular that if \( A \in \mathcal{G}_r(\Gamma) \) is a gratis event and we fix any representative \( S \in \mathcal{S} \) and consider the Arrow-Debreu type security \( 1_A \) in the currency unit determined by \( S \), i.e. the contingent claim \( F : S \to L^0_+ (\mathcal{F}_\tau) \) from Proposition 2.19 satisfying \( F(S) = 1_A \), then its limit quantile superreplication price in the currency unit determined by \( S \) is 0.

(b) The notion of gratis events can be seen as a numéraire independent version of the notion of cheap thrills introduced by Loewenstein & Willard (2000), Definition 2. Indeed, fix as above any representative \( S \in \mathcal{S} \) and any set \( A \in \mathcal{G}_r(\Gamma) \), and let \( F : S \to L^0_+ (\mathcal{F}_\tau) \) be the contingent claim from Proposition 2.19 satisfying \( F(S) = 1_A \). Then \( \Pi^*(F|\Gamma) = 0 \), and the homogeneity of limit quantile superreplication prices gives

\[
\Pi^*(nF|\Gamma) = 0, \quad n \in \mathbb{N}.
\]

Let \( C : \mathcal{S} \to \mathbb{R}_{++} \) be the strictly positive contingent claim (at time 0) from Proposition 2.19 satisfying \( C(S) = 1 \). Then for each \( n \in \mathbb{N} \), Corollary 2.30 gives an undefaultable strategy \( \vartheta_n \) satisfying

\[
V_0(\vartheta_n)(S) \leq \frac{1}{n} \quad \text{and} \quad P[V_T(\vartheta_n)(S) \geq n1_A] \geq 1 - \frac{1}{n}.
\]

Hence, on a gratis event \( A \), we can in any currency unit get arbitrarily much with probability almost 1 with arbitrarily low initial investment using strategies having nonnegative wealth processes.

(c) If we fix \( S \in \mathcal{S} \), \( \tau \in \mathcal{T}_{[0,T]} \) and \( \Gamma \) and then define the generalised coherent risk measure \( \rho_r : L^0_+ (\mathcal{F}_\tau) \to [0,\infty] \) by \( \rho_r(X) := \Pi^*(F|\Gamma)(S) \), where \( F : \mathcal{S} \to L^0_+ (\mathcal{F}_\tau) \) is the contingent claim from Proposition 2.19 satisfying \( F(S) = -X \), then \( \mathcal{G}_r(\Gamma) = \emptyset \) is equivalent to saying that the generalised risk measure \( \rho_r \) is relevant or sensitive (see Föllmer & Schied (2004), Definition 4.32). This kind of connection between generalised risk measures and absence of arbitrage deserves a more careful analysis, which we postpone to future research.

The following corollary provides a characterisation of gratis events solely in terms of strategies.

**Corollary 3.6.** Let \( S \) be a market, \( \Gamma \subset \mathcal{U}(\mathcal{S}) \) a strategy cone, \( \tau \in \mathcal{T}_{[0,T]} \) and \( F : \mathcal{S} \to L^0_+ (\mathcal{F}_\tau) \) a contingent claim. Then sup F \( \in \mathcal{G}_r(\Gamma) \) if and only if for all \( \epsilon \in (0,1) \), all \( \delta > 0 \) and all strictly positive contingent claims \( C : \mathcal{S} \to \mathbb{R}_{++} \) (at time 0), there exists \( \vartheta \in \Gamma \) satisfying

\[
V_0(\vartheta) \leq \delta C \quad \text{and} \quad P[V_T(\vartheta) \geq F] \geq 1 - \epsilon.
\]

**Proof.** First, suppose that sup \( F \in \mathcal{G}_r(\Gamma) \). Then \( \Pi^*_r(F|\Gamma) = 0 \) by Proposition 3.4 and the stated condition follows from Corollary 2.30. Conversely, assume the stated condition. Letting first \( \delta \to 0 \) establishes \( \Pi^*_r(F|\Gamma) = 0 \) for all \( \epsilon \in (0,1) \), and letting then \( \epsilon \to 0 \) yields \( \Pi^*_r(F|\Gamma) = 0 \). This gives sup sup \( F \in \mathcal{G}_r(\Gamma) \).

We proceed to show that the set \( \mathcal{G}_r(\Gamma) \) is also closed under countable unions.

**Lemma 3.7.** Let \( S \) be a market, \( \Gamma \subset \mathcal{U}(\mathcal{S}) \) a strategy cone and \( \tau \in \mathcal{T}_{[0,T]} \) a stopping time. Then \( \mathcal{G}_r(\Gamma) \) is closed under countable unions.

**Proof.** We may assume without loss of generality that \( \mathcal{G}_r(\Gamma) \neq \emptyset \). Let \( (A_n)_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{G}_r(\Gamma) \) and set \( A := \bigcup_{n \in \mathbb{N}} A_n \). Moreover, let \( F : \mathcal{S} \to L^0_+ (\mathcal{F}_\tau) \) be a strictly positive contingent claim (at time \( \tau \)). For \( n \in \mathbb{N} \), define \( B_n := \bigcup_{1 \leq k \leq n} A_k \). Then \( (B_n)_{n \in \mathbb{N}} \) is a sequence in \( \mathcal{G}_r(\Gamma) \) by Proposition 3.3 (a) and \( (F1_{B_n})_{n \in \mathbb{N}} \) is a monotone increasing sequence of contingent claims satisfying

\[
\lim_{n \to \infty} F1_{B_n} = F1_A.
\]

Thus, Proposition 3.4 and Lemma 2.32 imply that

\[
\Pi^*(F1_A|\Gamma) = \lim_{n \to \infty} \Pi^*(F1_{B_n}|\Gamma) = 0.
\]
Lemma 3.7 has far reaching consequences when studying the finer structure of markets having gratis events. For the purposes of this paper, however, we content ourselves with deducing that gratis events can also be characterized by classical superreplication prices.

**Corollary 3.8.** Let $S$ be a market, $\Gamma \subseteq \mathcal{U}(S)$ a strategy cone, $\tau \in \mathcal{T}_{[0,T]}$ and $A \in \mathcal{F}_{\tau} \setminus \mathcal{N}$. Then $A \in \mathcal{G}_\tau(\Gamma)$ if and only if for all $\epsilon \in (0,1)$, there exists a contingent claim $F^{(\epsilon)} : S \to L^0_\mathcal{F}(\mathcal{F}_{\tau})$ with

$$P[A \setminus \text{supp } F^{(\epsilon)}] \leq \epsilon \quad \text{and} \quad \Pi(F^{(\epsilon)} | \Gamma) = 0.$$  

**Proof.** First, assume that $A \in \mathcal{G}_\tau(\Gamma)$ and let $\epsilon \in (0,1)$ be given. By the definition of gratis events, there exists a contingent claim $F : S \to L^0_\mathcal{F}(\mathcal{F}_{\tau})$ with

$$A = \text{supp } F \quad \text{P-a.s.} \quad \text{and} \quad \Pi(F | \Gamma) = 0.$$  

By Proposition 2.29, there exists $B \in \mathcal{F}_\tau$ with $P[B] \geq 1 - \epsilon$ such that

$$\Pi(F 1_B | \Gamma) \leq \Pi(F | \Gamma) = 0.$$  

Now $F^{(\epsilon)} := F 1_B$ does the job.

Conversely, assume that for all $n \in \mathbb{N}$, there exists a contingent claim $F_n : S \to L^0_\mathcal{F}(\mathcal{F}_{\tau})$ with

$$P[A \setminus \text{supp } F_n] \leq \frac{1}{n} \quad \text{and} \quad \Pi(F_n | \Gamma) = 0.$$  

(3.2)

Then $\Pi(F_n | \Gamma) = 0$ for all $n \in \mathbb{N}$, and $A_n := A \cap \text{supp } F_n \in \mathcal{G}_\tau(\Gamma)$ for all $n \in \mathbb{N}$ sufficiently large (for small $n$, possibly $A_n \in \mathcal{N}$). Moreover, $A = \bigcup_{n \in \mathbb{N}} A_n$ P-a.s. since $A_n \subseteq A$ and $P[A \setminus A_n] \leq 1/n$ for all $n \in \mathbb{N}$. Now Lemma 3.7 and Proposition 3.3 (b) yield that $A \in \mathcal{G}_\tau(\Gamma)$. \qed

**Remark 3.9.** Note that Corollary 3.8 implies in particular that $\mathcal{G}_\tau(\Gamma) = \emptyset$ if and only if for all non-zero contingent claims $F : S \to L^0_\mathcal{F}(\mathcal{F}_{\tau})$, we have $\Pi(F | \Gamma) > 0$.

### 3.2 Gratis events: dynamic properties

So far we have looked at the concept of arbitrage from a static perspective. Now we want to adopt a more dynamic point of view, i.e. we want to consider the sets $\mathcal{G}_\tau(\Gamma)$ as a function of $\tau \in \mathcal{T}_{[0,T]}$. In order to get interesting results, we have to assume that $S$ is a numéraire market and that $\Gamma$ is rich enough to allow us at each stopping time to switch to some numéraire strategy.

**Definition 3.10.** Let $S$ be a numéraire market and $\Gamma$ a strategy cone. Then $\Gamma$ is said to allow for switching to numéraires if for all $\tau \in \mathcal{T}_{[0,T]}$, there exists a numéraire strategy $\eta$, called (a) switching numéraire strategy at time $\tau$, which may depend on $\tau$ and which is such that for all $\vartheta \in \Gamma$,

$$\vartheta 1_{[[0,\tau]]} + V_\tau(\vartheta)(S^{(\eta)})\eta 1_{(\tau,T]} \in \Gamma.$$  

**Remark 3.11.** It is not difficult to check that if $S$ is any numéraire market, then $L^f(S)$ and $\mathcal{U}(S)$ allow for switching to numéraires. Indeed, any numéraire strategy can be taken as switching numéraire strategy at all stopping times.

If $S$ is a bounded numéraire market, then $bL^f(S)$ and $b\mathcal{U}(S)$ allow for switching to numéraires. Indeed, any bounded numéraire strategy with a bounded numéraire representative can be taken as switching numéraire strategy at all stopping times. In particular, if $S$ is a nonnegative numéraire market, then $L^f(S)$, $\mathcal{U}(S)$, $bL^f(S)$ and $b\mathcal{U}(S)$ allow for switching to numéraires, and $\eta := (1, \ldots, 1)$ can be taken as switching numéraire strategy at all stopping times.

The following result shows that for strategy cones which allow for switching to numéraires, the notion of gratis events is time-consistent in the sense that gratis events propagate forward in time.

**Proposition 3.12.** Let $S$ be a numéraire market, $\Gamma \subseteq \mathcal{U}(S)$ a strategy cone allowing for switching to numéraires and $\tau_1, \tau_2 \in \mathcal{T}_{[0,T]}$ with $\tau_1 \leq \tau_2$ P-a.s. Then

$$\mathcal{G}_{\tau_1}(\Gamma) \cap \mathcal{G}_{\tau_2}(\Gamma).$$

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Proof. We may assume without loss of generality that there is \( A \in \mathcal{F}_T \); otherwise there is nothing to prove. By the definition of gratis events, there exists a contingent claim \( F_1 : S \rightarrow \mathbb{L}^1_{\infty} (\mathcal{F}_T) \) with \( \text{supp} \ F_1 = A \ P \text{-a.s.} \) Let \( \eta \) be a switching numéraire strategy at time \( \tau \) and \( F_2 : S \rightarrow \mathbb{L}^1_{\infty} (\mathcal{F}_T) \) the contingent claim (at time \( \tau_2 \)) from Proposition 2.19 satisfying \( F_2(S^{(u)}) = F_1(S^{(u)}) \) \( P \text{-a.s.} \) Note that \( F_2(S^{(u)}) \) is indeed \( \mathcal{F}_{\tau_1} \text{-measurable, since it is } \mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2} \). Let \( \epsilon \in (0,1), \delta > 0 \) and \( C : S \rightarrow \mathbb{R}_{++} \) be a strictly positive contingent claim (at time \( 0 \)). By Corollary 3.6, there exists \( \vartheta \in \Gamma \) such that

\[
V_0(\vartheta) \leq \delta C \quad \text{and} \quad P[V_{\tau_1}(\vartheta) \geq F_1] \geq 1 - \epsilon.
\]

Set \( \tilde{\vartheta} := \vartheta I_{[0,\tau_1]} + V_{\tau_1}(\vartheta)(S^{(u)}) \eta \mathds{1}_{[\tau_1,\tau]} \in \Gamma \). Then \( V_{\tau_2}(\tilde{\vartheta})(S^{(u)}) = V_{\tau_1}(\vartheta)(S^{(u)}) \) \( P \text{-a.s.} \), which together with \( F_2(S^{(u)}) = F_1(S^{(u)}) \) gives \( V_0(\tilde{\vartheta}) \leq \delta C \) and

\[
P[V_{\tau_2}(\tilde{\vartheta}) \geq F_2] = P[V_{\tau_2}(\tilde{\vartheta})(S^{(u)}) \geq F_2(S^{(u)})] = P[V_{\tau_1}(\vartheta)(S^{(u)}) \geq F_1(S^{(u)})] \geq 1 - \epsilon.
\]

Since \( \epsilon, \delta \) and \( C \) were arbitrary, the claim follows from Corollary 3.6.

\( \square \)

Remark 3.13. A refinement of the above result, showing that \( \mathcal{G}_{\tau_1}(\Gamma) \cap \{ \tau_1 \leq \tau_2 \} \subset \mathcal{G}_{\tau_2}(\Gamma) \cup \mathcal{N} \) for arbitrary stopping times \( \tau_1, \tau_2 \in \mathcal{T}_{[0,T]} \), is the cornerstone for studying finer properties of markets admitting gratis events, i.e. failing to satisfy NGE.

An immediate consequence of Proposition 3.12 is the fact that a numéraire market satisfies NGE if and only if there are no gratis events at the time horizon \( T \).

Corollary 3.14. Let \( S \) be a numéraire market and \( \Gamma \subset \mathcal{U}(S) \) a strategy cone allowing for switching to numéraires. Then \( S \) satisfies NGE for \( \Gamma \) if and only if \( \mathcal{G}_T(\Gamma) = \emptyset \).

4 Maximal strategies

In this section, we address the question which strategies are “reasonable investments” in a market \( S \) for a given strategy cone \( \Gamma \). We do this in full generality and do not assume that the market satisfies NGE or any other no-arbitrage condition. Notwithstanding, the existence of different kinds of maximal strategies is intimately linked to notions of no-arbitrage. This connection will be studied in detail in the next section.

To motivate the notion of maximal strategies, suppose that we want to invest a part of our wealth at time \( 0 \) in the market and trade until time \( \tau \in \mathcal{T}_{[0,T]} \) by choosing a strategy \( \vartheta \in \Gamma \). Then independently of our personal preferences, \( \vartheta \) can be considered a “reasonable investment” only if the following two conditions are satisfied:

(1) We cannot create more wealth at time \( \tau \) with the same initial investment.

(2) We cannot create the same wealth at time \( \tau \) with a lower initial investment.

Inspired by Delbaen & Schachermayer (1995), strategies satisfying the latter two properties will be called maximal. Depending on how we make conditions (1) and (2) mathematically precise, we are led to the notions of weakly maximal and strongly maximal strategies. Note that our setup (apart from the numéraire independent approach) differs from Delbaen & Schachermayer (1995) insofar as we do not assume that the market satisfies NFLVR or any other no-arbitrage condition. Moreover, what we call “weakly maximal strategies” is simply called “maximal strategies” in Delbaen & Schachermayer (1995), whereas the notion of “strongly maximal strategies” seems to be new. We shall see in Section 7, however, that both notions coincide under the assumption NGE and a fortiori also under the stronger assumption NFLVR.
4.1 Weakly maximal strategies

**Definition 4.1.** Let $S$ be a market, $\Gamma$ a strategy cone and $\tau \in \mathcal{T}_{[0,T]}$ a stopping time. A strategy $\tilde{\vartheta} \in \Gamma$ is called

- **weakly maximal at time $\tau$ for $\Gamma$** if for all $\tilde{\vartheta} \in \Gamma$ satisfying

$$V_0(\tilde{\vartheta}) \leq V_0(\tilde{\vartheta}) \quad \text{and} \quad V_\tau(\tilde{\vartheta}) \geq V_\tau(\tilde{\vartheta}) \quad \mathbb{P}\text{-a.s.,}$$

we have

$$V_0(\tilde{\vartheta}) = V_0(\tilde{\vartheta}) \quad \text{and} \quad V_\tau(\tilde{\vartheta}) = V_\tau(\tilde{\vartheta}) \quad \mathbb{P}\text{-a.s.}$$

- **weakly maximal for $\Gamma$** if it is weakly maximal at each time $\tau \in \mathcal{T}_{[0,T]}$ for $\Gamma$.

It there is no danger of confusion, we omit the specification “for $\Gamma$”.

The following easy but fundamental result shows that the zero strategy plays a crucial role when studying weakly maximal strategies.

**Proposition 4.2.** Let $S$ be a market, $\Gamma$ a strategy cone and $\tau \in \mathcal{T}_{[0,T]}$. Then $\vartheta \in \Gamma$ is weakly maximal at time $\tau$ only if $0$ is weakly maximal at time $\tau$.

**Proof.** By contraposition, suppose that $0$ is not weakly maximal at time $\tau$. Then there exists $\tilde{\vartheta} \in \Gamma$ such that

$$V_0(\tilde{\vartheta}) \leq 0 \quad \text{and} \quad V_\tau(\tilde{\vartheta}) \geq 0 \quad \mathbb{P}\text{-a.s.}$$

and

$$V_0(\tilde{\vartheta}) < 0 \quad \text{or} \quad \mathbb{P}[V_\tau(\tilde{\vartheta}) > 0] > 0.$$ 

Let $\vartheta \in \Gamma$ and set $\tilde{\vartheta} := \vartheta + \tilde{\vartheta} \in \Gamma$. Comparing the strategy $\tilde{\vartheta}$ to $\vartheta$ shows that $\vartheta$ is not weakly maximal at time $\tau$. \hfill $\square$

The following example shows that the converse of Proposition 4.2 does not hold.

**Example 4.3.** Let $W = (W_t)_{t \in [0,1]}$ be a Brownian motion on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq 1}, \mathbb{P})$. Define stopping times $\tau_1, \tau_2$ by

$$\tau_1 := \inf \left\{ t \in [0,1] : \int_0^t \frac{1}{1-s} dW_s = -\frac{1}{4} \right\}, \quad \tau_2 := \inf \left\{ t \in [0,1] : \int_0^t \frac{1}{1-s} dW_s = -\frac{1}{2} \right\}.$$ 

It is a standard exercise in stochastic analysis to check that $\tau_1 < \tau_2 < 1 \mathbb{P}$-a.s. Define the process $S = (S^1_t, S^2_t)_{t \in [0,1]}$ by

$$S^1_t := 1 + \int_0^{\tau_1 \wedge t} \frac{2}{1-s} dW_s \quad \text{and} \quad S^2_t := 1 + \int_0^{\tau_2 \wedge t} \frac{1}{1-s} dW_s, \quad (4.1)$$

and let $\mathcal{S}$ be the market corresponding to $S$. Note that $S^1$ and $S^2$ are (strict) local $\mathbb{P}$-martingales with $S^1_0 = S^2_0 = \frac{1}{2} \mathbb{P}$-a.s. We claim that $0$ is weakly maximal at time 1 for $\mathcal{U}(\mathcal{S})$ and that the numéraire strategy $\eta := (1,1)$ is **not** weakly maximal at time 1 for $\mathcal{U}(\mathcal{S})$.

To establish the first claim, by way of contradiction, suppose there exists $\tilde{\vartheta} \in \mathcal{U}(\mathcal{S})$ such that $V_0(\tilde{\vartheta}) = 0$ and $\mathbb{P}[V_1(\tilde{\vartheta}) > 0] > 0$. (Note that we cannot have $V_0(\tilde{\vartheta}) < 0$ because $\tilde{\vartheta} \in \mathcal{U}(\mathcal{S})$.) Since $S^1$ and $S^2$ are continuous local $\mathbb{P}$-martingales, $V(\tilde{\vartheta})(S) = \vartheta \cdot S$ is a nonnegative local $\mathbb{P}$-martingale and a $\mathbb{P}$-supermartingale. Hence we have $\mathbb{E}[V_1(\tilde{\vartheta})(S)] \leq V_0(\tilde{\vartheta})(S) = 0$, in contradiction to $V_1(\tilde{\vartheta})(S) \geq 0 \mathbb{P}$-a.s. and $\mathbb{P}[V_1(\tilde{\vartheta})(S) > 0] > 0$.

To establish the second claim, consider the strategy $\eta^* := (-2,4)1_{[0,\tau_1]} + (4,0)1_{(\tau_1,1]}$. Then $V_0(\eta^*)(S) = V_0(\eta)(S) = 2$ but $V_1(\eta)(S) = 1$ and $V_1(\eta^*)(S) = 2$. In fact, it can be shown that $\eta^*$ is a **dominating maximal strategy** for $\eta$ (see Definition 6.1 and Lemma 7.11, and check that $V(\eta^*)(S)$ is a true martingale).
The next result shows that strategies which are not weakly maximal always fail to satisfy condition (1) of a “reasonable investment” (see above), provided that $S$ is a numéraire market and $\Gamma$ contains a numéraire strategy.

**Proposition 4.4.** Let $S$ be a numéraire market, $\tau \in T_{[0,T]}$ and $\Gamma$ a strategy cone containing a numéraire strategy. Suppose that $\vartheta \in \Gamma$ is not weakly maximal at time $\tau$. Then there exists $\tilde{\vartheta} \in \Gamma$ such that

$$V_{0}(\tilde{\vartheta}) = V_{0}(\vartheta), \quad V_{\tau}(\tilde{\vartheta}) \geq V_{\tau}(\vartheta) \text{ P-a.s. \quad and} \quad P[V_{\tau}(\tilde{\vartheta}) > V_{\tau}(\vartheta)] > 0.$$ 

**Proof.** Since $\vartheta \in \Gamma$ is not maximal at time $\tau$, there exists $\hat{\vartheta} \in \Gamma$ such that

$$V_{0}(\hat{\vartheta}) \leq V_{0}(\vartheta) \quad \text{and} \quad V_{\tau}(\hat{\vartheta}) \geq V_{\tau}(\vartheta) \quad \text{P-a.s.}$$

and

$$V_{0}(\hat{\vartheta}) < V_{0}(\vartheta) \quad \text{or} \quad P[V_{\tau}(\hat{\vartheta}) > V_{\tau}(\vartheta)] > 0.$$ 

We may assume without loss of generality that $V_{0}(\hat{\vartheta}) < V_{0}(\vartheta)$, for otherwise we can simply take $\tilde{\vartheta} := \hat{\vartheta}$. Let $\eta \in \Gamma$ be a numéraire strategy and set

$$\tilde{\vartheta} := \hat{\vartheta} + V_{0}(\vartheta - \hat{\vartheta})(S^{(n)})\eta \in \Gamma.$$ 

Then $\tilde{\vartheta} \in \Gamma$ satisfies $V_{0}(\tilde{\vartheta}) = V_{0}(\vartheta)$ and

$$V_{\tau}(\tilde{\vartheta}) = V_{\tau}(\hat{\vartheta}) + V_{\tau}(\vartheta - \hat{\vartheta})(S^{(n)})V_{\tau}(\eta) > V_{\tau}(\hat{\vartheta}) \geq V_{\tau}(\vartheta) \quad \text{P-a.s.}$$

Our next goal is to show that the property of being a weakly maximal strategy is time-consistent. To this end, we have to assume that $S$ is a numéraire market and $\Gamma$ is rich enough to allow at each stopping time to switch to a dominated strategy. This property is in some sense a weak analogue of predictable convexity of the set of (constrained) admissible strategies considered in the standard framework.

**Definition 4.5.** Let $S$ be a numéraire market and $\Gamma$ a strategy cone. Then $\Gamma$ is said to allow for switching to dominated strategies if for all $\tau \in T_{[0,T]}$, there exists a numéraire strategy $\eta$, called (a) switching numéraire strategy at time $\tau$, which may depend on $\tau$ and which is such that for all $\vartheta^{(1)}, \vartheta^{(2)} \in \Gamma$,

$$\vartheta^{(1)}I_{[0,\tau]} + \left(1 - I_{\{V_{\tau}(\vartheta^{(1)}) < V_{\tau}(\vartheta^{(2)})\}}\right)\vartheta^{(1)}I_{(\tau,T]} + \left(1 - I_{\{V_{\tau}(\vartheta^{(1)}) \geq V_{\tau}(\vartheta^{(2)})\}}\right)\left(\vartheta^{(2)} + V_{\tau}(\vartheta^{(1)} - \vartheta^{(2)})(S^{(n)})\eta\right)I_{(\tau,T]} \in \Gamma.$$ 

**Remark 4.6.**

(a) It is not difficult to check that if $S$ is any numéraire market, then $L^\infty(S)$ and $\mathcal{U}(S)$ allow for switching to dominated strategies. Indeed, any numéraire strategy can be taken as switching numéraire strategy at all stopping times.

If $S$ is a bounded numéraire market, then $\mathbf{b}L^\infty(S)$ and $\mathbf{b}\mathcal{U}(S)$ allow for switching to dominated strategies. Indeed, any bounded numéraire strategy with a bounded numéraire representative can be taken as switching numéraire strategy at all stopping times. In particular, if $S$ is a nonnegative numéraire market, then $L^\infty(S)$, $\mathcal{U}(S)$, $\mathbf{b}L^\infty(S)$ and $\mathbf{b}\mathcal{U}(S)$ allow for switching to numéraires, and $\eta := (1, \ldots, 1)$ can be taken as switching numéraire strategy at all stopping times.

(b) If $S$ is a numéraire market and $\Gamma \subset \mathcal{U}(S)$ allows for switching to dominated strategies, then $\Gamma$ also allows for switching to numéraires. Indeed, take $\vartheta^{(1)} := \vartheta$ and $\vartheta^{(2)} := 0$. Therefore it is justified to call $\eta$ in both cases a “switching numéraire strategy”.

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Proposition 4.7. Let $S$ be a numéraire market, $\tau_1 \leq \tau_2 \in \mathcal{T}_{[0,T]}$ stopping times and $\Gamma$ a strategy cone allowing for switching to dominated strategies. If $\vartheta \in \Gamma$ is weakly maximal at time $\tau_2$, then it is also weakly maximal at time $\tau_1$.

Proof. By contraposition, suppose that $\vartheta \in \Gamma$ is not weakly maximal at time $\tau_1$. Then there exists $\vartheta'$ such that

$$V_0(\vartheta') = V_0(\vartheta) \quad \text{and} \quad V_{\tau_1}(\vartheta') \geq V_{\tau_1}(\vartheta) \quad \mathbb{P}\text{-a.s.}$$

and

$$V_0(\vartheta') < V_0(\vartheta) \quad \text{or} \quad \mathbb{P}[V_{\tau_1}(\vartheta') > V_{\tau_1}(\vartheta)] > 0.$$ 

Let $\eta$ be a switching numéraire strategy at time $\tau$, and set

$$\tilde{\vartheta} := \vartheta I_{[0,\tau_1]} + (\vartheta + V_{\tau_1}(\vartheta - \vartheta)(S^{(\eta)})\eta)I_{(\tau_1,\tau]} \in \Gamma,$$

Then

$$V_0(\tilde{\vartheta}) = V_0(\vartheta) \quad \text{and} \quad V_{\tau_2}(\tilde{\vartheta}) = V_{\tau_2}(\vartheta) + (V_{\tau_1}(\vartheta - \vartheta)(S^{(\eta)})\eta)V_{\tau_2}(\eta) \quad \mathbb{P}\text{-a.s.}$$

Since either $V_0(\tilde{\vartheta}) < V_0(\vartheta)$ or $\mathbb{P}[V_{\tau_1}(\vartheta - \vartheta)(S^{(\eta)})\eta > 0] > 0$, this contradicts the hypothesis that $\vartheta$ is weakly maximal at time $\tau_2$. \qed

Corollary 4.8. Let $S$ be a numéraire market and $\Gamma$ a strategy cone allowing for switching to dominated strategies. Then $\vartheta \in \Gamma$ is weakly maximal if and only if it is weakly maximal at time $T$.

Finally, we show that if a weakly maximal strategy is dominated by another strategy at some (stopping) time, then it is dominated at all earlier (stopping) times, too.

Proposition 4.9. Let $S$ be a numéraire market, $\Gamma$ a strategy cone allowing for switching to dominated strategies and $\tau \in \mathcal{T}_{[0,T]}$. Suppose that $\vartheta \in \Gamma$ is weakly maximal at time $\tau$ and $\tilde{\vartheta} \in \Gamma$ satisfies $V_{\tau}(\tilde{\vartheta}) \geq V_{\tau}(\vartheta)$ $\mathbb{P}$-a.s. Then

$$V_{\tau}(\tilde{\vartheta}) \geq V_{\tau}(\vartheta) \quad \mathbb{P}\text{-a.s. on } [[0,\tau]]$$

Proof. Seeking a contradiction, suppose there is a stopping time $\sigma \leq \tau$ with $\mathbb{P}[V_{\sigma}(\tilde{\vartheta}) < V_{\sigma}(\vartheta)] > 0$. Let $\eta$ be a switching numéraire strategy at time $\sigma$, and set

$$\tilde{\vartheta} := \vartheta I_{[0,\sigma]} + I_{\{V_{\sigma}(\vartheta) < V_{\sigma}(\tilde{\vartheta})\}} \vartheta I_{(\sigma,\tau]} + I_{\{V_{\sigma}(\vartheta) \geq V_{\sigma}(\tilde{\vartheta})\}}(\vartheta + V_{\sigma}(\vartheta - \tilde{\vartheta})(S^{(\eta)})\eta)I_{(\sigma,\tau]} \in \Gamma,$$

Then $V_0(\tilde{\vartheta}) = V_0(\vartheta)$ and since $V_{\tau}(\tilde{\vartheta}) \geq V_{\tau}(\vartheta)$ $\mathbb{P}$-a.s. by assumption,

$$V_{\tau}(\tilde{\vartheta}) = I_{\{V_{\sigma}(\vartheta) < V_{\sigma}(\tilde{\vartheta})\}} V_{\tau}(\vartheta) + I_{\{V_{\sigma}(\vartheta) \geq V_{\sigma}(\tilde{\vartheta})\}}(V_{\tau}(\tilde{\vartheta}) + V_{\sigma}(\vartheta - \tilde{\vartheta})(S^{(\eta)})V_{\sigma}(\eta))$$

$$\geq I_{\{V_{\sigma}(\vartheta) < V_{\sigma}(\tilde{\vartheta})\}} V_{\tau}(\vartheta) + I_{\{V_{\sigma}(\vartheta) \geq V_{\sigma}(\tilde{\vartheta})\}}(V_{\tau}(\vartheta) + V_{\sigma}(\vartheta - \tilde{\vartheta})(S^{(\eta)})V_{\sigma}(\eta))$$

$$= V_{\tau}(\vartheta) + I_{\{V_{\sigma}(\vartheta) > V_{\sigma}(\tilde{\vartheta})\}} V_{\sigma}(\vartheta - \tilde{\vartheta})(S^{(\eta)})V_{\sigma}(\eta) \quad \mathbb{P}\text{-a.s.}$$

Since $\mathbb{P}[V_{\sigma}(\vartheta) > V_{\sigma}(\tilde{\vartheta})] > 0$, this is a contradiction to the hypothesis that $\vartheta$ is weakly maximal at time $\tau$. \qed

Corollary 4.10. Let $S$ be a numéraire market and $\tau \in \mathcal{T}_{[0,T]}$. Suppose that $\vartheta^{(1)}, \vartheta^{(2)} \in \mathcal{U}(S)$ are weakly maximal strategies at time $\tau$ for $\mathcal{U}(S)$. Then $\vartheta^{(1)} + \vartheta^{(2)}$ is weakly maximal at time $\tau$, too. As a consequence, if $\vartheta^{(1)}, \vartheta^{(2)} \in \mathcal{U}(S)$ are weakly maximal strategies, then $\vartheta^{(1)} + \vartheta^{(2)}$ is weakly maximal, too.
Proof. The second claim follows immediately from the first one. To establish the first claim, by way of contradiction, we may assume by Proposition 4.4 that there exists \( \hat{\vartheta} \in \mathcal{U}(\mathcal{S}) \) such that

\[
V_0(\hat{\vartheta}) = V_0(\vartheta^{(1)} + \vartheta^{(2)}), \\
V_\tau(\hat{\vartheta}) \geq V_\tau(\vartheta^{(1)} + \vartheta^{(2)}) \quad \text{P-a.s. and } \quad \mathbb{P}[V_\tau(\tilde{\vartheta}) > V_\tau(\vartheta^{(1)} + \vartheta^{(2)})] > 0.
\]

Define

\[
\vartheta := \hat{\vartheta} - \vartheta^{(2)} \mathbb{I}_{[0, \tau]} + \left(V_\tau(\tilde{\vartheta} - \vartheta^{(2)})(S^{(\eta)})\right) \mathbb{I}_{(\tau, T]},
\]

where \( \eta \in \mathcal{U}(\mathcal{S}) \) is some numéraire strategy. Since \( V_\tau(\tilde{\vartheta}) \geq V_\tau(\vartheta^{(2)}) \) P-a.s. and \( \vartheta^{(2)} \) is weakly maximal at time \( \tau \), it follows by Proposition 4.9 that \( \vartheta \in \mathcal{U}(\mathcal{S}) \). Moreover,

\[
V_0(\vartheta) = V_0(\vartheta^{(1)}), \quad V_\tau(\vartheta) \geq V_\tau(\vartheta^{(1)}) \quad \text{P-a.s. and } \quad \mathbb{P}[V_\tau(\vartheta) > V_\tau(\vartheta^{(1)})] > 0.
\]

This contradicts the hypothesis that \( \vartheta^{(1)} \) is weakly maximal at time \( \tau \).

\[\square\]

4.2 Strongly maximal strategies

Definition 4.11. Let \( \mathcal{S} \) be market, \( \Gamma \) a strategy cone and \( \tau \in \mathcal{T}_{[0, T]} \) a stopping time. A strategy \( \vartheta \in \Gamma \) is called

- **strongly maximal in the classical sense at time \( \tau \)** for \( \Gamma \) if for all non-zero contingent claims \( F : \mathcal{S} \to \mathbb{L}^0_\mathfrak{F} \),

\[
\Pi(V_\tau(\vartheta) + F|\Gamma) > V_0(\vartheta),
\]

- **strongly maximal in the limit quantile sense at time \( \tau \)** for \( \Gamma \) if for all non-zero contingent claims \( F : \mathcal{S} \to \mathbb{L}^0_\mathfrak{F} \),

\[
\Pi^*(V_\tau(\vartheta) + F|\Gamma) > V_0(\vartheta),
\]

- **strongly maximal in the classical/limit quantile sense for \( \Gamma \)** if it is strongly maximal in the classical/limit quantile sense at each time \( \tau \in \mathcal{T}_{[0, T]} \) for \( \Gamma \).

If there is no danger of confusion, we omit the specification “for \( \Gamma \)”.

Remark 4.12. It follows immediately from Remark 2.27 (a) that strategies which are strongly maximal in the limit quantile sense are also strongly maximal in the classical sense.

It is clear from the definition that strongly maximal strategies satisfy condition (1) of a “reasonable investment” (see above). The next result shows that they also satisfy condition (2) there, provided that \( \mathcal{S} \) is a numéraire market and \( \Gamma \) contains a numéraire strategy.

Proposition 4.13. Let \( \mathcal{S} \) be a numéraire market, \( \tau \in \mathcal{T}_{[0, T]} \) and \( \Gamma \subseteq \mathcal{U}(\mathcal{S}) \) a strategy cone containing a numéraire strategy.

(a) Suppose that \( \vartheta \in \Gamma \) is strongly maximal in the classical sense at time \( \tau \). Then

\[
\Pi(V_\tau(\vartheta)|\Gamma) = V_0(\vartheta).
\]

(b) Suppose that \( \vartheta \in \Gamma \) is strongly maximal in the limit quantile sense at time \( \tau \). Then

\[
\Pi^*(V_\tau(\vartheta)|\Gamma) = V_0(\vartheta).
\]
Proof. We only prove part (b); part (a) follows by an analogous argument. First, by Remark 2.27 (a) and the definition of classical superreplication prices, it suffices to show that
\[ \Pi^*(V_\tau(\vartheta) \mid \Gamma) \geq V_0(\vartheta). \]
Let \( \eta \in \Gamma \) be a numéraire strategy and \( \delta > 0 \) arbitrary. Then the (strictly positive) contingent claim \( F : \mathcal{S} \to \mathcal{L}^{0+}_+(\mathcal{F}_\tau) \), defined by \( F := \delta V_\tau(\eta) \), satisfies
\[ \Pi^*(F \mid \Gamma) \leq \Pi(F \mid \Gamma) \leq \delta V_0(\vartheta). \tag{4.2} \]
Since \( \vartheta \) is strongly maximal in the limit quantile sense at time \( \tau \), subadditivity of limit quantile superreplication prices and (4.2) give
\[ V_0(\vartheta) < \Pi^*(V_\tau(\vartheta) + F \mid \Gamma) \leq \Pi^*(V_\tau(\vartheta) \mid \Gamma) + \Pi^*(F \mid \Gamma) \leq \Pi^*(V_\tau(\vartheta) \mid \Gamma) + \delta V_0(\vartheta). \]
Letting \( \delta \to 0 \) establishes the claim. \( \square \)

The next result shows that strongly maximal strategies are also weakly maximal, provided that \( \mathcal{S} \) is a numéraire market and \( \Gamma \) contains a numéraire strategy.

**Proposition 4.14.** Let \( \mathcal{S} \) be a numéraire market, \( \tau \in \mathcal{T}_{[0,T]} \) a stopping time and \( \Gamma \subset \mathcal{U}(\mathcal{S}) \) a strategy cone containing a numéraire strategy. If \( \vartheta \in \Gamma \) is strongly maximal in the classical/limit quantile sense at time \( \tau \), then it is also weakly maximal at time \( \tau \).

**Proof.** Seeking a contradiction, suppose there exists \( \tilde{\vartheta} \in \Gamma \) which is strongly maximal in the classical sense but not weakly maximal at time \( \tau \). Then by Proposition 4.4, there exists \( \tilde{\vartheta} \in \Gamma \) such that
\[ V_0(\tilde{\vartheta}) = V_0(\vartheta), \quad V_\tau(\tilde{\vartheta}) \geq V_\tau(\vartheta) \text{ P-a.s. and } \mathbb{P}[V_\tau(\tilde{\vartheta}) > V_\tau(\vartheta)] > 0. \]
Define the non-zero contingent claim \( F : \mathcal{S} \to \mathcal{L}^0_+(\mathcal{F}_\tau) \) by
\[ F := V_\tau(\tilde{\vartheta}) - V_\tau(\vartheta). \]
Then
\[ \Pi(V_\tau(\vartheta) + F \mid \Gamma) = \Pi(V_\tau(\tilde{\vartheta}) \mid \Gamma) \leq V_0(\tilde{\vartheta}) = V_0(\vartheta). \]
Thus, we arrive at a contradiction. \( \square \)

The following example shows that the converse of Proposition 4.14 does not hold. It is a more elaborate variant of an example given in Herdegen (2012).

**Example 4.15.** Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space supporting a random variable \( \tau \), valued in \( \mathbb{N} \cup \{+\infty\} \) and satisfying \( \mathbb{P}[\tau = n] > 0 \) for all \( n \in \mathbb{N} \cup \{+\infty\} \), and an independent sequence \((Y_n)_{n \in \mathbb{N}}\) of independent random variables, where
\[ Y_n = \begin{cases} 1 + \frac{1}{(1+n)^r} & \text{with probability } \frac{n}{n+1}, \\ 1 - \frac{1}{(1+n)^r} & \text{with probability } \frac{1}{n+1}. \end{cases} \]
For \( n \in \mathbb{N}_0 \cup \{\infty\} \), define
\[ X_n := \prod_{k=1}^{n} Y_k \]
so that \( X_0 = 0 \). It follows from elementary analysis that for all \( n \in \mathbb{N} \),
\[ \frac{1}{3} < \exp \left( -\sum_{k=1}^{\infty} \frac{2/(1+k)^3}{(1+k)^3} \right) \leq X_n \leq \exp \left( \sum_{k=1}^{\infty} \frac{1/(1+k)^2}{(1+k)^2} \right) < 3. \]
This together with the fact that \( Y_n \to 1 \) as \( n \to \infty \) shows that \( X_\infty = \lim_{n \to \infty} X_n \) is well defined\(^1\).

The crucial feature of the process \( X = (X_n)_{n \in \mathbb{N} \cup \{ \infty \}} \) is that, due to the presence of \( \tau \), it may stop fluctuating at any time \( n \in \mathbb{N} \), and if it has not stopped before time \( n \), i.e. on the event \{ \( X_n \neq X_{n-1} \) \}, we cannot infer any further information about \( \tau \). More precisely define the\(^2\) discrete-time filtration \((\mathcal{F}_n)_{n \in \mathbb{N}} \) by \( \mathcal{F}_n := \sigma(X_1, \ldots, X_n), \ n \in \mathbb{N} \). We claim that for all \( n \in \mathbb{N}, \mathcal{F}_n \) is conditionally independent of \( \tau \) given \{ \( \tau \geq n \) \}. Indeed, fix \( n \in \mathbb{N} \) and let \( x_1, \ldots, x_n \in \mathbb{R}_+ \) and \( m \in \mathbb{N} \). Then by independence of \((Y_n)_{n \in \mathbb{N}} \) and \( \tau \),

\[
\mathbb{P}[X_1 = x_1, \ldots, X_n = x_n, \tau = m \mid \tau \geq n] = \mathbb{P}[Y_1 = x_1, \ldots, \prod_{k=1}^n Y_k = x_n, \tau = m \mid \tau \geq n] = \mathbb{P}[Y_1 = x_1, \ldots, Y_k = x_k \mid \tau \geq n] \mathbb{P}[\tau = m \mid \tau \geq n] = \mathbb{P}[X_1 = x_1, \ldots, X_n = x_n \mid \tau \geq n] \mathbb{P}[\tau = m \mid \tau \geq n].
\]

Define the continuous-time stochastic process \((S^1_t, S^2_t)_{t \in [0,1]} \) by

\[
S^1_t := 1 \quad \text{and} \quad S^2_t := \sum_{k=0}^{\infty} X_k \mathbb{1}_{[[1-2^{-k},1-2^{-(k+1)})]} + X_\infty \mathbb{1}_{[1]},
\]

and let \( \mathcal{S} \) be the market corresponding to \( S \). Note that \( S^2 \) is left-continuous in \( t = 1 \) because \( X_\infty = \lim_{n \to \infty} X_n \).

Define the continuous-time filtration \((\mathcal{F}_t)_{t \in [0,1]} \) by \( \mathcal{F}_t := \sigma(S^2_r, \ r \leq t) \cap \mathcal{N}, \ t \in [0,1] \), where \( \mathcal{N} \) denotes the \( \mathbb{P} \)-null sets in \( \sigma(S^2_r, \ r \leq 1) \). Now if \( \zeta \in \mathcal{L}(S) \) is left-continuous and adapted to \((\mathcal{F}_t)_{t \in [0,1]} \), it follows immediately from the definitions of \((\mathcal{F}_t)_{t \in [0,1]} \) and \((\mathcal{F}_n)_{n \in \mathbb{N}} \) that for all \( k \in \mathbb{N}, \zeta_{1-2^k} \) is \( \mathcal{F}_{1-2^{-(k+1)}} \)-measurable and there exists an \( \mathbb{R}^2 \)-valued \( \mathcal{F}_{k-1} \)-measurable random variable \((\rho_k, \xi_k) \) satisfying \( \zeta_{1-2^k} = (\rho_k, \xi_k) \) \( \mathbb{P} \)-a.s. Moreover, there exists \((\rho_0, \xi_0) \in \mathbb{R}^2 \) such that \( \zeta_0 = (\rho_0, \xi_0) \). By the monotone class theorem, all this extends to all \( \zeta \in \mathcal{L}(S) \). In particular, for all \( \vartheta \in \mathcal{U}(S) \), there exists a \((\mathcal{G}_k)_{k \in \mathbb{N}_0} \)-predictable process \((\rho_k, \xi_k)_{k \in \mathbb{N}_0} \) such that

\[
V_0(\vartheta)(S) = \vartheta_0 \cdot S_0 = \rho_0 + \xi_0,
\]

\[
V_1(\vartheta)(S) = \vartheta_0 \cdot S_0 + \sum_{k=1}^{\infty} \vartheta_{1-2^{-k}} \Delta S_{1-2^{-k}} = \rho_0 + \xi_0 + \sum_{k=1}^{\infty} \xi_k \Delta X_k
\]

\[
= \rho_0 + \xi_0 + \sum_{k=1}^{\infty} \xi_k \Delta X_k \quad \mathbb{P}\text{-a.s.}
\]  

(4.3)

Note that in the above sums, due to the left-continuity of \( S^2 \) in \( t = 1 \), the index \( \infty \) is not part of the summation.

After these technical preparations, we come to the announced counterexample. We claim that the numéraire strategy \( \eta := (1,0) \) is weakly maximal but not strongly maximal (in the classical/limit quantile sense) at time 1 for \( \mathcal{U}(S) \). Note that \( \mathcal{S} = \mathcal{S}(\eta) \).

To establish that \( \eta \) is weakly maximal at time 1, let \( \vartheta \in \mathcal{U}(S) \) be such that

\[
V_0(\vartheta) = V_0(\eta) \quad \text{and} \quad V_1(\vartheta) \geq V_1(\eta) \quad \mathbb{P}\text{-a.s.}
\]  

(4.4)

By Proposition 4.4, it suffices to show that the inequality in (4.4) is an equality. By the representation (4.3), it suffices to show that if \((\xi_k)_{k \in \mathbb{N}_0} \) is a \((\mathcal{G}_k)_{k \in \mathbb{N}_0} \)-predictable process satisfying

\[
\sum_{k=1}^{\infty} \xi_k \Delta X_k \geq 0 \quad \mathbb{P}\text{-a.s.}
\]  

(4.5)

---

\(^1\) A calculation in Mathematica\textsuperscript{©} suggests that \( X_\infty \) is valued in \( \left( \frac{\cosh(\sqrt{\pi}/2)}{3\pi}, \frac{\sinh(\sqrt{\pi})}{2\pi} \right) \approx (0.81, 1.84) \).

---
then the inequality is an equality. So let $(\xi_k)_{k \in \mathbb{N}}$ be a $(\mathcal{G}_k)_{k \in \mathbb{N}_0}$-predictable process satisfying (4.5). We show by induction on $n$ that for all $n \in \mathbb{N}$,

$$\xi_n \Delta X_n = 0 \ \text{P-a.s.} \quad (4.6)$$

**Induction basis** ($n = 1$). First, we show that

$$P[\xi_1 \Delta X_1 \geq 0] = 1. \quad (4.7)$$

Indeed, by the (conditional) independence of $\mathcal{G}_1$ and $\tau$ (given $\{\tau \geq 1\} = \Omega$) and (4.5),

$$P[\xi_1 \Delta X_1 \geq 0] = P[\xi_1 \Delta X_1 \geq 0 | \tau = 1] = P\left[ \sum_{k=1}^{\tau} \xi_k \Delta X_k \geq 0 \bigg| \tau = 1 \right] = 1.$$

Next,

$$\xi_1 \Delta X_1 = \xi_1 (Y_1 - 1) \ \text{P-a.s.}$$

Since $\xi_1$ is $\mathcal{G}_0$-measurable and constant and $Y_1 - 1$ takes positive and negative values both with positive probability, (4.7) first implies $\xi_1 = 0$ and then $\xi_1 \Delta X_1 = 0 \ \text{P-a.s.}$

**Induction step** ($n \to n + 1$). First, we show that

$$P[\xi_{n+1} \Delta X_{n+1} \geq 0] = 1. \quad (4.8)$$

Since $\Delta X_{n+1} = 0$ on $\{\tau \leq n\}$, this is equivalent to showing that

$$P[\xi_{n+1} \Delta X_{n+1} \geq 0 | \tau \geq n + 1] = 1.$$

Indeed, by the induction hypothesis (yielding $\xi_k \Delta X_k = 0 \ \text{P-a.s.}$ for $k = 1, \ldots, n$), conditional independence of $\{\sum_{k=1}^{n+1} \xi_k \Delta X_k \geq 0\} \in \mathcal{G}_{n+1}$ and $\{\tau = n + 1\}$ given $\{\tau \geq n + 1\}$, a standard result on conditional independence (see Kallenberg (2002), Proposition 6.6) and (4.5),

$$P[\xi_{n+1} \Delta X_{n+1} \geq 0 | \tau \geq n + 1] = P\left[ \sum_{k=1}^{n+1} \xi_k \Delta X_k \geq 0 \bigg| \tau \geq n + 1 \right]$$

$$= P\left[ \sum_{k=1}^{n+1} \xi_k \Delta X_k \geq 0 \bigg| \tau \geq n + 1, \tau = n + 1 \right]$$

$$= P\left[ \sum_{k=1}^{n} \xi_k \Delta X_k \geq 0 \bigg| \tau = n + 1 \right]$$

$$= P\left[ \sum_{k=1}^{\tau} \xi_k \Delta X_k \geq 0 \bigg| \tau = n + 1 \right] = 1.$$

Next,

$$\xi_{n+1} \Delta X_{n+1} = \xi_{n+1} X_{n+1} I_{\{\tau \geq n + 1\}} (Y_{n+1} - 1).$$

Since $\xi_{n+1} X_{n+1} I_{\{\tau \geq n + 1\}}$ is $\sigma(X_1, \ldots, X_n, \tau)$-measurable, it is independent of $(Y_{n+1} - 1)$. Given that the latter takes positive and negative values both with positive probability, (4.8) first implies $\xi_{n+1} \Delta X_{n+1} = 0 \ \text{P-a.s.}$ and then $\xi_{n+1} \Delta X_{n+1} = 0 \ \text{P-a.s.}$

To establish that $\eta$ is not strongly maximal (in the classical/limit quantile sense) at time 1, by Proposition 4.16 below it suffices to show that the zero strategy is not strongly maximal (in the classical/limit quantile sense) at time 1.

To show that 0 is not strongly maximal in the limit quantile sense at time 1, we have to find a non-zero contingent claim $F : S \to L^0_\mathcal{U}(\mathcal{F}_1)$ with

$$\Pi^*(F | \mathcal{U}(S)) = 0. \quad (4.9)$$
Consider the contingent claim $F : S \to L^0_\infty(\mathcal{F}_1)$ from Proposition 2.19 satisfying $F(S) = 1_{\{\tau = \infty\}}$. Then $F$ is non-zero since $P[\tau = \infty] > 0$. Let $C : S \to \mathbb{R}_{++}$ be the contingent claim (at time 0) from Proposition 2.19 satisfying $C(S) = 1$, and let $\epsilon \in (0, 1)$ and $\delta > 0$ be arbitrary. Choose $N \in \mathbb{N}$ large enough that $\frac{1}{N+1} \leq \min(\delta, \epsilon)$ and define the investment process $\vartheta^{(N)}$ for $S$ by

$$
\vartheta^{(N)} := \left( \frac{1}{N+1}, 0 \right) I_{[0,1-2^{-(N-1)}]} + \left( \frac{1}{N+1} - (N+1)^2 \frac{(N+1)^2}{X_{N-1}} \right) I_{[1-2^{-(N-1)},1-2^{-N}]} + \left( \frac{1}{N+1} + (N+1)^2 \frac{\Delta X_N}{X_{N-1}}, 0 \right) I_{(1-2^{-N},1]}.
$$

Then $\vartheta^{(N)}$ is a self-financing strategy. Indeed, using that $\Delta S_{1-2^{-N}}^2 = \Delta X_N$, it is easy to check that

$$
\vartheta^{(N)} \cdot S = \begin{cases} 
\frac{1}{N+1} & \text{if } t \in [0,1-2^{-N}), \\
\frac{1}{N+1} + (N+1)^2 \frac{\Delta X_N}{X_{N-1}} & \text{if } t \in [1-2^{-N},1]
\end{cases}
\vartheta^{(N)} \cdot S = \vartheta^{(N)} \cdot S_0.
$$

(4.10)

Next,

$$
\frac{\Delta X_N}{X_{N-1}} = (Y_N - 1) 1_{\{\tau \geq N\}} = \begin{cases} 
0 & \text{if } \tau < N, \\
1/(N+1)^2 & \text{if } \tau \geq N, Y_N = 1 + 1/(N+1)^2, \\
-1/(N+1)^3 & \text{if } \tau \geq N, Y_N = 1 - 1/(N+1)^3.
\end{cases}
$$

This together with (4.10) implies that $\vartheta^{(N)}$ is undefaultable. Moreover, by the definition of $F$, independence of $Y_N$ and $\tau$ and the above,

$$
V_0(\vartheta^{(N)})(S) = \frac{1}{N+1} \leq \delta C(S),
$$

$$
P[V_1(\vartheta^{(N)})(S) \geq F(S)] = P[\tau < \infty] + P[\tau = \infty, Y_N = 1 + 1/(N+1)^2]
\geq P[\tau < \infty] + P[\tau = \infty]P[Y_N = 1 + 1/(N+1)^2]
\geq 1 - \epsilon.
$$

Letting first $\delta \to 0$ shows that $P^*(F|\mathcal{U}(S)) = 0$ for all $\epsilon \in (0,1)$, and letting then $\epsilon \to 0$ gives (4.9).

Finally, in order to show that $0$ is not strongly maximal in the classical sense at time 1, consider $F$ as above. Then by Proposition 2.29 there exists $A \in \mathcal{F}_1$ such that $F := F|A$ is again a non-zero contingent claim and satisfies $\Pi(F|\mathcal{U}(S)) \leq \Pi^*(F|\mathcal{U}(S)) = 0$.

As in the case of weakly maximal strategies, the zero strategy plays a fundamental role when studying strongly maximal strategies.

**Proposition 4.16.** Let $S$ be a market, $\Gamma \subset \mathcal{U}(S)$ a strategy cone and $\tau \in \mathcal{T}_{[0,T]}$ a stopping time. Then $\vartheta \in \Gamma$ is strongly maximal in the classical/limit quantile sense at time $\tau$ only if $0$ is strongly maximal in the classical/limit quantile sense at time $\tau$.

**Proof.** We only establish the classical case, the argument for the limit quantile case being analogous. By contraposition, suppose that $0 \in \Gamma$ is not strongly maximal in the classical sense at time $\tau$ and let $\vartheta \in \Gamma$. Then there exists a non-zero contingent claim $F : S \to L^0_\infty(\mathcal{F}_\tau)$ with $\Pi(F|\Gamma) = 0$. This yields

$$
\Pi(V_\tau(\vartheta + F|\Gamma) \leq \Pi(V_\tau(\vartheta) | \Gamma) + \Pi(F | \Gamma) = \Pi(V_\tau(\vartheta) | \Gamma) \leq V_0(\vartheta).
$$

Thus, $\vartheta$ is not strongly maximal in the classical sense at time $\tau$. \qed
The following example shows that the converse of Proposition 4.16 does not hold.

**Example 4.17.** Consider the setup of Example 4.3. We claim that 0 is strongly maximal in the classical and the limit quantile sense at time 1 for \( \mathcal{U}(S) \) whereas the numéraire strategy \( \eta = (1, 1) \) is not strongly maximal, neither in the classical nor in the limit quantile sense, at time 1 for \( \mathcal{U}(S) \).

The second claim follows immediately from the fact that \( \eta \) is not even weakly maximal at time 1 for \( \mathcal{U}(S) \) (see Example 4.3) and Proposition 4.14.

To establish the first claim, suppose by contradiction that there exists a non-zero contingent claim \( F : S \rightarrow L^0_+(\mathcal{F}_1) \) with \( \Pi^*(F | \mathcal{U}(S)) = 0 \). Then by Proposition 2.29 there exists \( A \in \mathcal{F}_1 \) such that \( \bar{F} := F \mathbb{1}_A \) is again a non-zero contingent claim and satisfies \( \Pi^*(\bar{F} | \mathcal{U}(S)) \leq \Pi^*(F | \mathcal{U}(S)) = 0 \). Choose \( 0 < \delta < \mathbb{E}[\bar{F}(S)] \). Proposition 2.23 with \( C = V_0(\eta) \) gives \( \tilde{\theta} \in \mathcal{U}(S) \) satisfying

\[
V_0(\tilde{\theta}) \leq \delta V_0(\eta) \quad \text{and} \quad V_1(\tilde{\theta}) \geq \bar{F} \text{ P-a.s.}
\]

Since \( S^1, S^2 \) are continuous local \( \mathbb{P} \)-martingales, \( V(\tilde{\theta}) = \tilde{\theta}_0 \cdot S_0 + \tilde{\theta} \bullet S \) is a nonnegative local \( \mathbb{P} \)-martingale and a \( \mathbb{P} \)-supermartingale. Hence,

\[
V_0(\tilde{\theta})(S) \geq \mathbb{E}[V_1(\tilde{\theta})(S)] \geq \mathbb{E}[\bar{F}(S)] > \delta = \delta V_0(\eta) \geq V_0(\tilde{\theta}).
\]

Thus, we arrive at a contradiction.

Next, we show that as in the case of weakly maximal strategies, the property of being a strongly maximal strategy is consistent over time, provided that \( S \) is a numéraire market and \( \Gamma \) contains a numéraire strategy.

**Proposition 4.18.** Let \( S \) be a numéraire market, \( \tau_1 \leq \tau_2 \in \mathcal{T}_{[0,T]} \) stopping times and \( \Gamma \subset \mathcal{U}(S) \) a strategy cone allowing for switching to dominated strategies. If \( \tilde{\theta} \in \Gamma \) is strongly maximal in the classical/limit quantile sense at time \( \tau_2 \), then it is also strongly maximal in the classical/limit quantile sense at time \( \tau_1 \).

**Proof.** We only establish the limit quantile case, the argument for the classical case being analogous. By contraposition, suppose that \( \tilde{\theta} \in \Gamma \) is not strongly maximal in the limit quantile sense at time \( \tau_1 \). Then there exists a non-zero contingent claim \( F_1 : S \rightarrow L^0_+(\mathcal{F}_{\tau_1}) \) (at time \( \tau_1 \)) such that

\[
\Pi^*(V_{\tau_1}(\tilde{\theta}) + F_1 | \Gamma) \leq V_0(\tilde{\theta}).
\]

Let \( \eta \) be a switching numéraire strategy at time \( \tau_1 \) and \( F_2 : S \rightarrow L^0_+(\mathcal{F}_{\tau_2}) \) the contingent claim (at time \( \tau_2 \)) from Proposition 2.19 satisfying

\[
F_2(S^{(\eta)}) = F_1(S^{(\eta)}) \text{ P-a.s.}
\]

Let \( \epsilon \in (0, 1) \), \( \delta > 0 \) and \( C : S \rightarrow \mathbb{R}_{++} \) be a strictly positive contingent claim (at time 0). Then by the proof of Corollary 2.30, there exist \( \tilde{\theta} \in \Gamma \) and \( A \in \mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2} \) with \( \mathbb{P}[A] \geq 1 - \epsilon \) such that

\[
V_0(\tilde{\theta}) \leq V_0(\tilde{\theta}) + \delta C \quad \text{and} \quad V_{\tau_1}(\tilde{\theta}) \geq (V_{\tau_1}(\tilde{\theta}) + F_1) \mathbb{1}_A \text{ P-a.s.}
\]

Define \( \tilde{\theta} \in \Gamma \) by

\[
\tilde{\theta} := \tilde{\theta}_0 \mathbb{1}_{[0, \tau_1]} + \mathbb{1}_{\{V_{\tau_1}(\tilde{\theta}) < V_{\tau_1}(\theta)\}} \tilde{\theta}_1 \text{ } \mathbb{1}_{(\tau_1, T]} + \mathbb{1}_{\{V_{\tau_1}(\tilde{\theta}) \geq V_{\tau_1}(\theta)\}} \left( \tilde{\theta} + V_{\tau_1}(\tilde{\theta} - \tilde{\theta})(S^{(\eta)}) \eta \right) \mathbb{1}_{(\tau_1, T]}.
\]

Then on the one hand,

\[
V_0(\tilde{\theta}) = V_0(\tilde{\theta}) \leq V_0(\tilde{\theta}) + \delta C,
\]

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and on the other hand, using that \( A \cap \{ V_{\tau_1}(\tilde{\theta}) < V_{\tau_1}(\theta) \} \) is a \( \mathbb{P} \)-null set,

\[
V_{\tau_2}(\tilde{\theta})(S^{(n)}) \geq V_{\tau_1}(\tilde{\theta})(S^{(n)}) \mathbb{1}_A \\
= (V_{\tau_2}(\theta)(S^{(n)}) + V_{\tau_1}(\tilde{\theta} - \theta)(S^{(n)})) \mathbb{1}_A \\
\geq (V_{\tau_2}(\theta)(S^{(n)}) + F_1(S^{(n)})) \mathbb{1}_A \\
= (V_{\tau_2}(\theta)(S^{(n)}) + F_2(S^{(n)})) \mathbb{1}_A \ \mathbb{P}\text{-a.s.}
\]

Thus,

\[
\Pi^*(V_{\tau_2}(\theta) + F_2|\Gamma) \leq V_0(\theta) + \delta C.
\]

Letting \( \delta, \epsilon \to 0 \) shows that \( \vartheta \) is not strongly maximal in the limit quantile sense at time \( \tau_2 \).

\textbf{Corollary 4.19.} Let \( S \) be a numéraire market and \( \Gamma \subset \mathcal{U}(S) \) a strategy cone allowing for switching to dominated strategies. Then \( \vartheta \in \Gamma \) is strongly maximal in the classical/limit quantile sense if and only if it is strongly maximal in classical sense/limit quantile sense at time \( T \).

Finally, in the special case \( \Gamma = \mathcal{U}(S) \), we show that any weakly maximal strategy is strongly maximal in the classical sense provided that the zero strategy is strongly maximal in the classical sense.

\textbf{Proposition 4.20.} Let \( S \) be a numéraire market, and suppose that \( 0 \) is strongly maximal in the classical sense for \( \mathcal{U}(S) \). Then every weakly maximal strategy \( \vartheta \in \mathcal{U}(S) \) is strongly maximal in the classical sense, too.

\textit{Proof.} Let \( \vartheta \in \mathcal{U}(S) \) be weakly maximal. Seeking a contradiction, suppose that \( \vartheta \) is not strongly maximal in the classical sense. Then by Corollary 4.19, it is not strongly maximal at time \( T \) and so there exists a non-zero contingent claim \( F : S \to \mathbb{L}_+^0(\mathcal{F}_T) \) with

\[
\Pi^*(V_T(\vartheta) + F|\mathcal{U}(S)) \leq V_0(\vartheta).
\]

Let \( \delta > 0 \) and \( C : S \to \mathbb{R}_+^+ \) be a strictly positive contingent claim (at time 0). Then by Proposition 2.23, there exists \( \tilde{\vartheta} \in \mathcal{U}(S) \) such that

\[
V_0(\tilde{\vartheta}) \leq V_0(\vartheta) + \delta C \quad \text{and} \quad V_T(\tilde{\vartheta}) \geq V_T(\vartheta) + F \ \mathbb{P}\text{-a.s.}
\]

Since \( \vartheta \) is weakly maximal, Proposition 4.9 shows that

\[
\tilde{\vartheta} := \tilde{\vartheta} - \vartheta \in \mathcal{U}(S).
\]

Moreover,

\[
V_0(\tilde{\vartheta}) \leq \delta C \quad \mathbb{P}\text{-a.s.} \quad \text{and} \quad V_T(\tilde{\vartheta}) \geq F \quad \mathbb{P}\text{-a.s.}
\]

But this implies

\[
\Pi^*(F|\mathcal{U}(S)) \leq \delta C \quad \mathbb{P}\text{-a.s.}
\]

Letting \( \delta \to 0 \) shows that \( \Pi^*(F|\mathcal{U}(S)) = 0 \), in contradiction to the hypothesis that \( 0 \) is strongly maximal in the classical sense.

5 Maximal strategies and no-arbitrage

In this section, we introduce six notions of no-arbitrage based upon weakly and strongly maximal strategies. We compare these notions with the no-arbitrage concept \( \text{NGE} \) introduced in Section 3 and with classical notions of no-arbitrage including \( \text{NA} \) and \( \text{NFLVR} \).

First, we introduce three \textit{numéraire independent} notions of no-arbitrage based upon weakly and strongly maximal strategies.
Definition 5.1. Let $S$ be a market. Then $S \in S$ is said to satisfy
- $wm(0)$ if $0$ is weakly maximal for $U(S)$.
- $sm(0)$ if $0$ is strongly maximal in the classical sense for $U(S)$.
- $sm^*(0)$ if $0$ is strongly maximal in the limit quantile sense for $U(S)$.

Remark 5.2. The above notions are numéraire independent; if one of them is satisfied by some $S \in S$, then it is satisfied by all $S \in S$. Moreover, these concepts can be defined and make sense even if $S$ fails to be a numéraire market.

The above notions are closely linked to the no-arbitrage concept NGE introduced in Section 3.

Proposition 5.3. Let $S$ be a numéraire market. Then the following are equivalent:
- (a) $S$ satisfies NGE for $U(S)$.
- (b) $S$ satisfies $sm^*(0)$.
- (c) $S$ satisfies $sm(0)$.

Proof. Recall that in a numéraire market $S$, the strategy cone $U(S)$ allows for switching to numéraires and to dominated strategies. First, assume (a). Then it follows by the definition of NGE that $G_T = \emptyset$, i.e. all non-zero contingent claims $F : S \rightarrow L_+^0(T)$ satisfy $\Pi^*(F | U(S)) > 0$. This means that $0$ is strongly maximal in the limit quantile sense at time $T$ for $U(S)$, and Corollary 4.19 gives (b). The implication (b) $\implies$ (c) follows from Remark 4.12. Finally, assume (c). Then Remark 3.9 implies that $G_T = \emptyset$, and Corollary 3.14 gives (a).

Next, we introduce three numéraire dependent notions of no-arbitrage based again upon weakly and strongly maximal strategies.

Definition 5.4. Let $S$ be a numéraire market and $\eta$ a numéraire strategy. Then $S^{(\eta)}$ is said to satisfy property
- $wm(\eta)$ if $\eta$ is weakly maximal for $U(S)$.
- $sm(\eta)$ if $\eta$ is strongly maximal in the classical sense for $U(S)$.
- $sm^*(\eta)$ if $\eta$ is strongly maximal in the limit quantile sense for $U(S)$.

Remark 5.5. The above notions are numéraire dependent; they only make sense if $S$ is a numéraire market and they only hold for one (numéraire) representative $S^{(\eta)} \in S$.

The following result studies the implication structure between the six notions of no-arbitrage for a numéraire representative $S^{(\eta)}$, and the subsequent remark shows that this is conclusive.

Proposition 5.6. Let $S$ be a numéraire market and $\eta$ a numéraire strategy. Then
$$sm^*(\eta) \iff sm(\eta) \implies wmn(\eta) \downarrow \downarrow \downarrow$$
$$sm^*(0) \iff sm(0) \implies wmn(0).$$

Moreover,
$$\left( wmn(\eta) \& sm(0) \right) \iff sm(\eta).$$

Proof. The implications $sm^*(\eta) \implies sm^*(0)$, $sm(\eta) \implies sm(0)$ and $wmn(\eta) \implies wmn(0)$ follow from Propositions 4.16 and 4.2, respectively. The implications $sm(\eta) \implies wmn(\eta)$ and $sm(0) \implies wmn(0)$ follow from Proposition 4.14. The equivalence $sm^*(0) \iff sm(0)$ follows from Proposition 5.3. The implication $sm^*(\eta) \implies sm(\eta)$ follows from Remark 4.12. The implication $sm(\eta) \implies sm^*(\eta)$ will be proved in Corollary 7.12 below. Moreover, the equivalence $(w(\eta) \& s(0)) \iff s(\eta)$ follows from Proposition 4.20. \qed
Remark 5.7. Examples 4.3 and 4.17 illustrate that $wm(0) \not\Rightarrow wm(\eta)$, $sm(0) \not\Rightarrow sm(\eta)$ and $sm^*(0) \not\Rightarrow sm^*(\eta)$. In addition, Example 4.15 gives $wm(\eta) \not\Rightarrow sm(\eta)$ and $wm(0) \not\Rightarrow sm(0)$.

We proceed to link the above notions of no-arbitrage to classical ones in the literature, including NA and NFLVR which are linked to topological concepts. To recall these concepts, we follow – mutatis mutandis – the notation in Delbaen & Schachermayer (1994) and Kabanov (1997).

Definition 5.8. Let $S$ be a numéraire market and $\eta$ a numéraire strategy. Set

$$K_0(S^{(\eta)}) := \{(\vartheta \bullet S^{(\eta)}): \vartheta \in L^{ad}(S^{(\eta)})\},$$

$$K_0^1(S^{(\eta)}) := \{(\vartheta \bullet S^{(\eta)}): \vartheta \in L^{ad}(S^{(\eta)}, 1)\},$$

$$C(S^{(\eta)}) := \{g \in L^\infty(\mathcal{F}_T): g \leq f \text{ for some } f \in K_0(S^{(\eta)})\}$$

Then $S^{(\eta)}$ is said to satisfy

- NA if $K_0(S^{(\eta)}) \cap L^0_+(\mathcal{F}_T) = \{0\}$,
- BK if $K_0^1(S^{(\eta)})$ is bounded in probability,
- NFLVR if $\overline{C(S^{(\eta)})} \cap L^\infty_+(\mathcal{F}_T) = \{0\}$, where $\overline{C(S^{(\eta)})}$ denotes the closure of $C(S^{(\eta)})$ in the norm topology of $L^\infty(\mathcal{F}_T)$.

Remark 5.9. The above notions are a good generalisation of the corresponding notions in the standard framework. Indeed, if we interpret $S^{(\eta),1}, \ldots, S^{(\eta),d}$ (where $d$ is the dimension of $S$) as $d$ discounted “risky” assets in a standard framework (with riskless asset $S^{(\eta),0} \equiv 1$ in the background), then from a purely mathematical point of view, we recover the classical definitions (see also Remark 2.14).

The classical notions of no-arbitrage are equivalent to some of the numéraire dependent and independent notions introduced above. This kind of result is well known if $\eta := (1, 0, \ldots, 0)$ is a numéraire strategy for $S$ and if one works with its numéraire representative $S^{(\eta)}$. For the convenience of the reader we give a full proof.

Proposition 5.10. Let $S$ be a market and $\eta$ a numéraire strategy. Then $S^{(\eta)}$ satisfies

(a) NA if and only if it satisfies $wm(\eta)$.
(b) BK if and only if it satisfies $sm(0)$.
(c) NFLVR if and only if it satisfies $sm(\eta)$.

Proof. (a) We argue by contraposition. First, assume that $S^{(\eta)}$ fails $wm(\eta)$. Then by Corollary 4.8 and Proposition 4.4, there exists a strategy $\tilde{\vartheta} \in \mathcal{U}(S)$ such that

$$V_0(\tilde{\vartheta}) = V_0(\eta),$$

$$V_T(\tilde{\vartheta}) \geq V_T(\eta) \text{ P-a.s. and } \mathbb{P}[V_T(\tilde{\vartheta}) > V_T(\eta)] > 0.$$ 

Set $\zeta := (\tilde{\vartheta} - \eta)1_{[0,T]}$. Since $V(\eta)(S^{(\eta)}) \equiv 1$, we have $\zeta \in L^{ad}(S^{(\eta)})$ and

$$\zeta \bullet S_T^{(\eta)} = V_T(\tilde{\vartheta} - \eta)(S^{(\eta)}) \in K_0(S^{(\eta)}) \cap (L^0_+(\mathcal{F}_T) \setminus \{0\}).$$

Thus $S^{(\eta)}$ fails NA.

Conversely, assume that $S^{(\eta)}$ fails NA. Then there exists $\zeta \in L^{ad}(S^{(\eta)})$ such that

$$\zeta \bullet S_T^{(\eta)} \geq 0 \text{ P-a.s. and } \mathbb{P}[\zeta \bullet S_T^{(\eta)} > 0] > 0.$$
By Proposition 2.15, there exist \( \tilde{\vartheta} \in \mathcal{U}(S) \) and \( a \geq 0 \) such that
\[
V(\tilde{\vartheta} - a\eta)(S^{(n)}) = \zeta \cdot S^{(n)} \quad \mathbb{P}\text{-a.s.}
\] (5.1)

Set \( \tilde{\vartheta} = \frac{1}{a} \vartheta \in \mathcal{U}(S) \). Then \( \zeta \cdot S^{(n)}_0 = 0 \) and \( \zeta \cdot S^{(n)}_T \geq 0 \) \( \mathbb{P}\text{-a.s.} \) together with (5.1) give
\[
V_0(\tilde{\vartheta}) = V_0(\eta), \quad V_T(\tilde{\vartheta}) \geq V_T(\eta) \quad \mathbb{P}\text{-a.s.} \quad \text{and} \quad \mathbb{P}[V_T(\tilde{\vartheta}) > V_T(\eta)] > 0.
\]
Thus \( S^{(n)} \) fails \( sm(\eta) \).

(b) Again, we argue by contraposition. First, suppose that \( S^{(n)} \) fails \( sm(0) \). By Corollary 4.19, there exists a non-zero contingent claim \( F : S \to \mathbf{L}^1_0(\mathcal{F}_T) \) with \( \mathbb{P}(F|\mathcal{U}(S)) = 0 \). By Proposition 2.23 with \( C = V_0(\eta) \), for all \( n \in \mathbb{N} \), there exists \( \vartheta^{(n)} \in \mathcal{U}(S) \) such that
\[
V_0(\vartheta^{(n)}) \leq \frac{1}{n} V_0(\eta) \quad \text{and} \quad V_T(\vartheta^{(n)}) \geq F \quad \mathbb{P}\text{-a.s.}
\]
Since \( \mathbb{P}[F(S^{(n)}) > 0] > 0 \), there exist \( \varepsilon > 0 \) and \( N \in \mathbb{N} \) such that \( \mathbb{P}[NF(S^{(n)}) \geq 2 \geq \varepsilon] \). Define
\[
\zeta^{(n)} := (nN \vartheta^{(n)} - \eta) \mathbf{1}_{[0,T]} \quad \text{and} \quad g_n := (\zeta^{(n)} \cdot S^{(n)})_T, \quad n \in \mathbb{N}.
\]
Then for \( n \in \mathbb{N} \),
\[
\begin{align*}
\zeta^{(n)} \cdot S^{(n)} &= V(nN \vartheta^{(n)} - \eta)(S^{(n)}) - V_0(nN \vartheta^{(n)} - \eta)(S^{(n)}) \\
&\geq V(nN \vartheta^{(n)} - \eta)(S^{(n)}) - V(\eta)(S^{(n)}) = -1 \quad \mathbb{P}\text{-a.s.}, \\
\zeta^{(n)} \cdot S^{(n)}_T &= V_T(nN \vartheta^{(n)} - \eta)(S^{(n)}) \geq nNF(S^{(n)}) - 1 \quad \mathbb{P}\text{-a.s.}
\end{align*}
\]
In particular, for each \( n \in \mathbb{N} \), \( g_n \in K^1_0(S^{(n)}) \) and
\[
\mathbb{P}[g_n \geq n] = \mathbb{P}[\zeta^{(n)} \cdot S^{(n)}_T \geq n] \geq \mathbb{P}[nNF(S^{(n)}) - 1 \geq n] \\
\geq \mathbb{P}[nNF(S^{(n)}) \geq 2n] = \mathbb{P}[NF(S^{(n)}) \geq 2] \geq \varepsilon.
\]
Thus, the sequence \( \{g_n\}_{n \in \mathbb{N}} \) in \( K^1_0(S^{(n)}) \) is not bounded in probability, and \( S^{(n)} \) fails BK.

Conversely, assume that \( S^{(n)} \) fails BK. Then there exist \( \varepsilon > 0 \) and a sequence \( \{g_n\}_{n \in \mathbb{N}} \) in \( K^1_0(S^{(n)}) \) satisfying \( \mathbb{P}[g_n \geq n - 1] \geq \varepsilon \). By the definition of \( K^1_0(S^{(n)}) \), for all \( n \in \mathbb{N} \), there exists \( \zeta^{(n)} \in L^a|_{S^{(n)}}, 1 \) such that \( g_n = \zeta^{(n)} \cdot S^{(n)}_T \). Hence by Proposition 2.15, for all \( n \in \mathbb{N} \), there exists \( \vartheta^{(n)} \in \mathcal{U}(S) \) such that
\[
V(\vartheta^{(n)} - \eta)(S^{(n)}) = \zeta^{(n)} \cdot S^{(n)} \quad \mathbb{P}\text{-a.s.}
\]
For \( n \in \mathbb{N} \), set \( h_n := \frac{g_n}{n} + \frac{1}{n} \). Then for all \( n \in \mathbb{N} \),
\[
V_T(\vartheta^{(n)})(S^{(n)}) = \frac{1}{n} V_T(n\vartheta^{(n)} - \eta)(S^{(n)}) + \frac{1}{n} = \frac{g_n}{n} + \frac{1}{n} = h_n,
\]
\[
\mathbb{P}[h_n \geq 1] = \mathbb{P}[g_n \geq n - 1] \geq \varepsilon.
\]
By a convexification argument (see Delbaen & Schachermayer (2006), Lemma 9.8.1), there exists a sequence \( \{\tilde{h}_n\}_{n \in \mathbb{N}} \) with \( \tilde{h}_n \in \text{conv}\{h_n, h_{n+1}, \ldots\} \) which converges \( \mathbb{P}\)-a.s. to some \([0, \infty]\)-valued \( \mathcal{F}_T\)-measurable random variable \( \tilde{h} \) satisfying \( \mathbb{P}[\tilde{h} > 0] > 0 \). By Egorov’s theorem, there exists \( h \in \mathbf{L}^1_0(\mathcal{F}_T) \setminus \{0\} \) such that \( \tilde{h}_n \geq h \) \( \mathbb{P}\)-a.s. for all sufficiently large \( n \in \mathbb{N} \).

Let \( \delta > 0 \) be given. Choose \( N \in \mathbb{N} \) large enough that \( \frac{1}{N} \leq \delta \) and \( \tilde{h}_N \geq h \). Then there exist indices \( n_1 < \cdots < n_k \in \{N, N + 1, \ldots\} \) and nonnegative weights \( \alpha_1, \ldots, \alpha_k \) summing to 1 such that \( \tilde{h}_N = \sum_{j=1}^k \alpha_j h_n, \) \( \mathbb{P}\)-a.s. Define
\[
\tilde{\vartheta}^N := \sum_{j=1}^k \alpha_j \vartheta^{(n_j)} \in \mathcal{U}(S).
\]
Then,
\[
V_0(\tilde{\theta}^{(N)}) = \sum_{j=1}^{k} \alpha_j V_0(\tilde{\theta}^{(\eta_j)}) = \sum_{j=1}^{k} \alpha_j \frac{1}{N} V_0(\eta) \leq \sum_{j=1}^{k} \frac{1}{N} V_0(\eta) = \frac{1}{N} V_0(\eta) \leq \delta V_0(\eta),
\]
\[
V_T(\tilde{\theta}^{(N)})(S^{(\eta)}) = \sum_{j=1}^{k} \alpha_j V_T(\tilde{\theta}^{(\eta_j)})(S^{(\eta)}) = \sum_{j=1}^{h} \alpha_j h_{n_j} = \hat{h}_N \geq h \text{ P-a.s.}
\]

Let \( F : \mathcal{S} \to L^0_{+}(\mathcal{F}_T) \) be the contingent claim from Proposition 2.19 satisfying \( F(S^{(\eta)}) = h \text{ P-a.s.} \)

Then,
\[
\Pi(F | \mathcal{U}(S)) \leq \delta V_0(\eta).
\]

Letting \( \delta \to 0 \) shows that \( S^{(\eta)} \) fails \( sm(0) \).

(c) This follows from parts (a) and (b) by the fact that \( s(\eta) \) is equivalent to \( (wm(\eta) \& sm(0)) \)
by Proposition 5.6 and that NFLVR is equivalent to \( (NA \& BK) \) by Lemma 2.2 in Kabanov (1997).

\[\square\]

**Remark 5.11.** The above result shows in particular that the most standard no-arbitrage notions NA and NFLVR are numéraire dependent. This is not surprising in view of the fact that the definitions of NA and NFLVR involve the chosen numéraire via the notion of admissible investment processes. By contrast, the BK property is numéraire independent, which is at first glance not obvious.

## 6 Dominating maximal strategies

In this section, we prove that if the zero strategy is strongly maximal in the classical sense for \( \mathcal{U}(S) \) in a numéraire market \( S \), then for every \( \vartheta \in \mathcal{U}(S) \), there exists a strategy \( \vartheta^* \in \mathcal{U}(S) \) which is strongly maximal in the classical sense and whose value process equals the value process of \( \vartheta \) at time 0 and dominates it at time \( T \). In particular, for any criterion which is monotonic with respect to final wealth, it is enough in an NGE market to optimise over strongly maximal strategies.

**Definition 6.1.** Let \( S \) be a numéraire market and \( \vartheta \in \mathcal{U}(S) \). A strategy \( \vartheta^* \in \mathcal{U}(S) \) which is strongly maximal in the classical sense for \( \mathcal{U}(S) \) and satisfies
\[
V_0(\vartheta^*) = V_0(\vartheta) \quad \text{and} \quad V_T(\vartheta^*) \geq V_T(\vartheta) \text{ P-a.s.}
\]
is called a dominating maximal strategy for \( \vartheta \).

**Remark 6.2.** We shall see below that a dominating maximal strategy is even strongly maximal in the limit quantile sense (see Lemma 7.11).

**Theorem 6.3.** Let \( S \) be a numéraire market and \( \vartheta \in \mathcal{U}(S) \). Suppose that 0 is strongly maximal in the classical sense for \( \mathcal{U}(S) \). Then there exists a dominating maximal strategy \( \vartheta^* \in \mathcal{U}(S) \) for \( \vartheta \).

The proof of Theorem 6.3 strongly relies upon arguments used in Delbaen & Schachermayer (1994) and Kabanov (1997). Therefore we reformulate Theorem 6.3 in the language of admissible investment processes.

**Lemma 6.4.** Let \( S \) be a numéraire market, \( \eta \) a numéraire strategy and \( \zeta \in L^a(S^{(\eta)}, 1) \). Suppose that \( S^{(\eta)} \) satisfies property BK. Then there exists \( \zeta^* \in L^a(S^{(\eta)}, 1) \) such that
\[
\zeta^* \cdot S_T \geq \zeta \cdot S_T \text{ P-a.s.}
\]

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and there is no \( \zeta \in L^0(S^{(n)},1) \) satisfying

\[
\zeta \bullet S_T \geq \zeta^* \bullet S_T \quad \text{P.-a.s. and } \quad \mathbb{P} [ \zeta \bullet S_T > \zeta^* \bullet S_T ] > 0.
\]

In order to see that Theorem 6.3 and Lemma 6.4 are indeed equivalent, first note that by Definition 5.1 and Proposition 6.10 (b), the hypothesis of Theorem 6.3 that 0 is strongly maximal in the classical sense for \( U(S) \) is equivalent to the hypothesis of Lemma 6.4 that \( S^{(n)} \) satisfies property BK. Next, note that by Proposition 4.20, it suffices to show in Theorem 6.3 that \( \vartheta^* \) is weakly maximal. Finally, note that by Proposition 2.15, we can identify \( \zeta \) and \( \zeta^* \) in Lemma 6.3 with \( \vartheta - \eta \) and \( \vartheta^* - \eta \), where \( \vartheta \) and \( \vartheta^* \) are as in Theorem 6.3.

The idea for proving Lemma 6.4 is to start with \( \zeta \) and to “improve” it successively, yielding a sequence \( (\zeta_n)_{n \in \mathbb{N}} \) in \( L^0(S^{(n)},1) \) such that for all \( n \), \( \zeta_n \) dominates \( \zeta \), and as \( n \) grows, \( \zeta_n \) becomes more and more “similar” to a weakly maximal strategy. The hope is then that the sequence \( (\zeta_n)_{n \in \mathbb{N}} \) “converges” to some strategy \( \zeta^* \in L^0(S^{(n)},1) \) which is weakly maximal and dominates \( \zeta \).

First, we make the idea of successively “improving” the strategy \( \zeta \) precise.

**Definition 6.5.** Let \( S \) be a numéraire market, \( \eta \) a numéraire strategy and \( \zeta \in L^0(S^{(n)},1) \). A sequence \( (\zeta_n)_{n \in \mathbb{N}} \) in \( L^0(S^{(n)},1) \) is called a maximalising sequence for \( \zeta \) if for all \( n \in \mathbb{N} \),

1. \( \zeta_{n+1} \bullet S_T^{(n)} \geq \zeta_n \bullet S_T^{(n)} \geq \zeta \bullet S_T^{(n)} \) \( \text{P.-a.s.} \)
2. there is no \( \zeta_n \in L^0(S^{(n)},1) \) satisfying

\[
\zeta_n \bullet S_T^{(n)} \geq \zeta_n \bullet S_T^{(n)} \quad \text{P.-a.s. and } \quad \mathbb{P} [ \zeta_n \bullet S_T^{(n)} > \zeta_n \bullet S_T^{(n)} + 2^{-n} ] \geq 2^{-n}.
\]

Next, we show that the good notion of convergence for a maximalising sequence \( (\zeta_n)_{n \in \mathbb{N}} \) for \( \zeta \) is the convergence of the sequence of integral processes \( (\zeta_n \bullet S)_{n \in \mathbb{N}} \) in the semimartingale topology. For the definition and properties of the latter, we refer to Mémin (1980).

**Proposition 6.6.** Let \( S \) be a numéraire market, \( \eta \) a numéraire strategy, \( \zeta \in L^0(S^{(n)},1) \) and \( (\zeta_n)_{n \in \mathbb{N}} \) in \( L^0(S^{(n)},1) \) a maximalising sequence for \( \zeta \). Suppose that the sequence of integral processes \( (\zeta_n \bullet S^{(n)})_{n \in \mathbb{N}} \) converges in the semimartingale topology to some càdlàg adapted process \( \Xi \). Then there exists \( \zeta^* \in L^0(S^{(n)},1) \) such that \( \Xi = \zeta^* \bullet S^{(n)} \) and

1. \( \zeta^* \bullet S_T^{(n)} \geq \zeta \bullet S_T^{(n)} \) \( \text{P.-a.s.} \)
2. there is no \( \zeta \in L^0(S^{(n)},1) \) satisfying

\[
\zeta \bullet S_T^{(n)} \geq \zeta^* \bullet S_T^{(n)} \quad \text{P.-a.s. and } \quad \mathbb{P} [ \zeta \bullet S_T^{(n)} > \zeta^* \bullet S_T^{(n)} ] > 0.
\]

**Proof.** First, by Mémin’s theorem (see Mémin (1980), Theorem V.4), there exists \( \zeta^* \in L(S^{(n)}) \) with \( \zeta_0^* = 0 \) such that \( \Xi = \zeta^* \bullet S^{(n)} \) \( \text{P.-a.s.} \). Since all the \( \zeta_n \) are in \( L^0(S^{(n)},1) \) and convergence in the semimartingale topology implies uniform convergence in probability (on any finite time horizon), we have \( \Xi \geq -1 \) \( \text{P.-a.s.} \), which implies \( \zeta^* \in L^0(S^{(n)},1) \). Next, convergence in the semimartingale topology implies convergence in probability at time \( T \), and since the sequence \( (\zeta_n \bullet S_T^{(n)})_{n \in \mathbb{N}} \) is in addition increasing, it follows that for all \( n \in \mathbb{N} \),

\[
\zeta^* \bullet S_T^{(n)} \geq \zeta_n \bullet S_T^{(n)} \geq \zeta \bullet S_T \quad \text{P.-a.s.}
\]

(6.1)

Finally, seeking a contradiction, suppose that there exists \( \zeta \in L^0(S^{(n)},1) \) such that

\[
\zeta \bullet S_T^{(n)} \geq \zeta^* \bullet S_T^{(n)} \quad \text{P.-a.s. and } \quad \mathbb{P} [ \zeta \bullet S_T^{(n)} > \zeta^* \bullet S_T^{(n)} ] > 0.
\]

Then there exists \( N \in \mathbb{N} \) such that

\[
\mathbb{P} [ \zeta \bullet S_T^{(n)} \geq \zeta^* \bullet S_T^{(n)} + 2^{-N} ] \geq 2^{-N}.
\]
Since $\zeta^* \cdot S_T^{(n)} \geq \zeta_N \cdot S_T^{(n)}$ P-a.s. by (6.1), we have

$$\zeta^* \cdot S_T^{(n)} \geq \zeta_N \cdot S_T^{(n)} \quad \text{P-a.s.} \quad \text{and} \quad \mathbb{P}[\zeta^* \cdot S_T^{(n)} \geq \zeta_N \cdot S_T^{(n)} + 2^{-N}] \geq 2^{-N}.$$ 

This contradicts the hypothesis that $(\zeta_n)_{n \in \mathbb{N}}$ is a maximising sequence for $\zeta$. \hfill \Box

We proceed to prove Lemma 6.4, postponing three technical results which will be proved afterwards.

**Proof of Lemma 6.4.** Firstly, by Proposition 6.7 (a) below, there exists a maximising sequence $(\zeta_n)_{n \in \mathbb{N}}$ in $L^{ad}(S^{(0)}, 1)$ for $\zeta$.

Secondly, Proposition 6.8 below yields a subsequence, called again $(\zeta_n)_{n \in \mathbb{N}}$, such that the sequence of integral processes $(\zeta_n \cdot S^{(n)})_{n \in \mathbb{N}}$ converges P-a.s. in the uniform topology to some càdlàg adapted process $\Xi$.

Thirdly, Proposition 6.9 below gives a sequence $(\tilde{\zeta}_n)_{n \in \mathbb{N}}$ with $\tilde{\zeta}_n \in \text{conv}\{\zeta_n, \zeta_{n+1}, \ldots\}$ such that the sequence of integral processes $(\tilde{\zeta}_n \cdot S^{(n)})_{n \in \mathbb{N}}$ converges to $\Xi$ in the semimartingale topology.

Finally, by Proposition 6.7 (b) below, $(\tilde{\zeta}_n)_{n \in \mathbb{N}}$ has a subsequence, called again $(\tilde{\zeta}_n)_{n \in \mathbb{N}}$, which is a maximising sequence for $\zeta$. Then the claim follows from Proposition 6.6. \hfill \Box

The following three propositions contain the technical results which where postponed in the proof of Lemma 6.4

**Proposition 6.7.** Let $S$ be a numéraire market, $\eta$ a numéraire strategy, $\zeta_0 \in L^{ad}(S^{(0)}, 1)$, and suppose that $S^{(0)}$ satisfies BK. Then:

(a) There exists a maximising sequence for $\zeta_0$.

(b) If $(\zeta_n)_{n \in \mathbb{N}}$ is a maximising sequence for $\zeta_0$ and $(\tilde{\zeta}_n)_{n \in \mathbb{N}}$ is any other sequence satisfying $\tilde{\zeta}_n \in \text{conv}\{\zeta_n, \zeta_{n+1}, \ldots\}$ for all $n \in \mathbb{N}$, then $(\zeta_n)_{n \in \mathbb{N}}$ has a subsequence which is a maximising sequence for $\zeta_0$.

**Proof.** (a) First, for any $\zeta \in L^{ad}(S^{(0)}, 1)$ and $n \in \mathbb{N}$, set

$$B^{(n)}(\zeta) := \left\{ \tilde{\zeta} \in L^{ad}(S^{(0)}, 1) : \tilde{\zeta} \cdot S_T^{(n)} \geq \zeta \cdot S_T^{(n)} \quad \text{P-a.s.} \right\},$$

$$B^{(n)}_n(\zeta) := \left\{ \tilde{\zeta} \in B^{(n)}(\zeta) : \mathbb{P}[\tilde{\zeta} \cdot S_T^{(n)} \geq \zeta \cdot S_T^{(n)} + 2^{-n}] \geq 2^{-n} \right\}.$$ 

Note that $\zeta \in B^{(0)}(\zeta_0) \neq \emptyset$ for all $\zeta \in L^{ad}(S^{(0)}, 1)$. It is straightforward to check that a sequence $(\zeta_n)_{n \in \mathbb{N}}$ in $L^{ad}(S^{(0)}, 1)$ is a maximising sequence for $\zeta_0$ if and only if for all $n \in \mathbb{N}$,

$$\zeta_n \in B^{(n)}(\zeta_{n-1}) \quad \text{and} \quad B^{(n)}_n(\zeta_n) = \emptyset. \quad (6.2)$$

Next, for any $\zeta \in L^{ad}(S^{(0)}, 1)$ and $n \in \mathbb{N}$, set

$$A^{(n)}_n(\zeta) := \left\{ \tilde{\zeta} \in B^{(n)}(\zeta) : B^{(n)}_n(\tilde{\zeta}) = \emptyset \right\}.$$ 

We claim that $A^{(n)}_n(\zeta) \neq \emptyset$ for all $\zeta \in L^{ad}(S^{(0)}, 1)$ and all $n \in \mathbb{N}$. By way of contradiction, suppose there exist $\tilde{\zeta}_0 \in L^{ad}(S^{(0)}, 1)$ and $N \in \mathbb{N}$ such that $A^{(n)}_n(\tilde{\zeta}_0) = \emptyset$. First, we construct by induction a sequence $(\tilde{\zeta}_n) \in L^{ad}(S^{(0)}, 1)$ satisfying

$$A^{(n)}_n(\tilde{\zeta}_n) = \emptyset \quad \text{and} \quad \tilde{\zeta}_n \in B^{(n)}_n(\tilde{\zeta}_{n-1}), \quad n \in \mathbb{N}. \quad (6.3)$$

**Induction basis** ($n = 1$). Since $A^{(0)}_N(\tilde{\zeta}_0) = \emptyset$ and $\tilde{\zeta}_0 \in B^{(0)}(\tilde{\zeta}_0)$, there exists $\tilde{\zeta}_1 \in B^{(0)}_N(\tilde{\zeta}_0) \subset B^{(0)}(\tilde{\zeta}_0)$. This yields $B^{(0)}(\tilde{\zeta}_1) \subset B^{(0)}(\tilde{\zeta}_0)$ and therefore $A^{(0)}_N(\tilde{\zeta}_1) \subset A^{(0)}_N(\tilde{\zeta}_0) = \emptyset$.

**Induction step** ($n \to n + 1$). Since $A^{(n)}_N(\tilde{\zeta}_n) = \emptyset$ by induction hypothesis, and $\tilde{\zeta}_n \in B^{(n)}(\tilde{\zeta}_n)$, arguing as in the induction basis yields $\tilde{\zeta}_{n+1} \in B^{(n)}_N(\tilde{\zeta}_n)$ satisfying $A^{(n)}_N(\tilde{\zeta}_{n+1}) = \emptyset$.  

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Next, for $n \in \mathbb{N}$, set $a_n := \tilde{\zeta}_n \cdot S_T^{(n)} - \tilde{\zeta}_{n-1} \cdot S_T^{(n)}$ and $A := \limsup_{n \to \infty} \{a_n \geq 2^{-N}\}$. Then $P[A] \geq 2^{-N}$ by elementary probability theory since $P[a_n \geq 2^{-N}] \geq 2^{-N}$ for all $n \in \mathbb{N}$ because $\tilde{\zeta}_n \in B_N^u(\tilde{\zeta}_{n-1})$. In addition,

$$
\lim_{n \to \infty} \tilde{\zeta}_n \cdot S_T^{(n)} = \tilde{\zeta}_0 \cdot S_T^{(n)} + \sum_{k=1}^{\infty} a_k \geq -1 + \infty = \infty \text{ P-a.s. on } A.
$$

But this implies that $S^{(n)}$ fails BK, and we arrive at a contradiction.

Finally, using that $A'_n(\zeta) \neq \emptyset$ for all $\zeta \in L^d(S^{(n)}, 1)$ and all $n \in \mathbb{N}$, we can recursively construct a sequence $(\zeta_n)_{n \in \mathbb{N}} \in L^d(S^{(n)}, 1)$ such that $\zeta_n \in A'_n(\zeta_{n-1})$ for all $n \in \mathbb{N}$. Thus $(\zeta_n)_{n \in \mathbb{N}}$ satisfies (6.2) and therefore is a maximalising sequence for $\zeta_0$.

(b) For $n \in \mathbb{N}$, define

$$h(n) := \min \{k \geq n : \tilde{\zeta}_n \in \text{conv}\{\zeta_n, \ldots, \zeta_k\}\}.
$$

Set $n_1 := 1$ and define recursively $n_{k+1} := h(n_k) + 1$. Since the sequence $(\zeta_n \cdot S_T^{(n)})_{n \in \mathbb{N}}$ is P-a.s. increasing and $\zeta_n \in \text{conv}\{\zeta_n, \zeta_{n+1}, \ldots\}$, it follows on the one hand that for all $n \in \mathbb{N}$,

$$
\tilde{\zeta}_n \cdot S_T^{(n)} \geq \zeta_n \cdot S_T^{(n)} \geq \zeta \cdot T \text{ P-a.s.,}
$$

(6.4)

and on the other hand, by the definition of $h$, that for all $k \in \mathbb{N}$,

$$
\tilde{\zeta}_{n_{k+1}} \cdot S_T^{(n)} \geq \zeta_{h(n_{k+1})} \cdot S_T^{(n)} \geq \tilde{\zeta}_{n_k} \cdot S_T^{(n)} \text{ P-a.s.}
$$

(6.5)

Since $(\zeta_{n_k})_{k \in \mathbb{N}}$ is a fortiori a maximalising sequence for $\zeta$, (6.4) and (6.5) imply that $(\tilde{\zeta}_{n_k})_{k \in \mathbb{N}}$ is a maximalising sequence for $\zeta$, too. □

**Proposition 6.8.** Let $S$ be a numéraire market, $\eta$ a numéraire strategy, $\zeta \in L^d(S^{(n)}, 1)$ and $(\zeta_n)_{n \in \mathbb{N}}$ in $L^d(S^{(n)}, 1)$ a maximalising sequence for $\zeta$. Then the sequence of integral processes $(\zeta_n \cdot S_T^{(n)})_{n \in \mathbb{N}}$ converges uniformly in probability to some adapted càdlàg process $\Xi$.

**Proof.** Since on a finite time horizon, the space of adapted càdlàg processes with the topology of uniform convergence in probability is complete (see Emery (1979), Proposition 2), it suffices to show that the sequence $(\zeta_n \cdot S^{(n)})_{n \in \mathbb{N}}$ is uniformly in $t \in [0, T]$ a Cauchy sequence in probability. So let $\varepsilon > 0$ be given and choose $N \in \mathbb{N}$ large enough that $\varepsilon > 2^{-N}$. It suffices to show that for all $m, n \geq N$,

$$
P \left[ \sup_{0 \leq t \leq T} \left| \zeta_n \cdot S_T^{(n)} - \zeta_m \cdot S_T^{(n)} \right| \geq \varepsilon \right] < 2\varepsilon.
$$

Seeking a contradiction, suppose there exist $m, n \geq N$ such that (after possibly exchanging $m$ and $n$)

$$
P \left[ \sup_{0 \leq t \leq T} \left( \zeta_n \cdot S_T^{(n)} - \zeta_m \cdot S_T^{(n)} \right) \geq \varepsilon \right] \geq \varepsilon.
$$

Define the stopping time

$$
\tau := \inf \left\{ t \in [0, T] : \zeta_n \cdot S_T^{(n)} - \zeta_m \cdot S_T^{(n)} \geq \varepsilon \right\}.
$$

Then $P[\tau < \infty] \geq \varepsilon \geq 2^{-N},$ and if we set $\tilde{\zeta} := \zeta_n \mathbb{I}_{[0, \varepsilon]} + \zeta_m \mathbb{I}_{(\tau, T]}$, then P-a.s. for all $0 \leq t \leq T$,

$$
\tilde{\zeta} \cdot S_T^{(n)} = \zeta_n \cdot S_T^{(n)} \mathbb{I}_{[\tau \geq t]} + \left( (\zeta_n - \zeta_m) \cdot S_T^{(n)} + \zeta_m \cdot S_T^{(n)}\right) \mathbb{I}_{(\tau < t)}
$$

$$
\geq -1 \times \mathbb{I}_{[\tau \geq t]} + (\varepsilon - 1) \times \mathbb{I}_{(\tau < t)} \geq -1.
$$

This shows that $\tilde{\zeta} \in L^d(S^{(n)}, 1)$. In addition, since $(\zeta_n \cdot S_T^{(n)})_{n \in \mathbb{N}}$ is P-a.s. increasing,

$$
(\tilde{\zeta} \cdot S_T^{(n)})_T \geq (\zeta_N \cdot S_T^{(n)})_T + \varepsilon \mathbb{I}_{[\tau < \infty]} \geq (\zeta_N \cdot S_T^{(n)})_T + 2^{-N} \mathbb{I}_{[\tau < \infty]} \text{ P-a.s.}
$$

This contradicts the hypothesis that $(\zeta_n)_{n \in \mathbb{N}}$ is a maximalising sequence for $\zeta$. □
The final technical result is unlike the two previous ones mathematically very deep. It has been established in this form² by Kabanov (1997).

**Proposition 6.9.** Let $S$ be a numéraire market, $\eta$ a numéraire strategy and $(\xi_n)_{n \in \mathbb{N}}$ a sequence in $L^d(S^{(n)}, 1)$ with $(\xi_n \cdot S^{(n)})_{n \in \mathbb{N}}$ converging $\mathbb{P}$-a.s. uniformly in $t \in [0, T]$ to some adapted càdlàg process $\Xi$. Suppose that $S^{(n)}$ satisfies BK. Then there exists a sequence $(\hat{\xi}_n)_{n \in \mathbb{N}}$ with $\hat{\xi}_n \in \text{conv}\{\xi_n, \xi_{n+1}, \ldots\}$, such that $(\hat{\xi}_n \cdot S^{(n)})_{n \in \mathbb{N}}$ converges to $\Xi$ in the semimartingale topology.

**Remark 6.10.** Note that in the above result we do not assume that $(\xi_n)_{n \in \mathbb{N}}$ is a maximalising sequence for some $\xi \in L^d(S^{(n)}, 1)$.

**Proof.** First, set

$$\xi := \sup_{n \in \mathbb{N}} \sup_{0 \leq t \leq T} |(\xi_n \cdot S^{(n)})_t|.$$ 

Since $(\xi_n \cdot S)_{n \in \mathbb{N}}$ converges $\mathbb{P}$-a.s. to $\Xi$ uniformly in $t \in [0, T]$, since $\Xi$ has $\mathbb{P}$-a.s. càdlàg paths, and since suprema of càdlàg functions on compact intervals are finite, it follows that $\xi < \infty$ $\mathbb{P}$-a.s. Define the probability measure $\mathbb{Q} \approx \mathbb{P}$ on $\mathcal{F}_T$ by

$$d\mathbb{Q} := \frac{\exp(-\xi)}{\mathbb{E}_\mathbb{P}[\exp(-\xi)]}d\mathbb{P}.$$ 

Then $\xi \in L^2(\mathcal{F}_T, \mathbb{Q})$. Observing that the BK property, the topology of uniform convergence on $[0, T]$ in probability and the semimartingale topology are all invariant under an equivalent change of measure, we see that all hypotheses of Lemma 2.8 in Kabanov (1997) are fulfilled, setting there $X^n := \xi_n \cdot S^{(n)}$ and working under $\mathbb{Q}$ instead of $\mathbb{P}$. Thus we get a sequence $(\hat{\xi}_n)_{n \in \mathbb{N}}$ with $\hat{\xi}_n \in \text{conv}\{\xi_n, \xi_{n+1}, \ldots\}$ such that the sequence $(\hat{\xi}_n)_{n \in \mathbb{N}}$ converges in the semimartingale topology, where $\hat{\xi}_n$ denotes the local $\mathbb{Q}$-martingale part of the special $\mathbb{Q}$-semimartingale $\xi_n \cdot S^{(n)}$.

Finally, since $(\xi_n \cdot S^{(n)})_{n \in \mathbb{N}}$ converges $\mathbb{P}$-a.s. and hence $\mathbb{Q}$-a.s. to $\Xi_T$, the same is true for $(\hat{\xi}_n \cdot S_T)_{n \in \mathbb{N}}$. Therefore, all hypotheses of Lemma 3.3 in Kabanov (1997) are satisfied, setting there $h_0 := \Xi_T \cdot X^n := \hat{\xi}_n \cdot S^{(n)}$ and working under $\mathbb{Q}$ instead of $\mathbb{P}$. Thus, we may conclude that the sequence $(\hat{\xi}_n \cdot S^{(n)})_{n \in \mathbb{N}}$ converges in the semimartingale topology to $\Xi$. \hfill $\square$

### 7 No gratis events markets

In this section, we study numéraire markets which satisfy the no gratis events (NGE) condition for (a subcone of) undefaultable strategies (see Section 3). We prove that a numéraire market $S$ satisfies NGE for $\mathcal{U}(S)$ if and only if there exists a representative $S \in \mathcal{S}$ which is a $\sigma$-martingale under the physical measure $\mathbb{P}$. Moreover, we show that in this case there exist “sufficiently many” numéraire representatives which satisfy NFLVR and have equivalent $\sigma$-martingale measures. This result is a numéraire independent version of the celebrated *fundamental theorem of asset pricing* by Delbaen & Schachermayer (1998), Theorem 1.1. We proceed to establish that all the different notions of maximal strategies are equivalent in numéraire markets satisfying NGE for $\mathcal{U}(S)$, and show that bounded numéraire markets satisfying NGE for $\mathcal{U}(S)$ have true martingale representatives if and only if every bounded undefaultable strategy is maximal. Next, we show that limit quantile and classical superreplication prices coincide in numéraire markets satisfying NGE for $\mathcal{U}(S)$ and can be computed in a dual way using $\sigma$-martingale representatives. Finally, we provide a dual characterisation of maximal and attainable contingent claims. In summary, we see that numéraire markets satisfying NGE for $\mathcal{U}(S)$ have in essence the same desirable properties as a marked model in the standard framework satisfying NFLVR.

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²The very careful reader, however, will observe that in essence the result also follows from Delbaen & Schachermayer (1994), Lemma 4.7 et seq.
7.1 Existence of strongly maximal numéraire strategies

Recall that a self-financing strategy \( \vartheta \) for a market \( S \) is called undefaultable, written as \( \vartheta \in U(S) \), if its value process \( V(\vartheta) \) is nonnegative. Notwithstanding, \( V(\vartheta) \) may become 0 or get infinitesimally close to 0 at or just before some point in time.

**Definition 7.1.** Let \( S \) be a market and \( \vartheta \in U(S) \). The stopping time

\[
a_\vartheta := \inf\{ t \in [0, T] : V_t(\vartheta) = 0 \text{ or } V_t^- (\vartheta) = 0 \}
\]

is called the absorption time of \( \vartheta \).

Note that \( a_\vartheta \) is \( \mathbb{P} \)-a.s. well defined, i.e. does not depend on the chosen representative \( S \in S \), and is valued in \( [0, T] \cup \{ \infty \} \).

The next result explains the name “absorption time”.

**Proposition 7.2.** Let \( S \) be a numéraire market and \( \Gamma \subset U(S) \) a strategy cone allowing for switching to dominated strategies and containing a numéraire strategy. Suppose that \( S \) satisfies NGE for \( \Gamma \). Then for all \( \vartheta \in \Gamma \),

\[
V(\vartheta) \equiv 0 \text{ on } [a_\vartheta, T] \quad \mathbb{P}\text{-a.s.}
\]

**Proof.** Let \( \eta \in \Gamma \) be a numéraire strategy and \( \vartheta \in \Gamma \). For \( c > 0 \), define the stopping time

\[
\tau_c := \inf\{ t \in [a_\vartheta, T] : V_t(\vartheta) \geq cV_t(\eta) \}.
\]

It suffices to show that for all \( c > 0 \), \( \tau_c = +\infty \) \( \mathbb{P}\)-a.s. Seeking a contradiction, suppose there exists \( c > 0 \) such that \( \mathbb{P}[\tau_c < \infty] > 0 \). Define the non-zero contingent claim \( F : S \to \mathbb{L}^0_+ (\mathcal{F}_{\tau_c \wedge T}) \) by

\[
F := cV_{\tau_c}(\eta) 1_{\{\tau_c < \infty\}}
\]

and for \( n \in \mathbb{N} \), define the stopping time

\[
\sigma_n := \inf\{ t \in [0, T] : V_t(\vartheta) \leq \frac{1}{n} V_t(\eta) \} \land T.
\]

Then \( \sigma_n \leq a_\vartheta \leq \tau_c \mathbb{P}\text{-a.s. for all } n \in \mathbb{N} \). Let \( \delta > 0 \) be given. Choose \( N \) large enough that \( \frac{1}{N} < \min(\delta, c) \). Then by right-continuity of each representative \( S \in S \),

\[
\sigma_N < \tau_c \quad \mathbb{P}\text{-a.s. on } \{\tau_c < \infty\}.
\]

Denote by \( \tilde{\eta} \) a switching numéraire strategy at time \( \sigma_N \) and set

\[
\tilde{\vartheta} := \frac{1}{N} \eta 1_{[0, \sigma_N]} + \left( \vartheta + V_{\sigma_N} \left( \frac{1}{N} \eta - \vartheta \right) (S(\tilde{\eta})) \right) 1_{(\sigma_N, T]} \in \Gamma.
\]

Then

\[
V_{\tilde{\vartheta}}(\tilde{\vartheta}) = \frac{1}{N} V_0(\eta) \leq \delta V_0(\eta),
\]

\[
V_{\tau_c \wedge T}(\tilde{\vartheta}) \geq V_{\tau_c}(\tilde{\vartheta}) 1_{\{\tau_c < \infty\}} = \left( V_{\tau_c}(\vartheta) + V_{\sigma_N} \left( \frac{1}{N} \eta - \vartheta \right) (S(\tilde{\eta})) V_{\tau_c}(\tilde{\eta}) \right) 1_{\{\tau_c < \infty\}}
\]

\[
\geq cV_{\tau_c}(\eta) 1_{\{\tau_c < \infty\}} = F \quad \mathbb{P}\text{-a.s.}
\]

Since \( \delta > 0 \) was arbitrary, we may conclude that

\[
\Pi_{\tau_c \wedge T}(F | \Gamma) \leq \Pi_{\tau_c \wedge T}(F | \Gamma) = 0.
\]

This implies that \( \mathcal{G}_{\tau_c \wedge T}(\Gamma) \neq \emptyset \), in contradiction to the hypothesis that \( S \) satisfies NGE for \( \Gamma \). \( \square \)
Remark 7.3. Once Theorem 7.10 below has been established, the above result could be proved more easily using the minimum principle for nonnegative supermartingales (see Kallenberg (2002), Theorem 7.32). But since the above result is needed for the proof of Theorem 7.10, the above direct argument is really necessary.

An immediate but important consequence of the above result is the fact that in numéraire markets satisfying NGE for \( \mathcal{U}(S) \), an undefaultable strategy is a numéraire strategy if and only if its value process is strictly positive at the time horizon \( T \).

**Corollary 7.4.** Let \( \mathcal{S} \) be a numéraire market and \( \Gamma \subset \mathcal{U}(\mathcal{S}) \) a strategy cone allowing for switching to dominated strategies and containing a numéraire strategy. Suppose that \( \mathcal{S} \) satisfies NGE for \( \Gamma \). Then \( \eta \in \Gamma \) is a numéraire strategy if and only if

\[
V_T(\eta) > 0 \text{ P-a.s.}
\]

We proceed to prove one of the key results of this paper, namely that in any numéraire market satisfying NGE for undefaultable strategies, there exist “sufficiently many” strongly maximal numéraire strategies whose representatives satisfy NFLVR.

**Theorem 7.5.** Let \( \mathcal{S} \) be a numéraire market satisfying NGE for \( \mathcal{U}(\mathcal{S}) \). Then for each numéraire strategy \( \eta \in \mathcal{U}(\mathcal{S}) \), there exists a numéraire strategy \( \eta^* \) which is a dominating maximal strategy for \( \eta \), and whose numéraire representative \( \mathcal{S}(\eta^*) \) satisfies NFLVR.

**Proof.** Fix a numéraire strategy \( \eta \in \mathcal{U}(\mathcal{S}) \). Since \( \mathcal{S} \) satisfies NGE for \( \mathcal{U}(\mathcal{S}) \), Proposition 5.3 shows that 0 is strongly maximal in the classical sense for \( \mathcal{U}(\mathcal{S}) \). Hence by Theorem 6.3, there exists a dominating maximal strategy \( \eta^* \in \mathcal{U}(\mathcal{S}) \) for \( \eta \) satisfying in particular

\[
V_T(\eta^*) \geq V_T(\eta) > 0 \text{ P-a.s.}
\]

Corollary 7.4 shows that \( \eta^* \) is a numéraire strategy, and Proposition 5.10 (c) yields that \( \mathcal{S}(\eta^*) \) satisfies NFLVR.

\( \square \)

### 7.2 \( \sigma \)-martingale representatives

**Definition 7.6.** Let \( \mathcal{S} \) be a \( d \)-dimensional market, \( \mathcal{S} = (S_1, \ldots, S_d) \in \mathcal{S} \) a representative and \( Q \cong P \) on \( \mathcal{T}_d \) an equivalent probability measure. Then \( \mathcal{S} \) is called a \( Q \)-\( \sigma \)-martingale representative or, more generally, an equivalent \( \sigma \)-martingale representative if \( S_1, \ldots, S_d \) are \( \sigma \)-martingales under \( Q \). We denote the set of all \( Q \)-\( \sigma \)-martingale representatives of \( \mathcal{S} \) by \( \mathcal{M}_Q \). For convenience, we set \( \mathcal{M} := \mathcal{M}_P \) and call \( S \in \mathcal{M} \) a \( \sigma \)-martingale representative.

The following result is easy but useful.

**Proposition 7.7.** Let \( \mathcal{S} \) be a market. Then the following are equivalent:

(a) \( \mathcal{M} \neq \emptyset \).

(b) There exists \( Q \cong P \) on \( \mathcal{T}_d \) such that \( \mathcal{M}_Q \neq \emptyset \).

(c) \( \mathcal{M}_Q \neq \emptyset \) for all \( Q \cong P \) on \( \mathcal{T}_d \).

**Proof.** The implications “(c) \( \Rightarrow \) (a)” and “(a) \( \Rightarrow \) (b)” are trivial. To establish the implication “(b) \( \Rightarrow \) (c)”, let \( \tilde{Q} \cong P \) on \( \mathcal{T}_d \) be arbitrary. By hypothesis, there exist \( Q \cong P \) on \( \mathcal{T}_d \) and \( S \in \mathcal{S} \) such that \( \mathcal{S} \) is a \( Q \)-\( \sigma \)-martingale. Then \( \tilde{Q} \cong Q \) on \( \mathcal{T}_d \). Denote by \( Z \) the density process of \( \tilde{Q} \) with respect to \( Q \). Then \( Z \) is a strictly positive \( Q \)-martingale and a fortiori an exchange rate process. Set \( \tilde{S} := Z^{-1}S \). Then \( Z \) and \( Z\tilde{S} = S \) are \( Q \)-\( \sigma \)-martingales and hence \( \tilde{S} \) is a \( \tilde{Q} \)-\( \sigma \)-martingale representative of \( \tilde{S} \) (see Kallsen (2004), Proposition 5.1). \( \square \)

**Remark 7.8.** If \( \mathcal{S} \) is a market, \( Q \cong P \) on \( \mathcal{T}_d \), \( S \in \mathcal{M}_Q \) and \( \vartheta \in \mathcal{U}(\mathcal{S}) \), then \( V(\vartheta)(S) \) is a nonnegative local \( Q \)-martingale and a \( Q \)-supermartingale. Indeed, this follows from Ansel & Stricker (1994), Corollaire 3.5, and Fatou’s lemma as \( V(\vartheta)(S) = \vartheta_0 \cdot S_0 + \vartheta \bullet S \geq 0 \text{ Q-a.s.} \) (using that \( Q \cong P \)). We use both facts in the sequel without further comments.
7.3 A numéraire independent FTAP

Our next goal is to prove a a numéraire independent version of the celebrated fundamental theorem of asset pricing by Delbaen and Schachermayer. First we recall this classical result in the setup of this paper.

**Theorem 7.9.** Let \( S \) be a numéraire market and \( \eta \) a numéraire strategy. Then the following are equivalent:

(a) \( \eta \) is strongly maximal in the classical sense for \( \mathcal{U}(S) \).

(b) \( S^{(\eta)} \) satisfies NFLVR.

(c) \( S^{(\eta)} \) is an equivalent \( \sigma \)-martingale representative.

**Proof.** The equivalence “(a) \( \iff \) (b)” is Proposition 5.10 (c), and the equivalence “(b) \( \iff \) (c)” is Delbaen & Schachermayer (1998), Theorem 1.1.

We proceed to prove a numéraire independent version of Theorem 7.9.

**Theorem 7.10.** Let \( S \) be a numéraire market. Then the following are equivalent:

(a) \( 0 \) is strongly maximal in the classical sense for \( \mathcal{U}(S) \).

(b) \( S \) satisfies NGE for \( \mathcal{U}(S) \).

(c) For each numéraire strategy \( \eta \), there exists a numéraire strategy \( \eta^* \) which is a dominating maximal strategy for \( \eta \), and whose numéraire representative \( S^{(\eta^*)} \) satisfies NFLVR and is an equivalent \( \sigma \)-martingale representative.

(d) There exists \( Q \approx P \) on \( \mathcal{F}_T \) such that \( \mathcal{M}_Q \neq \emptyset \).

(e) For all \( Q \approx P \) on \( \mathcal{F}_T \), we have \( \mathcal{M}_Q \neq \emptyset \).

**Proof.** “(a) \( \implies \) (b)”. This follows from Proposition 5.3.

“(b) \( \implies \) (c)”. This follows immediately from Theorems 7.5 and 7.9 (the latter applied to \( \eta^* \)).

“(c) \( \implies \) (d)”. Since \( S \) is a numéraire market, there exists a numéraire strategy \( \eta \), and therefore by hypothesis also a numéraire strategy \( \eta^* \) such that \( S^{(\eta^*)} \) is an equivalent \( \sigma \)-martingale representative, i.e. there exists \( Q \approx P \) on \( \mathcal{F}_T \) with \( S^{(\eta^*)} \in \mathcal{M}_Q \neq \emptyset \).

“(d) \( \implies \) (e)” This follows from Proposition 7.7.

“(e) \( \implies \) (a)”. Seeking a contradiction, we may assume by Corollary 4.19 that there exists a non-zero contingent claim \( F : \mathcal{S} \to \mathbb{L}_+^2(\mathcal{F}_T) \) with

\[
\Pi(F|\mathcal{U}(S)) = 0.
\]

By hypothesis there exists \( S \in \mathcal{M} \neq \emptyset \). Let \( C : \mathcal{S} \to \mathbb{R}_{++} \) be the contingent claim (at time \( 0 \)) from Proposition 2.19 satisfying \( C(S) = 1 \) and \( \delta := E[F(S)]/2 > 0 \). By Proposition 2.23, there exists \( \vartheta \in \mathcal{U} \) such that

\[
V_0(\vartheta) \leq \delta C \quad \text{and} \quad V_T(\vartheta) \geq F \quad \text{P-a.s.}
\]

The (P-)supermartingale property of \( V(\vartheta)(S) \) gives

\[
V_0(\vartheta)(S) \geq E[V_T(\vartheta)(S)] \geq E[F(S)] = 2\delta > \delta C(S),
\]

and we arrive at a contradiction.
7.4 Maximal strategies in NGE markets

We proceed to show that if a numéraire market $S$ satisfies NGE for $U(S)$, then all notions of maximality for strategies in $U(S)$ coincide.

**Lemma 7.11.** Let $S$ be a numéraire market satisfying NGE for $U(S)$ and $\vartheta \in U(S)$. Then the following are equivalent:

(a) $\vartheta$ is weakly maximal for $U(S)$.

(b) $\vartheta$ is strongly maximal in the classical sense for $U(S)$.

(c) $\vartheta$ is strongly maximal in the limit quantile sense for $U(S)$.

(d) There exist $Q \approx P$ on $\mathcal{F}_T$ and a numéraire strategy $\eta^*$ (which is strongly maximal in the classical sense) such that $S^{(\eta^*)} \in \mathcal{M}_Q$ and $V(\vartheta)(S^{(\eta^*)})$ is a true $Q$-martingale uniformly bounded by 1.

(e) There exist $Q \approx P$ on $\mathcal{F}_T$ and $S \in \mathcal{M}_Q$ such that $V(\vartheta)(S)$ is a true $Q$-martingale.

(f) For each $Q \approx P$ on $\mathcal{F}_T$, there exists $S \in \mathcal{M}_Q$ such that $V(\vartheta)(S)$ is a true $Q$-martingale.

*Proof.* “(a) $\implies$ (d)”. Suppose that $\vartheta$ is weakly maximal for $U(S)$. Then by Theorem 7.5, there exists a numéraire strategy $\eta^* \in U(S)$ which is strongly maximal in the classical sense and a fortiori weakly maximal by Proposition 4.14. Set $\eta := \eta^* + \vartheta$. Then $\eta$ is a numéraire strategy and weakly maximal by Corollary 4.10. In the notation from Section 5, $S^{(\eta)}$ satisfies $sm(\eta)$. Proposition 5.3 yields that $S^{(\eta)}$ satisfies $sm(0)$, Proposition 5.6 shows that $S^{(\eta)}$ satisfies $sm(\eta)$, and Theorem 7.9 shows that there exists $Q \approx P$ on $\mathcal{F}_T$ such that $S^{(\eta)}$ is a $Q$-martingale with representative of $S$. Now the claim follows from the fact that the nonnegative local $Q$-martingale $V(\vartheta)(S^{(\eta)})$ is uniformly bounded by 1 and therefore a true $Q$-martingale. Indeed, since $V(\eta)(S) \geq 0$ Q-a.s.,

$$0 \leq V(\vartheta)(S^{(\eta)}) \leq V(\eta^* + \vartheta)(S^{(\eta)}) = V(\eta)(S^{(\eta)}) \equiv 1 \text{ Q-a.s.}$$

“(d) $\implies$ (e)”. Let $Q \approx P$ on $\mathcal{F}_T$ and $\eta^*$ be a numéraire strategy such that $S^{(\eta^*)} \in \mathcal{M}_Q$ and $V(\vartheta)(S^{(\eta^*)})$ is a true $Q$-martingale uniformly bounded by 1. For convenience set $S := S^{(\eta^*)}$. Recall from Proposition 2.33 that limit quantile superreplication prices are invariant under an equivalent change of measure, so that we can apply also under $Q$ all results that are formulated under $P$. Seeking a contradiction, we may assume by Corollary 4.19 that there exists a non-zero contingent claim $F : S \to \mathbb{L}^0_+(\mathcal{F}_T)$ satisfying

$$\Pi^*(V_T(\vartheta) + F | U(S)) \leq V_0(\vartheta). \quad (7.1)$$

If $V_T(\vartheta)(S) = 0$ Q-a.s., then the $Q$-martingale property of $V(\vartheta)(S)$ gives $V_0(\vartheta)(S) = 0$, and so (7.1) implies that $Q_T(U(S)) \neq \emptyset$, which is a contradiction to the hypothesis that $S$ satisfies NGE for $U(S)$. Therefore we may assume that $Q[V_T(\vartheta)(S) > 0] > 0$. By the $Q$-martingale property of $V(\vartheta)(S)$, this implies that $V_0(\vartheta)(S) > 0$. Set $K := E_Q[F(S)]/2 > 0$. By Proposition A.1, there exists $\epsilon \in (0, K/3)$ such that for all $A \in \mathcal{F}_T$ with $Q[A] \geq 1 - \epsilon$,

$$E_Q[F(S)1_A] \geq K = 3\epsilon. \quad (7.2)$$

Set $\delta := \epsilon/V_0(\vartheta)(S)$. Applying Corollary 2.30 with $C := V_0(\vartheta)$ gives $\tilde{\vartheta} \in U(S)$ and $A \in \mathcal{F}_T$ with $Q[A] \geq 1 - \epsilon$ such that

$$V_0(\tilde{\vartheta}) \leq (1 + \delta)V_0(\vartheta) \quad \text{and} \quad V_T(\tilde{\vartheta}) \geq (V_T(\vartheta) + F)1_A \text{ Q-a.s.}$$

The $Q$-supermartingale property of $V(\vartheta)(S)$, the fact that $V(\vartheta)(S)$ is a $Q$-martingale satisfying $V_T(\vartheta)(S) \leq 1$, and (7.1), (7.2) give

$$V_0(\tilde{\vartheta})(S) \geq \mathbb{E}_Q[V_T(\tilde{\vartheta})(S)] \geq \mathbb{E}_Q[(V_T(\vartheta)(S) + F(S))1_A]$$

$$\geq V_0(\vartheta)(S) - \mathbb{E}_Q[1_A] + 3\epsilon \geq V_0(\vartheta)(S) + 2\epsilon$$

$$= (1 + 2\delta)V_0(\vartheta)(S) > (1 + \delta)V_0(\vartheta)(S).$$

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Thus, we arrive at a contradiction.

‘(c) ’ ‘(b) ’ ‘(a) ’. This follows from Remark 4.12 and Proposition 4.14.

‘(d) ’ ‘(e) ’. This is trivial.

‘(f) ’ (a) ’. This is similar to but easier than the proof of ‘(d) ’ ‘(e) ’. Let $S := Z^{-1}E$. Then $Z$ and $ZV(\tilde{\vartheta})(S)$ are both true $Q$-martingales and hence $V(\tilde{\vartheta})(S)$ is a true $Q$-martingale.

‘(f) ’ (a) ’. This is similar to but easier than the proof of ‘(d) ’ ‘(e) ’. Let $S \in \mathcal{M}$ be such that $V(\vartheta)(S)$ is a true ($P$-)martingale. Seeking a contradiction, we may assume by Proposition 4.4 and Corollary 4.8 that there exists $\tilde{\vartheta} \in \mathcal{U}(S)$ satisfying

$$V_0(\tilde{\vartheta})(S) = V_0(\vartheta)(S), \quad V_T(\tilde{\vartheta})(S) \geq V_T(\vartheta)(S) \quad \text{P-a.s. and } \quad \mathbb{P}[V_T(\tilde{\vartheta})(S) > V_T(\vartheta)(S)] > 0.$$ 

But the supermartingale property of $V(\tilde{\vartheta})(S)$ and the martingale property of $V(\vartheta)(S)$ yield

$$V_0(\tilde{\vartheta})(S) \geq \mathbb{E}[V_T(\tilde{\vartheta})(S)] > \mathbb{E}[V_T(\vartheta)(S)] = V_0(\vartheta)(S),$$

and we arrive at a contradiction. \hfill \Box

The following corollary establishes the final, until now unproved claim of Proposition 5.6.

**Corollary 7.12.** Let $S$ be a numéraire market and $\eta$ a numéraire strategy. Then $sm(\eta)$ implies $sm^*(\eta)$.

**Proof.** The already proved implication “$sm(\eta)$ $\implies$ $sm(0)$” in Proposition 5.6 and Proposition 5.3 show that $S$ satisfies NGE for $\mathcal{U}(S)$. Now the claim follows immediately from Lemma 7.11. \hfill \Box

**Remark 7.13.** Since in numéraire markets satisfying NGE for $\mathcal{U}(S)$, there is no difference between the notions of weakly and strongly maximal strategies in the classical/limit quantile sense for $\mathcal{U}(S)$, we can and often do call such strategies simply maximal strategies for $\mathcal{U}(S)$ in the sequel.

### 7.5 True martingale representatives

Theorem 7.10 shows that in a numéraire market satisfying NGE for $\mathcal{U}(S)$, for each $Q \approx P$, the set of all $Q$-sigma-martingale representatives $\mathcal{M}_Q$ is nonempty. In this section, we address the question of when $\mathcal{M}_Q$ even contains a true $Q$-martingale representative. We show that in this case, every bounded undefaultable strategy is (strongly) maximal.

The following result is proved exactly as Proposition 7.7.

**Proposition 7.14.** Let $S$ be a market. Then the following are equivalent:

(a) There exists a true ($P$-)martingale representative $S \in \mathcal{M}$.

(b) There exist $Q \approx P$ on $\mathcal{F}_T$ and a true $Q$-martingale representative $S \in \mathcal{M}_Q$.

(c) For all $Q \approx P$ on $\mathcal{F}_T$, there exists a true $Q$-martingale representative $S \in \mathcal{M}_Q$.

**Lemma 7.15.** Let $S$ be a numéraire market. Suppose there exist $Q \approx P$ on $\mathcal{F}_T$ and a true $Q$-martingale representative $S \in \mathcal{M}_Q$. Then:

(a) For every bounded self-financing strategy $\vartheta \in bL^f(S)$, the value process $V(\vartheta)(S)$ is a true $Q$-martingale.

(b) Every bounded undefaultable strategy $\vartheta \in b\mathcal{U}(S)$ is strongly maximal in the limit quantile sense for $\mathcal{U}(S)$.

(c) $S^{(Q)}$ is an equivalent true martingale representative for every bounded numéraire strategy $\eta$. 

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Proof. Let $d$ be the dimension of the market and $\| \cdot \|_{\infty}$ denote the maximum norm in $\mathbb{R}^d$. By Proposition 7.14, we may assume without loss of generality that $Q = P$.

(a) Fix $\vartheta \in bL^\infty(S)$. Then $V(\vartheta)(S)$ is a local $(P)$-martingale, and it is a true martingale if it is of class $(D)$, i.e. if the family $(V_P(\vartheta)(S))_{\tau \in T_{[0,T]}}$ is uniformly integrable. The latter holds because

$$|V_P(\vartheta)(S)| = |\vartheta_T \cdot S_T| \leq \|\vartheta\|_{\infty} \sum_{i=1}^{d} |S^i_T|, \quad \tau \in T_{[0,T]},$$

and for each $i = 1, \ldots, d$, the family $(|S^i_T|)_{\tau \in T_{[0,T]}}$ is uniformly integrable because $S^i$ is a uniformly integrable martingale.

(b) Fix $\vartheta \in b\mathcal{U}(S)$. By Theorem 7.10, $S$ satisfies NGE for $\mathcal{U}(S)$. Since $V(\vartheta)(S)$ is a true martingale by part (a), it is maximal in every sense by Lemma 7.11.

(c) Let $\eta$ be a bounded numéraire strategy. Then by part (a), $V(\eta)(S)$ is a strictly positive true $(P)$-martingale. Define $Q \equiv P$ on $\mathcal{F}_T$ by

$$dQ = \frac{V_P(\eta)(S)}{V_Q(\eta)(S)} dP,$$

Then $S^{(\eta)}$ is a true $Q$-martingale representative. Indeed, by (2.5),

$$\frac{V_Q(\eta)(S)}{V_Q(\eta)(S)} S^{(\eta)} = \frac{S}{V_Q(\eta)(S)} \text{ P-a.s.},$$

and $S/V_Q(\eta)(S)$ is a true $P$-martingale representative because $V_Q(\eta)(S)$ is a constant. $\square$

Remark 7.16. It follows immediately from the proof that part (a) of the above lemma does not need the existence of a numéraire strategy. Indeed, it can be reformulated in an abstract stochastic analysis setting as follows: Let $X$ be a $d$-dimensional true martingale and $H$ a bounded predictable process satisfying

$$H \cdot X = H \cdot X - H_0 \cdot X_0. \quad (7.3)$$

Then the stochastic integral $H \cdot X$ is again a true martingale.

Note that without assumption (7.3) this is false in general.

If $S$ is even a bounded numéraire market (see Definition 2.10), then Lemma 7.15 can be strengthened.

Corollary 7.17. Let $S$ be a bounded numéraire market. Then the following are equivalent:

(a) There exists a true martingale representative of $S$.

(b) There exists an equivalent true martingale representative of $S$.

(c) Every bounded undefaultable strategy $\vartheta \in b\mathcal{U}(S)$ is strongly maximal in the classical sense for $\mathcal{U}(S)$.

(d) $S^{(\eta)}$ is an equivalent true martingale representative for every bounded numéraire strategy $\eta$.

(e) There exists a bounded numéraire strategy $\eta$ such that also $S^{(\eta)}$ is bounded and which is strongly maximal in the classical sense for $\mathcal{U}(S)$.

Moreover, if one of the above assertions holds, then $S$ satisfies NGE for $\mathcal{U}(S)$.

Proof. “(a) $\iff$ (b)”. This follows from Proposition 7.14.

“(a) $\implies$ (c) $\implies$ (d)”. This follows from Lemma 7.15 recalling that when assuming (a), all notions of maximality for strategies in $\mathcal{U}(S)$ coincide.

“(d) $\implies$ (e)”. The existence of a bounded numéraire strategy $\eta$ such that also $S^{(\eta)}$ is bounded follows by the assumption that $S$ is a bounded numéraire market, and maximality of $\eta$ follows by Theorem 7.9.
“(e) $\implies$ (b).” Let $\eta$ be as in (e). Then by assumption and Theorem (7.9), $S^{(n)}$ is a bounded equivalent $\sigma$-martingale representative and hence an equivalent true martingale representative.

The final assertion follows immediately from the fact that (a) implies $\mathcal{M} \neq \emptyset$ and from Theorem 7.10.

The following corollary characterises the existence of true martingale representatives in nonnegative numéraire markets (which are in particular bounded numéraire markets).

**Corollary 7.18.** Let $S$ be a (d-dimensional) nonnegative numéraire market. Then the following are equivalent:

(a) There exists a true martingale representative of $S$.

(b) For each $i = 1, \ldots, d$, the buy-and-hold strategy of the $i$-th asset $\eta^{(i)} := (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is at position $i$, is strongly maximal in the classical sense for $\mathcal{U}(S)$.

(c) The numéraire strategy $\eta := (1, \ldots, 1)$ is strongly maximal in the classical sense for $\mathcal{U}(S)$.

**Proof.** “(a) $\implies$ (b).” This follows immediately from Corollary 7.17.

“(b) $\implies$ (c).” Since by Proposition 4.14, $\eta^{(1)}, \ldots, \eta^{(d)}$ are all a fortiori weakly maximal for $\mathcal{U}(S)$ and $\eta := \sum_{i=1}^{d} \eta^{(i)}$, Corollary 4.10 shows that $\eta$ is weakly maximal for $\mathcal{U}(S)$. Moreover, the fact that $\eta^{(i)}$ is strongly maximal in the classical sense for $\mathcal{U}(S)$ and Proposition 4.16 imply that 0 is strongly maximal in the classical sense for $\mathcal{U}(S)$. Now Theorem 7.10 shows that $S$ satisfies NGE for $\mathcal{U}(S)$ and Lemma 7.11 establishes that $\eta$ is even strongly maximal in the classical sense for $\mathcal{U}(S)$.

“(c) $\implies$ (a).” This follows immediately from Corollary 7.17 because $\eta$ and $S^{(0)}$ are bounded.

**Remark 7.19.** Weak maximality (for $\mathcal{U}(S)$) of the buy-and-hold strategies of the primary assets has been called no-dominance (ND) in the literature (see e.g. Jarrow & Larsson (2012), Definition 2.2 and references therein), and recently studied in the context of bubbles as a possible requirement additional to NFLVR$^3$ (see e.g. Jarrow et al. (2010)). NFLVR$^3$ and ND together have been shown to be equivalent to the existence of a true equivalent martingale measure for $S^{(0)}$ (see Jarrow & Larsson (2012), Theorem 3.2). Our Corollary 7.18 can be seen as a numéraire independent version of the latter result because by Theorem 7.9, strong maximality in the classical sense (for $\mathcal{U}(S)$) of the buy-and-hold strategies of the primary assets (in particular of $\eta^{(1)}$) imply$^4$ NFLVR for $S^{(0)}$.

We believe that the relationship between maximal strategies, bubbles and market efficiency (see Jarrow & Larsson (2012)) in a numéraire independent context deserves a more careful analysis, which we postpone to future research.

### 7.6 Superreplication prices in NGE markets

The goal of this section is to show that if a numéraire market $S$ satisfies NGE for $\mathcal{U}(S)$, then limit quantile and classical superreplication prices for $\mathcal{U}(S)$ coincide and can be computed using (equivalent) $\sigma$-martingale representatives. First, we introduce some notation.

**Definition 7.20.** Let $S$ be a market, $Q \approx P$ on $\mathcal{F}_T$ and $s \in \mathbb{R}^d \setminus \{0\}$. Set

$$\mathcal{M}_Q(s) := \{ S \in \mathcal{M}_Q : S_0 = s \} \quad \text{and} \quad \mathcal{M}(s) := \mathcal{M}_P(s).$$

**Remark 7.21.**

(a) If there is no $S \in S$ with $S_0 = s$, then $\mathcal{M}_Q(s) = \emptyset$ even if $\mathcal{M}_Q \neq \emptyset$. On the other hand, if $S$ is a numéraire market and there exists $S \in S$ with $S_0 = s$, then Theorem 7.10 and Proposition 7.7 show that $\mathcal{M}_Q(s) \neq \emptyset$ if and only if $S$ satisfies NGE for $\mathcal{U}(S)$.

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$^3$for the numéraire representative of $\eta^{(1)}$

$^4$if $\eta^{(1)}$ is a numéraire strategy
(b) If $M_Q(s) \neq \emptyset$ and $c > 0$, then $M_Q(cs) \neq \emptyset$ and

$$M_Q(cs) = cM_Q(s). \quad (7.4)$$

In particular, if $S$ is a numéraire market satisfying NGE for $U(S)$, $S \in S$, $D$ is an exchange rate process, $\tau \in [0,T]$ and $F : S \rightarrow L^0_+ (\mathcal{F}_\tau)$ is a contingent claim, then

$$\sup_{\tilde{S} \in M_Q(L_0S_0)} E_Q[F(\tilde{S})] = D_0 \sup_{\tilde{S} \in M_Q(S_0)} E_Q[F(\tilde{S})]. \quad (7.5)$$

This uses of course the exchange rate consistency (2.8) of $F$ (for $D \equiv D_0$).

(c) If $\mathcal{M}(s) \neq \emptyset$ and $Z$ is the density process of $Q$ with respect to $\mathbb{P}$, then $M_Q(s) \neq \emptyset$ and

$$M_Q(s) = \frac{1}{Z} M(s). \quad (7.6)$$

Note that $Z_0 = 1$ because $\mathcal{F}_0$ is trivial. In particular, if $S$ is a numéraire market satisfying NGE for $U(S)$, $S \in S$, $\tau \in [0,T]$ and $F : S \rightarrow L^0_+ (\mathcal{F}_\tau)$ is a contingent claim, then Bayes’ formula and the exchange rate consistency (2.8) of $F$ (for $D = Z$) give

$$\sup_{\tilde{S} \in M_Q(S_0)} E_Q[F(\tilde{S})] = \sup_{\tilde{S} \in M(S_0)} E[F(\tilde{S})]. \quad (7.7)$$

The next result shows that in numéraire markets satisfying NGE for $U(S)$, the limit quantile superreplication price of a contingent claim in units of a $\sigma$-martingale representative is always greater than or equal to the expectation of the contingent claim in units of that same representative.

**Proposition 7.22.** Let $S$ be a numéraire market satisfying NGE for $U(S)$, $Q \approx \mathbb{P}$ on $\mathcal{F}_T$, $\tau \in [0,T]$ and $F : S \rightarrow L^0_+ (\mathcal{F}_\tau)$ a contingent claim. Then for all $S \in M_Q$,

$$\Pi(F | U(S))(S) \geq \Pi^*(F | U(S))(S) \geq E_Q[F(S)].$$

**Proof.** The first inequality follows from Remark 2.27 (a). To establish the second one, let $S \in M_Q$ and $C : S \rightarrow \mathbb{R}_{++}$ be the strictly positive contingent claim (at time 0) from Proposition 2.19 satisfying $C(S) = 1$. Seeking a contradiction, suppose that there exists $\tilde{K} < E_Q[F(S)]$ with

$$\Pi^*(F | U(S))(S) \leq \tilde{K}.$$

Pick $K$ with $\tilde{K} < K < E_Q[F(S)]$. Then Proposition A.1 gives $\epsilon \in (0,1)$ with $\epsilon \leq (K - \tilde{K})/2$ such that for all $A \in \mathcal{F}_T$ with $Q[A] \geq 1 - \epsilon$,

$$E_Q[F(S)1_A] \geq K.$$

By Corollary 2.30, there are $\vartheta \in U(S)$ and $A \in \mathcal{F}_T$ with $Q[A] \geq 1 - \epsilon$ such that

$$V_0(\vartheta) \leq \tilde{K} + \epsilon C \quad \text{and} \quad V_T(\vartheta) \geq F1_A \; \mathbb{Q}\text{-a.s.}$$

The $\mathbb{Q}$-supermartingale property of $V(\vartheta)(S)$ gives

$$V_0(\vartheta)(S) \geq E_Q[V_T(\vartheta)(S)] \geq E_Q[F(S)1_A] \geq K \geq \tilde{K} + 2\epsilon > \tilde{K} + \epsilon C(S).$$

Thus we arrive at a contradiction. \qed

**Corollary 7.23.** Let $S$ be a numéraire market satisfying NGE for $U(S)$, $Q \approx \mathbb{P}$ on $\mathcal{F}_T$, $\tau \in [0,T]$ and $F : S \rightarrow L^0_+ (\mathcal{F}_\tau)$ a contingent claim. Then for all $S \in S$,

$$\Pi(F | U(S))(S) \geq \Pi^*(F | U(S))(S) \geq \sup_{\tilde{S} \in M_Q(S_0)} E_Q[F(\tilde{S})].$$
Proof. Let \( S, \tilde{S} \in \mathcal{S} \) be arbitrary with \( \tilde{S}_0 = S_0 \). Then Remark 2.22 gives
\[
\Pi^*(F | \mathcal{U}(S))(S) = \Pi^*(F | \mathcal{U}(\tilde{S}))(\tilde{S}),
\]
and the claim follows from Proposition 7.22.

For the next result, we introduce some further notation.

Definition 7.24. Let \( \mathcal{S} \) be a numéraire market satisfying NGE for \( \mathcal{U}(S) \) and \( \eta^* \) a maximal numéraire strategy. Denote by \( Q(S^{(\eta^*)}) \) the set of all probability measures \( Q \approx P \) on \( \mathcal{F}_T \) such that \( S^{(\eta^*)} \) is a \( Q\sigma\)-martingale.

Remark 7.25.
(a) By Theorem 7.9, the set \( Q(S^{(\eta^*)}) \) is nonempty.
(b) Each \( Q \in Q(S^{(\eta^*)}) \) can be identified with \( ZS^{(\eta^*)} \in \mathcal{M}(S_0^{(\eta^*)}) \), where \( Z \) is the density process of \( Q \) with respect to \( P \). Doing this is motivated by the following observation: If \( \tau \in [0,T] \) is a stopping time and \( F : \mathcal{S} \to \mathbb{L}_0^1(\mathcal{F}_\tau) \) a contingent claim, then
\[
\mathbb{E}_Q[F(S^{(\eta^*)})] = \mathbb{E}[Z_T F(S^{(\eta^*)})] = \mathbb{E}[F(ZS^{(\eta^*)})].
\]

The next result recalls in the notation of this paper the standard result about the dual representation of superreplication prices

Proposition 7.26. Let \( \mathcal{S} \) be a numéraire market satisfying NGE for \( \mathcal{U}(S) \), \( \eta^* \) a maximal numéraire strategy, \( \tau \in [0,T] \) and \( g \in L_0^1(\mathcal{F}_\tau) \) with \( v := \sup_{Q \in Q(S^{(\eta^*)})} \mathbb{E}_Q[g] < \infty \). Then there exists \( \vartheta \in \mathcal{U}(S) \) such that
\[
V_0(\vartheta)(S^{(\eta^*)}) = v \quad \text{and} \quad V_\tau(\vartheta)(S^{(\eta^*)}) \geq g \quad P\text{-a.s.}
\]

Proof. By Proposition 4.2 in Kramkov (1996), there exists a process \( V = (V_t)_{t \in [0,T]} \) which is a \( Q\)-supermartingale for all \( Q \in Q(S^{(\eta^*)}) \) and for each fixed \( t \in [0,T] \),
\[
V_t = \text{ess sup}_{Q \in Q(S^{(\eta^*)})} \mathbb{E}_Q[g | \mathcal{F}_t] \quad P\text{-a.s.}
\]

In particular, for fixed \( t \in [0,T] \),
\[
V_t 1_{\{t \geq \tau\}} = g 1_{\{t \geq \tau\}} \quad P\text{-a.s.}
\]

Set \( v := V_0 > 0 \). By the optional decomposition theorem in the form of Föllmer & Kabanov (1998), Theorem 1, there exists a predictable process \( \zeta \in (S^{(\eta^*)}) \) with \( \zeta_0 = 0 \) such that
\[
v + \zeta \bullet S^{(\eta^*)} \geq V \quad P\text{-a.s.}
\]

Since \( V \geq 0 \) \( P\text{-a.s.} \), we have \( \zeta \in L^\infty(\mathcal{S}^{(\eta^*)}) \), and Corollary 2.16 yields \( \vartheta \in \mathcal{U}(S) \) such that
\[
V(\vartheta)(S^{(\eta^*)}) = v + \zeta \bullet S^{(\eta^*)} \quad P\text{-a.s.},
\]
which establishes the claim.

Corollary 7.27. Let \( \mathcal{S} \) be a numéraire market satisfying NGE for \( \mathcal{U}(S) \), \( \eta^* \) a maximal numéraire strategy, \( \tau \in [0,T] \) and \( F : \mathcal{S} \to \mathbb{L}_0^1(\mathcal{F}_\tau) \) a contingent claim with \( \sup_{S \in \mathcal{M}(S_0^{(\eta^*)})} \mathbb{E}[F(S)] < \infty \). Then there exists \( \vartheta \in \mathcal{U}(S) \) such that
\[
\Pi(F | \mathcal{U}(S))(S^{(\eta^*)}) \leq V_0(\vartheta)(S^{(\eta^*)}) \leq \sup_{S \in \mathcal{M}(S_0^{(\eta^*)})} \mathbb{E}[F(S)] < \infty,
\]
\[
V_\tau(\vartheta)(S^{(\eta^*)}) \geq F(S^{(\eta^*)}) \quad P\text{-a.s.}
\]
Proof. Set \( g := F(S^{(\sigma^*)}) \). Then (7.8) implies that for each \( Q \in \mathcal{Q}(S^{(\sigma^*)}) \),
\[
E_Q[g] = E_Q[F(S^{(\sigma^*)})] \leq \sup_{S \in \mathcal{M}(S^{(\sigma^*)})} E[F(S)].
\]
Now the claim follows from Proposition 7.26 and the definition of classical superreplication prices. \( \square \)

Finally, we can prove the main result about the dual representation of superreplication prices in numéraire markets satisfying NGE for \( \mathcal{U}(\mathcal{S}) \).

**Theorem 7.28.** Let \( \mathcal{S} \) be a numéraire market satisfying NGE for \( \mathcal{U}(\mathcal{S}) \), \( \tau \in \mathcal{T}_{[0,T]} \) a stopping time and \( F: \mathcal{S} \to L^0(\mathcal{F}_\tau) \) a contingent claim. Then for all \( S \in \mathcal{S} \) and all \( Q \approx \mathbb{P} \) on \( \mathcal{F}_T \),
\[
\Pi(F | \mathcal{U}(\mathcal{S}))(S) = \Pi^*(F | \mathcal{U}(\mathcal{S}))(S) = \sup_{\tilde{S} \in \mathcal{M}_Q(S)} E_Q[F(\tilde{S})].
\]
In addition, if \( \sup_{\tilde{S} \in \mathcal{M}_Q(S)} E_Q[F(\tilde{S})] < \infty \) for some \( Q \approx \mathbb{P} \), then there exists a maximal strategy \( \vartheta^* \in \mathcal{U}(\mathcal{S}) \) such that
\[
V_0(\vartheta^*) = \Pi(F | \mathcal{U}(\mathcal{S})) \quad \text{and} \quad V_\tau(\vartheta^*) \geq F \quad \mathbb{P}\text{-a.s.}
\]

**Proof.** Fix \( S \in \mathcal{S} \). Note that by Remark 7.21 (c), we may assume without loss of generality that \( Q = \mathbb{P} \) in both parts of the claim.

First, consider the case \( \sup_{\tilde{S} \in \mathcal{M}_Q(S)} E[F(\tilde{S})] = \infty \). Then Remark 2.27 (a) and Corollary 7.23 give
\[
\Pi(F | \mathcal{U}(\mathcal{S}))(S) \geq \Pi^*(F | \mathcal{U}(\mathcal{S}))(S) \geq \sup_{\tilde{S} \in \mathcal{M}(\mathcal{S})} E[F(\tilde{S})] = \infty.
\]
Next, consider the case \( \sup_{\tilde{S} \in \mathcal{M}_Q(S)} E[F(\tilde{S})] < \infty \). By Theorem 7.5, there exist a strongly maximal numéraire strategy \( \eta^* \) and \( Q \approx \mathbb{P} \) on \( \mathcal{F}_T \) such that \( S^{(\sigma^*)} \) is a \( Q\)-\( \sigma \)-martingale. Fix \( S \in \mathcal{S} \) and let \( D \) be the exchange rate process from Remark 2.3 (b) satisfying \( S^{(\sigma^*)} = DS \quad \mathbb{P}\text{-a.s.} \). Then Corollaries 7.23 and 7.27, (7.5) and (2.8) give
\[
\Pi(F | \mathcal{U}(\mathcal{S}))(S) \geq \Pi^*(F | \mathcal{U}(\mathcal{S}))(S) \geq \sup_{\tilde{S} \in \mathcal{M}(\mathcal{S})} E[F(\tilde{S})]
\]
\[
= \frac{1}{D_0} \sup_{\tilde{S} \in \mathcal{M}(S^{(\sigma^*)})} E[F(\tilde{S})] \geq \frac{1}{D_0} \Pi(F | \mathcal{U}(\mathcal{S}))(S^{(\sigma^*)})
\]
\[
= \Pi(F | \mathcal{U}(\mathcal{S}))(S).
\]
Therefore,
\[
\Pi(F | \mathcal{U}(\mathcal{S}))(S^{(\sigma^*)}) = \sup_{\tilde{S} \in \mathcal{M}(S^{(\sigma^*)})} E[F(\tilde{S})].
\]
This together with Corollary 7.27 gives \( \vartheta \in \mathcal{U}(\mathcal{S}) \) such that
\[
V_0(\vartheta) = \Pi(F | \mathcal{U}(\mathcal{S})) \quad \text{and} \quad V_\tau(\vartheta) \geq F \quad \mathbb{P}\text{-a.s.}
\]
Since the market \( \mathcal{S} \) satisfies NGE for \( \mathcal{U}(\mathcal{S}) \), Proposition 5.3 and Theorem 6.3 give a maximal strategy \( \vartheta^* \in \mathcal{U}(\mathcal{S}) \) satisfying
\[
V_0(\vartheta^*) = V_0(\vartheta) = \Pi(F | \mathcal{U}(\mathcal{S})) \quad \text{and} \quad V_\tau(\vartheta^*) \geq V_\tau(\vartheta) \geq F \quad \mathbb{P}\text{-a.s.} \quad \square
\]

**Remark 7.29.** Note that the supremum in (7.9) is in general **not** attained. We will see in the next section that it is attained if and only if the inequality in (7.10) is an equality, i.e. if the contingent claim \( F \) is **attainable**.
7.7 Maximal and attainable contingent claims

Finally, we address the question of attainability of contingent claims. We show that in a numéraire market $\mathcal{S}$ satisfying NGE for $\mathcal{U}(\mathcal{S})$ this is equivalent to the a priori more general property of the contingent claim being maximal. As we consider both concepts only in numéraire markets satisfying NGE for $\mathcal{U}(\mathcal{S})$, we do not define them in full generality here.

**Definition 7.30.** Let $\mathcal{S}$ be a numéraire market satisfying NGE for $\mathcal{U}(\mathcal{S})$, $\tau \in \mathcal{T}_{[0,T]}$ a stopping time and $F : \mathcal{S} \rightarrow \mathbf{L}^0_+ (\mathcal{F}_\tau)$ a contingent claim with $\Pi(F | \mathcal{U}(\mathcal{S})) < \infty$. Then $F$ is called

- maximal if for all non-zero contingent claims $G : \mathcal{S} \rightarrow \mathbf{L}^0_+ (\mathcal{F}_\tau)$,
  $$\Pi(F + G | \mathcal{U}(\mathcal{S})) > \Pi(F | \mathcal{U}(\mathcal{S})).$$
- attainable if there exists a maximal strategy $\vartheta^* \in \mathcal{U}(\mathcal{S})$ such that
  $$V_\tau(\vartheta^*) = F \text{ P-a.s.}$$

It follows directly from the definition of (strongly) maximal strategies that an attainable contingent claim is a fortiori maximal. The following result shows that also the converse is true. Moreover, it provides a dual characterisation of maximal/attainable contingent claims. This kind of result is well known if $\eta := (1, 0, \ldots, 0)$ is a maximal numéraire strategy for $\mathcal{S}$ and if one works with its numéraire representative $S^{(\eta)}$. For the convenience of the reader we give a full proof.

**Proposition 7.31.** Let $\mathcal{S}$ be a numéraire market satisfying NGE for $\mathcal{U}(\mathcal{S})$, $\tau \in \mathcal{T}_{[0,T]}$ a stopping time and $F : \mathcal{S} \rightarrow \mathbf{L}^0_+ (\mathcal{F}_\tau)$ a contingent claim with $\Pi(F | \mathcal{U}(\mathcal{S})) < \infty$. Then the following are equivalent:

1. $F$ is maximal.
2. $F$ is attainable.
3. There exist $Q \approx P$ on $\mathcal{F}_\tau$ and $S^* \in \mathcal{M}_Q$ such that $\Pi(F | \mathcal{U}(\mathcal{S}))(S^*) = E_Q[F(S^*)]$.
4. For all $S \in \mathcal{S}$, there exist $Q \approx P$ on $\mathcal{F}_\tau$ and a maximal numéraire strategy $\eta^*$ such that $S^{(\eta^*)} \in \mathcal{M}_Q(S_0)$ and
   $$\Pi(F | \mathcal{U}(\mathcal{S}))(S) = E_Q[F(S^{(\eta^*)})] = \sup_{\hat{S} \in \mathcal{M}_Q(S_0)} E_Q[F(\hat{S})].$$
5. For all $Q \approx P$ on $\mathcal{F}_\tau$ and all $S \in \mathcal{S}$, there exists $S^* \in \mathcal{M}_Q(S_0)$ such that
   $$\Pi(F | \mathcal{U}(\mathcal{S}))(S) = E_Q[F(S^*)] = \sup_{\hat{S} \in \mathcal{M}_Q(S_0)} E_Q[F(\hat{S})].$$

**Proof.** “(a) $\implies$ (b)”. By Theorem 7.28, there exists a strongly maximal strategy $\vartheta^* \in \mathcal{U}(\mathcal{S})$ such that
$$V_0(\vartheta^*) = \Pi(F | \mathcal{U}(\mathcal{S})) \quad \text{and} \quad V_\tau(\vartheta^*) \geq F \text{ P-a.s.}$$

By the definition of classical superreplication prices,
$$\Pi(V_\tau(\vartheta^*) | \mathcal{U}(\mathcal{S})) \leq V_0(\vartheta) \text{ P-a.s.}$$

Since $F$ is maximal and $V_\tau(\vartheta^*) \geq F$ P-a.s., this implies
$$V_\tau(\vartheta^*) = F \text{ P-a.s.}$$

“(b) $\implies$ (d)”. Let $\vartheta^* \in \mathcal{U}(\mathcal{S})$ be a maximal strategy satisfying $V_\tau(\vartheta^*) = F$ P-a.s. By Lemma 7.11, there exists $Q \approx P$ on $\mathcal{F}_\tau$ and a maximal numéraire strategy $\eta$ such that $S^{(\eta)} \in \mathcal{M}_Q$ and

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V(θ∗)(S(η)) is a true Q-martingale. Then Corollary 7.23, the Q-martingale property of V(θ∗)(S(η)) and the definition of classical superreplication prices give

\[ \Pi(F|U(S))(S(η)) \geq \mathbb{E}_Q[F(S(η))] = \mathbb{E}_Q[V_\gamma(θ∗)(S(η))] = V_0(θ∗)(S(η)) \]

\[ \geq \Pi(F|U(S))(S(η)). \]

Fix S ∈ S. Let D be the exchange rate process from Remark 2.3 (b) satisfying S = DS(η). Set η∗ := η/D0. Since D0 is a constant, it follows from the above and the definition of numéraire representatives that η∗ is again a maximal numéraire strategy, S(η∗) = D0S(η), V(θ∗)(S(η∗)) is a true Q-martingale and

\[ \Pi(F|U(S))(S(η∗)) = \mathbb{E}_Q[F(S(η∗))]. \]

Since S0(η∗) = D0S(η) = S, the claim follows by Remark 2.22 and Theorem 7.28.

“(d) ⇒ (c)” This is trivial.

“(c) ⇒ (e)” Fix Q ≈ P and S ∈ S. By assumption, there exists ̂Q ≈ P and ̂S ∈ M̂Q such that

\[ \Pi(F|U(S))(S^*) = \mathbb{E}_{\hat{Q}}[F(\hat{S})]. \quad (7.11) \]

Let Z be the density process of Q with respect to ̂Q and D the the exchange rate process from Remark 2.3 (b) satisfying S = DS. Set S∗ := Z−1D0S. Then S0∗ = D0S0 = S0 (since Z0 = 1 as F0 is P-trivial) and S∗ ∈ M̂Q(S0) by Remark 7.21 (b) and (c). Thus by Remark 2.22, the exchange rate consistence of contingent claims (2.8), Proposition 2.21 and (7.11),

\[ \Pi(F|U(S))(S) = D0\Pi(F|U(S))(\hat{S}) = D0\mathbb{E}_{\hat{Q}}[F(\hat{S})] = \mathbb{E}_{\hat{Q}}[Z_\gamma F(Z_\gamma^{-1}D0\hat{S})] = \mathbb{E}_Q[F(S^*)]. \]

Now the claim follows by Theorem 7.28.

“(e) ⇒ (a)” Let G : S → L1+ (F) be a non-zero contingent claim and Q ≈ P on F and S∗ ∈ M̂Q such that \( \Pi(F|U(S))(S^*) = \mathbb{E}_{\hat{Q}}[F(S^*)] \). Corollary 7.23 then gives

\[ \Pi(F + G|U(S))(S^*) \geq \mathbb{E}_Q[F(S^*) + G(S^*)] > \mathbb{E}_Q[F(S^*)] = \Pi(F|U(S))(S^*). \]

8 Comparison to other modelling frameworks

In this final section, we compare our numéraire independent approach of modelling financial markets and studying no-arbitrage to the standard and other recent approaches to these issues.

8.1 The standard modelling framework

First, the numéraire independent approach helps to get a deeper understanding of the standard modelling framework used throughout mathematical finance. A detailed comparison of the basic aspects of both frameworks has already been carried out in Herdegen (2012), Section 7. Hence we content ourselves here to compare the notion of no-arbitrage in the two frameworks.

As explained in Herdegen (2012), from the perspective of the numéraire independent modelling framework, one starts in the standard framework with a \( (d+1) \)-dimensional market \( S \) and assumes that the buy-and-hold strategy of the first\(^5\) asset \( η := (1, 0, \ldots, 0) \) is a numéraire strategy. One then works with the numéraire representative \( S(η) = S/S^1 \) and considers, for admissible trading, self-financing strategies of the form \( \theta = \tilde{α} - αη \), where \( \tilde{α} \in Υ(S) \) and \( α ≥ 0 \). The assumption that \( S(η) \) satisfies NA or NFLVR is then equivalent to assuming that \( S(η) \) satisfies wnm(η) or sm(η), i.e., in terms of maximal strategies, that \( η \) is weakly or strongly maximal in the classical sense for \( Υ(S) \) (see Proposition 5.10).

\(^5\)What we call \( S^1 \) here is usually denoted by \( S^0 \) in the standard framework, and the market is often called \( d \)-dimensional there because \( S^0 \) is assumed to be constant 1 (and hence does not seem to matter). But a precise formulation is that there are \( d + 1 \) traded assets (including the “bank account” \( S^0 \)) of which \( d \) are viewed as being “risky”. In contrast, our dimension is counting all traded assets.
If one interprets the first asset – as usually done in the standard framework – as bank account, NA or NFLVR therefore tacitly include the assumption that investing in the bank account is an *optimal* investment in some sense, as $\eta$ is maximal. This is in remarkable contrast to the financial folklore advice that one should not put all one’s money into one asset (let alone the bank account). The requirement that $S(\eta)$ satisfies NA or NFLVR therefore looks rather too restrictive. Moreover, even if $S(\eta)$ fails NA or NFLVR, one can make “money out of nothing without risk” only if $S$ fails NGE if one only allows undefaultable (as opposed to admissible) strategies for trading; otherwise the only thing one can (and definitely should) do is to invest one’s money better than into the bank account. With some anticipation, we remark that this last idea of having a better reference asset than the bank account has also come up in other papers; see in particular Section 8.3 below.

By contrast, the numéraire independent concept NGE, or its equivalent notions BK (see Kabanov (1997)) or NUPBR (see Kardaras (2012)), really captures the idea of forbidding to “make money out of nothing without risk”: If the market $S$ fails NGE for $U(S)$, then with some *fixed* positive probability, one can get as much as one likes in any currency unit one chooses and with as tiny an initial investment (again in the same arbitrary currency unit one chooses) as one likes (see Corollary 3.6). And if the (numéraire) market $S$ satisfies NGE for $U(S)$, there exist many “good” (i.e. strongly maximal for $U(S)$) numéraire strategies whose representatives satisfy NFLVR (see Theorems 7.10 and 7.9); the only but crucial caveat is that one cannot choose an arbitrary numéraire and expect to have good properties in that particular currency unit.

So assuming NFLVR for $S(\eta)$ for the particular choice $\eta = (1,0,\ldots,0)$ is too strong and artificial. As long as $S$ satisfies NGE for $U(S)$, there are always enough “good” investment opportunities (i.e. strongly maximal strategies for $U(S)$); but since $S^1,\ldots,S^{d+1}$ can be chosen almost arbitrary, it is too much to expect that one of these $d+1$ basic assets should automatically be so well-behaved. One still gets NFLVR from NGE, but one may have to choose a numéraire representative different from $S(\eta)$. Moreover, if one replaces $\eta = (1,0,\ldots,0)$ by a dominating maximal numéraire strategy $\eta^*$ (see Theorem 7.5), its numéraire representative $S(\eta^*)$ is even better than $S(\eta)$.

To sum up the above discussions: Picking one of the basic $d+1$ assets as numéraire can be too restrictive. But after choosing a “good” numéraire representative $S(\eta)$, one can proceed as in the standard framework using $V(\eta)(S(\eta))$ as “riskless asset” and $S(\eta)^1,\ldots,S(\eta)^{d+1}$ as “risky assets”, and apply in these units all results from mathematical finance that are formulated in the standard framework (with one dimension more). So with just a modicum of care in the choice of units, the standard modelling framework can be embedded into our numéraire independent approach.

### 8.2 The fair market framework of Yan

The numéraire independent modelling framework also ties up with the concepts of *fair markets* and *allowable strategies* developed and propagated by Yan (see Yan (1998) and Xia & Yan (2003)).

In Yan’s framework, one starts as in the standard setup with a $(d+1)$-dimensional semimartingale $S = (S^0,\ldots,S^d)$ describing the development of $d+1$ strictly positive risky assets\(^6\). One then considers *discounted* asset prices, where one either discounts by one of the basic assets $S^0,\ldots,S^d$ or by their sum $\sum_{i=0}^d S_i$. A market is called *fair* if the prices discounted by the first asset $S^0$ admit a *true* equivalent martingale measure (see Yan (1998), Definition 2.1). It is then shown that a market is fair if and only if the prices discounted by any of the assets $S^0,\ldots,S^d$ admits a true martingale measure (see Yan (1998), Theorem 2.2). Moreover, a (self-financing) strategy is called *allowable* if it is bounded from below by $-a \sum_{i=0}^d S_i$ for some $a > 0$ (see Yan (1998), Definition 2.4), and it is shown that a market is fair if and only if $S$ satisfies NFLVR for allowable strategies (see Yan (1998), Theorem 3.2).

In the language of our numéraire independent modelling framework, one starts with a $(d+1)$-dimensional nonnegative numéraire market $S$ satisfying $\inf_{t\in[0,T]} S_i^t > 0$ $P$-a.s. for all $S \in S$ and all $i = 1,\ldots,d+1$. (In other words, each basic asset $S^i$ is an exchange rate process.) One then considers the numéraire strategies $\eta^{(i)} = (0,\ldots,0,1,0,\ldots,0)$ where the 1 is at position $i$.

\(^6\)Note that $S^0$ is not assumed to be constant 1 here.
i = 1, \ldots, d + 1, and the numéraire strategy \( \eta = (1, \ldots, 1) \). The market is fair if \( S^{(\eta)} \) is a true equivalent martingale representative. By Corollaries 7.17 and 7.18, this is the case if and only if \( S^{(\eta)} \), \( i = 1, \ldots, d + 1 \), and \( S^{(\eta)} \) are true equivalent martingale representatives, which is a stronger requirement for \( S \) than NGE for \( \mathcal{U}(S) \). An allowable strategy is simply an admissible strategy given \( S^{(\eta)} \), and Theorem 7.9 and Corollary 7.18 imply that a market is fair if and only if \( S^{(\eta)} \) satisfies NFLVR. So we recover the results of Yan (1998).

\[
\begin{align*}
\sup_{\eta \text{ num.}} E \left[ \log \frac{V_T(\eta)(S^{(\eta)})}{V_0(\eta)(S^{(\eta)})} \right] < \infty, \quad (8.1)
\end{align*}
\]

**Note that in Karatzas & Kardaras (2007) trading strategies are parametrised in fractions of wealth rather than in number of shares.**

### 8.3 The numéraire portfolio and the benchmark framework

Next, the numéraire independent modelling framework sheds new light on the numéraire portfolio developed by various authors in increasing generality (see Karatzas & Kardaras (2007) and references therein) and on the closely related benchmark approach to modelling financial markets developed and propagated by Platen (see Platen & Heath (2006) and references therein).

In the benchmark framework, one starts out as in the standard framework with a \((d+1)\)-dimensional semimartingale \( S = (S_0, \ldots, S_d) \) describing the evolution of \( d + 1 \) risky assets (with \( S_0 \equiv 1 \)). One then looks for a self-financing strategy\(^7\) \( \rho \) with strictly positive wealth (i.e. value) process \( W^\rho \) and \( W^\rho_0 = 1 \) such that the relative wealth process \( W^\pi / W^\rho \) for every self-financing strategy \( \pi \) is a \( \mathbb{P} \)-supermartingale (see Karatzas & Kardaras (2007), Definition 3.1). Such a strategy \( \rho \) (or alternatively its wealth process \( W^\rho \)) is called numéraire portfolio or growth optimal portfolio (GOP) or benchmark portfolio. If it exists, it is unique and has several compelling features, e.g. in terms of utility maximisation; see Karatzas & Kardaras (2007), Remark 3.2. The numéraire portfolio exists if and only if it satisfies the so called NUPBR condition (see Karatzas & Kardaras (2007), Definition 4.11 and Theorem 4.12). Given the existence of the numéraire portfolio or GOP, the benchmark approach consists in discounting \( S \) by the GOP (called \( S^{\delta} \) in Platen & Heath (2006), Definition 10.2.1) and calling the resulting assets or portfolios (denoted by \( \hat{S}^{\delta} \) in Platen & Heath (2006)) benchmarked assets or portfolios. The benchmark framework is economically appealing for two main reasons:

1. Benchmarked securities and portfolios (the latter meaning the value processes of self-financing strategies) are \( \mathbb{P} \)-supermartingales, and hedging and pricing can be done under the physical or objective measure \( \mathbb{P} \). This has a clear economic meaning, in contrast to pricing in the standard framework under a risk-neutral measure \( \mathbb{Q} \), which lacks a straightforward economic interpretation.

2. The benchmark portfolio can also exist in markets “admitting arbitrage” in the sense that \( S \) fails NFLVR (but still satisfies NUPBR). Thus it is a bit more general that the standard approach.

In the language of our numéraire independent modelling framework, one starts out with a \((d+1)\)-dimensional numéraire market \( S \). The numéraire portfolio is then a (normalised) numéraire strategy \( \eta^* \) with the property that for each (normalised) numéraire strategy \( \eta \), the value process \( V(\eta)(S^{(\eta)}) \) is a \( \mathbb{P} \)-supermartingale. If a numéraire portfolio \( \eta^* \) exists, a similar argument as in the proof of Lemma 7.11 shows that \( \eta^* \) is a strongly maximal strategy in the classical/limit quantiles sense for \( \mathcal{U}(S) \). In particular, \( S^{(\eta^*)} \) satisfies NFLVR and is an equivalent martingale representative by Theorem 7.9. Hence \( S \) satisfies a fortiori NGE for \( \mathcal{U}(S) \) by Proposition 5.3.

The converse direction that NGE for \( \mathcal{U}(S) \) implies the existence of the numéraire portfolio (which by the equivalence of NGE and NUPBR follows from Karatzas & Kardaras (2007)), however, does not follow directly from the numéraire independent modelling framework. By using Becherer (2001) Theorem 4.5, it would be sufficient to show that if \( S \) satisfies NGE for \( \mathcal{U}(S) \), then there exists some maximal numéraire strategy \( \hat{\eta} \) such that

\[
\sup_{\eta \text{ num.}} E \left[ \log \frac{V_T(\eta)(S^{(\eta)})}{V_0(\eta)(S^{(\eta)})} \right] < \infty, \quad (8.1)
\]
where “num.” stands for “numéraire strategy”, and the denominator is needed for normalisation. Moreover, by Theorem 7.5, it even suffices to consider in (8.1) only maximal numéraire strategies. Therefore, using (2.5), (8.1) is equivalent to

\[
\inf_{\eta} \sup_{\max. \ num.} \ E \left[ \log \left( \frac{V_T(\eta)(S)}{V_T(\eta)(S)} \right) \right] < \infty, \tag{8.2}
\]

where “max. num.” stands for “maximal numéraire strategy” and \( S \in S \) is an arbitrary representative. At present, however, it is an open question whether (8.2) can be proved directly from NGE without excessive effort.

Finally, note that NGE for \( U(S) \) is the minimal necessary (and also sufficient) assumption on \( S \) to study utility maximisation in a meaningful way. This follows from the equivalence of NGE and NUPBR together with Karatzas & Kardaras (2007), Proposition 4.19. Note, however, that at the moment an appropriate numéraire independent formulation of the concept of “utility maximisation” seems out of reach, and we postpone this very interesting question to further research (see, however, Kardaras (2010)).

### 8.4 The equivalent local martingale deflator framework

Last but not least, our numéraire independent modelling framework can also be linked to the concept of equivalent local martingale deflators introduced by Kardaras (see Kardaras (2012) and references therein).

In Kardaras’ approach, one starts out as in the standard framework with a \((d+1)\)-dimensional\(^8\) semimartingale \( S = (S^0, \ldots, S^d) \) describing the evolution of \( d+1 \) risky assets, where it is assumed that \( S^0 \equiv 1 \). An equivalent local martingale deflator (ELMD) is then a strictly positive local martingale \( Z \) starting at 1 such that \( ZX \) is a local martingale for any nonnegative wealth process \( X \) (see Kardaras (2012), Sections 1.2 and 1.3). Using advanced tools from the general theory of stochastic processes, in particular predictable characteristics of semimartingales, it is shown that an ELMD exists if and only if the market does not admit arbitraging of the first kind which is equivalent to \( S \) satisfying the NUPBR condition, which in turn is equivalent to the BK property (see Kardaras (2012), Theorem 1.1 and Section 1.3).

In the language of our numéraire independent modelling framework, one starts with a \((d+1)\)-dimensional market \( S \), assumes as in Section 8.1 that \( \eta = (1, 0, \ldots, 0) \) is a numéraire strategy and considers the numéraire representative \( S^{(0)} \). An equivalent local martingale deflator is a (normalised) exchange rate process \( D \) with the property that \( DS^{(0)} \) is a local martingale representative. Note that since \( S^{(0)} \equiv 1 \), this property automatically implies that \( D \) is a local martingale. Since the BK property is equivalent to the NGE property for \( U(S) \) by Propositions 5.3 and 5.10, Theorem 1.1 in Kardaras (2012) follows from our Theorem 7.10, which also gives the additional characterisation (d).

#### Remark 8.1

After finishing this paper, we have been informed about the multidimensional extension of Kardaras’ result by Takaoka (2012), which uses a change of numéraire technique somewhat similar to some arguments used in our paper (even thought this might not be immediately apparent). In particular, the proof of Proposition 2.7 (i) in Takaoka (2012) is similar to the proof of our Proposition 7.2, and the first part of the proof of Theorem 2.6 in Takaoka (2012) uses similar arguments as the proof of our Theorem 6.3.

### A A fact from measure theory

#### Proposition A.1

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \( g \in L^0_\infty(\mathcal{F}) \). Then for all \( K < E[g] \), there exists \( \epsilon \in (0, 1) \) such that for all \( A \in \mathcal{F} \) with \( \mathbb{P}[A] \geq 1 - \epsilon \),

\[
E[g 1_A] \geq K.
\]

\(^8\)For notational and technical convenience, it is assumed that \( d = 1 \) in Kardaras (2012).
Remark A.2. In the above result the case $E[g] = +\infty$ is included. Moreover, it is straightforward to check that $\epsilon$ can always be chosen smaller than some arbitrary fixed $\delta > 0$.

Proof. Let $K < E[g]$. By monotone convergence, there exists $N \in \mathbb{N}$ such that

$$E[g\mathbb{1}_{\{g \leq N\}}] > K.$$

Set

$$\epsilon := \min\left(\frac{E[g\mathbb{1}_{\{g \leq N\}}] - K}{N}, \frac{1}{2}\right).$$

Then for all $A \in \mathcal{F}$ with $P[A] \geq 1 - \epsilon$,

$$E[g\mathbb{1}_A] \geq E[g\mathbb{1}_{\{g \leq N\}\cap A}] = E[g\mathbb{1}_{\{g \leq N\}}] - E[g\mathbb{1}_{\{g \leq N\}\cap A^c}]$$

$$\geq E[g\mathbb{1}_{\{g \leq N\}}] - E[N\mathbb{1}_A] \geq E[g\mathbb{1}_{\{g \leq N\}}] - (E[g\mathbb{1}_{\{g \leq N\}}] - K)$$

$$= K. \quad \Box$$

References


