FINRISK
National Centre of Competence in Research
Financial Valuation and Risk Management

National Centre of Competence in Research
Financial Valuation and Risk Management

Working Paper No. 840

Partial Hedging and Cash Requirement in Discrete Time

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First version: March 2013
Current version: June 2014

This research has been carried out within the NCCR FINRISK project on "Mathematical Methods in Financial Risk Management"

Partial Hedging and Cash Requirement in Discrete Time *

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June 4, 2014

Abstract

This paper develops a discrete-time version of the continuous-time model of Bouchard et al. for the problem of finding the minimal initial data for a controlled process to guarantee reaching a controlled target with probability one. An efficient numerical algorithm, based on dynamic programming, is proposed for the quantile hedging of standard call and put options, exotic options, and quantile hedging with portfolio constraints. The method is then extended to solve utility indifference pricing, good-deal bounds, and expected shortfall problems.

Keywords: mathematical finance, quantile hedging, stochastic optimal control, dynamic programming.

1 Introduction

In incomplete markets or markets with friction one can only determine no-arbitrage intervals, outside of which there is arbitrage. One is then faced with the issue of choosing an appropriate price among many. This choice depends crucially on the risk preferences of the individual investors and the real world measure. It is important to note that in a complete market the risk preferences do not play any role because, at least in theory, perfect hedging completely removes the risk associated with hedging as well as the opportunity to make a profit. However, in the other case, option pricing has to be based on individual's attitude toward risk which is modeled by a utility or loss function. For instance, a fully risk averse investor may choose to stay on the safe side by using a super-hedging strategy. The idea of super-hedging has been introduced to the literature by Bensaid et al. in discrete time and El-Karoui and Quenez in continuous time. The goal of super-hedging is to generate a final wealth that dominates the payoff of the contingent claim.

On the other hand if the investor is willing to take some risk, then there will be a reduction of the initial cost in the form of a risk premium. Similarly, if the investor wants to invest less capital than what is required for perfect or super-hedging of the liability, then some shortfall risk has to be accepted. In their seminal paper, Föllmer and Leukert use probability as a risk measure to quantify this shortfall risk. They describe quantile hedging as the optimal strategy when initial capital is less than the minimal super-hedging or perfect hedging cost. In addition, they determine the minimal amount of initial capital an investor can save by accepting a certain shortfall probability.

Spivak and Cvitanic, and the references therein, also studied quantile hedging and related problems in continuous time. The solution in complete markets was re-discovered using a duality

*The authors are grateful to H. Mete Soner, Selim Gökay for insightful discussions and comments. They also thank the two anonymous referees for their helpful comments and suggestions.

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approach in the context of utility maximization. The technique was further applied to solve the problem in a market with partial information and in markets in which the wealth process of the agent has a nonlinear drift. Krutchenko and Melnikov extend the work of Föllmer and Leukert to a jump-diffusion financial market model. Lindberg poses the quantile hedging problem as a knapsack problem and develops an efficient algorithm which works for European options in a discrete-time complete market model. Perez-Hernandez study the quantile hedging problem for American contingent claims in an infinite-state space setting from the perspective of the writer of the claim and Pinar investigates the same problem from the point of view of the buyer by minimizing the expected failure ratio. There are also a number of papers applying the quantile hedging approach to the insurance setting, e.g. Melnikov and Skornyakova.

In a recent paper, Bouchard et al. provide a different approach to quantile hedging. They consider the more general problem of finding the minimal initial data of a controlled process which guarantees reaching a controlled target with a given probability of success. As a special case, they focus on the quantile hedging problem and reproduce the explicit solution of Föllmer and Leukert in continuous time. We discretize their model and demonstrate how one can easily recover the solutions to quantile hedging, utility indifference pricing, good deal pricing, and expected shortfall problems. In addition, the discrete framework allows us to easily address quantile hedging with portfolio constraints.

Moreover, we prove the convergence and stability of our numerical scheme. Provided the continuous time limit problem satisfies a comparison principle (c.f. Crandall et al.), we show that the discrete-time solutions converge locally uniformly to the unique viscosity solution of the limiting equation. The method of proof is based on the procedure of Barles and Souganidis.

The paper is organized as follows: The next section presents a description of our financial market model. The general problem and its equivalent formulation are studied in Section 3. In Section 4, we derive the dynamic programming principle and propose the algorithm. The convergence and stability analysis is provided in Section 5. Section 6 includes an application of our method to quantile hedging. Finally, numerical results are provided in Section 7.

2 Complete Financial Market Setting

We consider a discrete-time complete market framework with one risk-free and one risky asset. We fix a time horizon, or equivalently a maturity, $T > 0$ and a time discretization $h := \frac{T}{n}$ for a large integer n. Denote

$$\mathcal{P}_n := \mathcal{P}_n([0,T]) = \{0, h, 2h \ldots, T-h, T\} \subset [0,T]$$

for the discretized time scale and define the order-preserving indexing map

$$\mathcal{I} : \mathcal{P}_n \to \{1, 2, \ldots, n+1\}.$$

We assume that a risky asset (discrete) price process X_t evolves according to

$$X_{t+h} = X_t \left(1 + \mu h + \xi_{t+h} \sigma \sqrt{h}\right), \tag{1}$$

where μ is the mean return rate, σ is the volatility, and $\{\xi_t\}_{t\geq 0}$ is an i.i.d. random sequence with

$$\mathbb{E}\left[\xi_t\right] = 0, \ \mathbb{E}\left[\xi_t^2\right] = 1.$$

We assume that ξ_t takes values in $\{-1, +1\}$ and hence we have a binomial tree structure for the risky asset. Then, the no-arbitrage condition

$$1 + \mu h - \sigma\sqrt{h} < 1 + rh < 1 + \mu h + \sigma\sqrt{h}, \tag{2}$$

is satisfied, for $h > 0$ sufficiently small. This will also be useful to speed up the computations in our model (it allows us to apply Lemma 4.2). We take interest rate $r = 0$ to simplify the calculations. For $t, s \in \mathcal{P}_n$ with $t \leq s$, we define the filtration

$$\mathcal{F}_t^s := \sigma(\xi_t, \ldots, \xi_s) = \sigma(X_t, \ldots, X_s). \tag{3}$$

to be generated by the spot prices $X_t, \ldots X_s$. Hence the sigma-algebra $\mathcal{F}_t^s$ contains the information from time t to s. We consider a self-financing *wealth process* Y_t following the dynamics

$$Y_{t+h} = Y_t + Z_t(X_{t+h} - X_t), \tag{4}$$

for a *portfolio process* $Z_\cdot$. We denote by $\mathcal{A}^t$ the set of all $\mathcal{F}_t^s$- adapted, portfolio (control) processes $Z_s^{t,z}$ for $t, s \in \mathcal{P}_n$, $t \leq s$, with $Z_t^{t,z} = z$. Given initial conditions (t, x) and (t, y) we employ the notation $X_\cdot^{t,x}$, $Y_\cdot^{t,y,Z_\cdot}$ (spot price and wealth processes, resp.) for the solutions to (1) and (4) with the initial conditions

$$X_t^{t,x} = x, \quad Y_t^{t,y,Z} = y.$$

The *liability* is assumed to be Markov with nonnegative payoff $g(X_T)$. We assume g is a continuous function which satisfies the linear growth condition $0 \leq g(s) < C(1 + s)$, for all $s > 0$.

3 The general problem and an equivalent formulation

We are given a non-decreasing and concave utility function $U : \mathbf{dom}\ U \subset \mathbb{R} \to \mathbb{R}$ which is defined on some connected subset of $\mathbb{R}$ (e.g. half-space or $\mathbb{R}$ itself, generally) and a *threshold* utility level u^*. We consider the problem of finding the minimal initial value of a controlled wealth process such that the expected utility from terminal wealth minus liability is guaranteed to surpass a pre-specified target utility level u^*

$$V(t, x) := \inf \left\{ y \in \mathbb{R} \ : \ Z \in \mathcal{A}^t, \ \mathbb{E}\left[U\left(Y_T^{t,y,Z} - g(X_T^{t,x}) \right) \right] \geq u^* \right\}. \tag{5}$$

Given an admissible portfolio process Z, we define

$$u_s := \mathbb{E}\left[U(Y_T^{t,y,Z} - g(X_T^{t,x})) \mid \mathcal{F}_t^s \right]. \tag{6}$$

It is clear that u_s depends on the initial conditions, as well as on the portfolio process, and in the sequel we will suppress these dependencies. First we note that

$$\mathbb{E}\left[u_{s+1} \mid \mathcal{F}_t^s \right] \;=\; \mathbb{E}\left[\mathbb{E}\left[U\left(Y_T^{t,y,Z} - g(X_T^{t,x}) \right) \mid \mathcal{F}_t^{s+1} \right] \mid \mathcal{F}_t^s \right] = u_s.$$

Hence, u_s is an $\mathcal{F}_t^s$-martingale. By the Martingale Representation Theorem, there exists an $\mathcal{F}_t^s$-adapted process α_s such that

$$u_{s+h} = u_s + \alpha_s \xi_{s+h}. \tag{7}$$

Moreover, in view the original problem (5)

$$u_t = \mathbb{E}\left[U\left(Y_T^{t,y,Z} - g(X_T^{t,x}) \right) \right] \geq u^*.$$

We expect this inequality to saturate. Therefore, we use the initial condition

$$u_t = u^*. \tag{8}$$

This process will be denoted by $u_s^{t,u^*,\alpha}$. Since

$$u_T^{t,u^*,\alpha} = U\left(Y_T^{t,y,Z} - g(X_T^{t,x})\right)^{1},$$

we formulate the following *super-replication* problem

$$\begin{aligned} v(t,x,u^*) := \inf\Big\{ \; y \in \mathbb{R} \; : \alpha \text{ is } \mathcal{F}_t^s\text{-adapted, } Z \in \mathcal{A}^t, \text{ and} \\ Y_T^{t,y,Z} \geq U^{-1}(u_T^{t,u^*,\alpha}) + g(X_T^{t,x}) \; \mathbb{P}\text{-a.s. } \Big\}, \end{aligned} \tag{9}$$

where

$$U^{-1}(u) = \inf\{y \in \textbf{dom } U \; : \; U(y) \geq u\}^{2}$$

Since U is concave, U^{-1} is convex.

Theorem 3.1. *For every $(t,x) \in [0,T] \times \mathbb{R}_+$, we have $V(t,x) = v(t,x,u^*)$.*

Proof. We first show $v \leq V$. Choose $Z^\epsilon \in \mathcal{A}_t^s$ such that

$$\mathbb{E}\left[U\left(Y_T^{t,y^\epsilon,Z^\epsilon} - g(X_T^{t,x})\right)\right] \geq u^*,$$

for some $y^\epsilon \leq V(t,x) + \epsilon$. Next, define u_s as in (6). By construction,

$$u_T \;\; = \;\; U\left(Y_T^{t,y^\epsilon,Z^\epsilon} - g(X_T^{t,x})\right), \quad \mathbb{P}\text{-a.s.}$$

By definition of U^{-1}, we see that $Y_T^{t,y^\epsilon,Z^\epsilon} \geq U^{-1}(u_T) + g(X_T^{t,x})$ holds almost surely. Then, $v(t,x,u^*) \leq y^\epsilon \leq V(t,x) + \epsilon$. Since ϵ was arbitrary, we have $v(t,x,u^*) \leq V(t,x)$.

To prove the reverse inequality let $Z^\epsilon \in \mathcal{A}_t^s$ and α^ϵ be super-replicating for problem (9) with initial wealth $y^\epsilon \leq v(t,x,u^*) + \epsilon$. Then,

$$\begin{aligned} & Y_T^{t,y^\epsilon,Z^\epsilon} \geq U^{-1}(u_T^{t,u^*,\alpha^\epsilon}) + g(X_T^{t,x}), \quad \mathbb{P}\text{-a.s.} \\ \Rightarrow \quad & U\left(Y_T^{t,y^\epsilon,Z^\epsilon} - g(X_T^{t,x})\right) \geq u_T^{t,u^*,\alpha^\epsilon}, \quad \mathbb{P}\text{-a.s.} \\ \Rightarrow \quad & \mathbb{E}\left[U\left(Y_T^{t,y^\epsilon,Z^\epsilon} - g(X_T^{t,x})\right)\right] \geq \mathbb{E}\left[u_T^{t,u^*,\alpha^\epsilon}\right] = u^*. \end{aligned}$$

It follows that $V(t,x) \leq y^\epsilon \leq v(t,x,u^*) + \epsilon$, by the definition of $V(t,x)$. Since ϵ was arbitrary, we conclude that $V(t,x) \leq v(t,x,u^*)$ and thus there is equality. □

4 Dynamic Programming

The classical super-hedging problem is approached via a dual formulation. The underlying assumptions are that the wealth dynamics are linear in the control variable and that trading strategies do not influence spot prices. However, more general dynamics and constraints (e.g. gamma constraints) cannot be treated by this approach. This motivated Soner and Touzi to introduce stochastic target methods to provide a PDE characterization of the super-hedging price of European claims under gamma constraints. Their main contribution is a dynamic programming principle which is written directly on the associated stochastic target problem. Bouchard et al. extends the work of Soner and Touzi to address pricing problems under risk constraints, such as quantile hedging. The following dynamic programming principle is easily proved by standard techniques (see Fleming and Soner and Soner and Touzi).

[1]If u is constrained to some interval I, we restrict to adapted (control) processes α_s for which $u_s^{t,u,\alpha} \in I$, for all $s \geq t$, $\mathbb{P}$-almost surely. This is achieved by the same arguments as in the case of quantile hedging (see Proposition 3.1 in Bouchard et al.).

[2]we use the convention that $\inf \emptyset = +\infty$

Theorem 4.1. *For any stopping time τ taking values in $\mathcal{P}_n$ with $t < \tau \leq T$, the minimum super-replication cost $v(t,x,u)$ satisfies*

$$v(t,x,u) = \inf \Big\{ \ y \in \mathbb{R} \ : \alpha \textit{ is } \mathcal{F}_t^s\textit{-adapted, } Z \in \mathcal{A}^t, \textit{ and} \\ Y_\tau^{t,y,Z} \geq v(\tau, X_\tau^{t,x}, u_\tau^{t,u,\alpha}) \ \mathbb{P}\textit{-a.s.} \ \Big\}. \tag{10}$$

By taking $\tau = t + h$, we conclude that $v(t,x,u) = \min y$ among all y satisfying

$$Y_{t+h}^{t,y,Z} \geq v(t+h, X_{t+h}^{t,x}, u_{t+h}^{t,u,\alpha}),$$

both when the risky asset moves up and when it moves down. Using the wealth equation (4), this can be rewritten as

$$y + Z_t(X_{t+h} - X_t) \geq v(t+h, X_{t+h}^{t,x}, u_{t+h}^{t,u,\alpha}).$$

In particular, setting

$$\begin{aligned} x_+ &:= x\left(1 + \mu h + \sigma\sqrt{h}\right), \\ x_- &:= x\left(1 + \mu h - \sigma\sqrt{h}\right), \end{aligned}$$

the dynamic programming equation becomes

$$\begin{aligned} v(t,x,u) &= \min y \quad \text{among all y satisfying} \\ & \quad y + z\left(x_+ - x\right) \geq v\left(t+h, x_+, u+\alpha\right), \\ & \quad y + z\left(x_- - x\right) \geq v\left(t+h, x_-, u-\alpha\right). \end{aligned}$$

Hence,

$$v(t,x,u) = \inf_{\alpha,z} \left\{ \max \left\{ \begin{array}{l} v\left(t+h, x_+, u+\alpha\right) - z\left(x_+ - x\right), \\ \\ v\left(t+h, x_-, u-\alpha\right) - z\left(x_- - x\right) \end{array} \right\} \right\}. \tag{11}$$

The above equation is complemented by the terminal condition

$$v(T,x,u) = U^{-1}(u) + g(x). \tag{12}$$

When there are state constraints on u, one needs to additionally state lateral boundary conditions. For instance, in the case of quantile hedging, where u represents a probability, clearly $u \in [0,1]$ and conditions at $u = 0, 1$ must be specified.

Lemma 4.2. *Let $A, B \subseteq \mathbb{R}$ be two non-empty closed intervals in $\mathbb{R}$ and $F, G : A \times B \to \mathbb{R}$ such that for fixed $\alpha \in A$, $z \mapsto F(\alpha, z)$ is non-increasing and $z \to G(\alpha, z)$ is non-decreasing. If for each $\alpha \in A$, there exists $z^*(\alpha) \in B$ such that $F(\alpha, z^*(\alpha)) = G(\alpha, z^*(\alpha))$, then the following holds:*

$$\inf_{z \in B} \left\{ \max\left\{ F(\alpha,z), G(\alpha,z) \right\} \right\} = F(\alpha, z^*(\alpha)) = G(\alpha, z^*(\alpha)). \tag{13}$$

Proof. Let $r > 0$ be arbitrary. Then since

$$F(\alpha, z^*(\alpha) - r) \geq F(\alpha, z^*(\alpha)) \geq F(\alpha, z^*(\alpha) + r)$$

and

$$G(\alpha, z^*(\alpha) + r) \geq G(\alpha, z^*(\alpha)) \geq G(\alpha, z^*(\alpha) - r)$$

and since the middle terms are equal, it is clear that

$$\max\{F(\alpha, z^*(\alpha) \pm r), G(\alpha, z^*(\alpha) \pm r)\} \geq \max\{F(\alpha, z^*(\alpha)), G(\alpha, z^*(\alpha))\}, \quad \forall r > 0.$$

This precisely states that $z^*(\alpha)$ is the global minimum. □

Remark 4.3. *Let $F, G : A \times B \to \mathbb{R}$ satisfy the hypotheses of Lemma 4.2. Then the following holds*

$$\inf_{\alpha \in A, z \in B} \{\max\{F(\alpha, z), G(\alpha, z)\}\} = \inf_{\alpha \in A} F(\alpha, z^*(\alpha)) = \inf_{\alpha \in A} G(\alpha, z^*(\alpha)). \tag{14}$$

Theorem 4.4. *$v(t, x, u)$ is convex in u for all x and fixed t.*

Proof. Let $u_1, u_2 \in \mathbf{dom}\ U$ be given. By definition of v, for every $\epsilon > 0$, there exist $y^{i,\epsilon}, Z^{i,\epsilon}, \alpha^{i,\epsilon}$, where $i = 1, 2$ such that

$$y^{i,\epsilon} < v(t, x, \alpha^{i,\epsilon}) + \epsilon,$$

and

$$Y_T^{t, y^{i,\epsilon}, Z^{i,\epsilon}} \geq U^{-1}(u_t^{t, u_i, \alpha^{i,\epsilon}}) + g(X_T^{t,x}).$$

Set

$$Z^\epsilon = \lambda Z^{1,\epsilon} + (1-\lambda) Z^{2,\epsilon}, \quad \alpha^\epsilon = \lambda \alpha^{1,\epsilon} + (1-\lambda)\alpha^{2,\epsilon}, \quad y^\epsilon = \lambda y^{1,\epsilon} + (1-\lambda) y^{2,\epsilon}.$$

Note that Z^ϵ and α^ϵ will satisfy portfolio and domain constraints (provided they are convex sets), respectively.

We claim that

$$Y_T^{t, y^\epsilon, Z^\epsilon} \geq U^{-1}\left(u_T^{t, \lambda u_1 + (1-\lambda) u_2, \alpha^\epsilon}\right) + g(X_T^{t,x}).$$

Indeed, since

$$\begin{aligned}
Y_T^{t, y^\epsilon, Z^\epsilon} &= y^\epsilon + \sum_{j=\mathcal{I}(t)}^{\mathcal{I}(T-h)} Z_j^\epsilon (X_{\mathcal{I}^{-1}(j+1)} - X_{\mathcal{I}^{-1}(j)}) \\
&= \lambda y^{1,\epsilon} + (1-\lambda) y^{2,\epsilon} + \sum_{j=\mathcal{I}(t)}^{\mathcal{I}(T-h)} \left(\lambda Z^{1,\epsilon}_{\mathcal{I}^{-1}(j)} + (1-\lambda) Z^{2,\epsilon}_{\mathcal{I}^{-1}(j)}\right)(X_{\mathcal{I}^{-1}(j+1)} - X_{\mathcal{I}^{-1}(j)}) \\
&= \lambda Y_T^{t, y^{1,\epsilon}, Z^{1,\epsilon}} + (1-\lambda) Y_T^{t, y^{2,\epsilon}, Z^{2,\epsilon}} \\
&\geq \lambda \left(U^{-1}(u_T^{t, u_1, \alpha^{1,\epsilon}}) + g(X_T^{t,x})\right) + (1-\lambda)\left(U^{-1}(u_T^{t, u_2, \alpha^{2,\epsilon}}) + g(X_T^{t,x})\right) \\
&\geq U^{-1}\left(u_T^{t, \lambda u_1 + (1-\lambda) u_2, \alpha^\epsilon}\right) + g(X_T^{t,x}),
\end{aligned}$$

where the last inequality follows by convexity of U^{-1}.

Since,

$$\lambda y^{1,\epsilon} < \lambda v(t, x, u_1) + \lambda \epsilon, \quad (1-\lambda) y^{2,\epsilon} < (1-\lambda) v(t, x, u_2) + (1-\lambda)\epsilon,$$

we obtain

$$y^\epsilon = \lambda y^{1,\epsilon} + (1-\lambda) y^{2,\epsilon} \leq \lambda v(t, x, u^1) + (1-\lambda) v(t, x, u^2) + \epsilon.$$

Finally, ϵ was chosen arbitrarily, therefore we may conclude

$$v\left(t, x, \lambda u_1 + (1-\lambda) u_2\right) \leq \lambda v(t, x, u_1) + (1-\lambda) v(t, x, u_2).$$

□

4.1 Derivation of the Algorithm

We are now in a position to present our algorithm. We will apply Lemma 4.2 with the functions

$$\begin{aligned} F(\alpha, z) &= v\left(t+h, x_+, u+\alpha\right) - z(x_+ - x), \\ G(\alpha, z) &= v\left(t+h, x_-, u-\alpha\right) - z(x_- - x). \end{aligned}$$

By setting $F(\alpha, z) = G(\alpha, z)$, we can solve for

$$z^*(\alpha) = \frac{v\left(t+h, x_+, u+\alpha\right) - v\left(t+h, x_-, u-\alpha\right)}{x_+ - x_-}. \tag{15}$$

Hence,

$$\begin{aligned} v(t,x,u) &= F(\alpha, z^*(\alpha)) = G(\alpha, z^*(\alpha)) \\ &= \inf_\alpha \left(\frac{(x-x_-)v\left(t+h, x_+, u+\alpha\right) + (x_+ - x)v\left(t+h, x_-, u-\alpha\right)}{x_+ - x_-} \right) \end{aligned}$$

Finally, writing $p^* = \frac{x-x_-}{x_+-x_-}$, we arrive at

$$v(t,x,u) = \inf_\alpha \{p^* v\left(t+h, x_+, u+\alpha\right) + (1-p^*) v\left(t+h, x_-, u-\alpha\right)\}. \tag{16}$$

Remark 4.5. *The algorithm* (16) *preserves convexity in u at each computational step. To see this, suppose that $\ell(x,u)$ is convex in u. Define $\ell_0(x,u,\alpha) := p^*\ell(x_+, u+\alpha) + (1-p^*)\ell(x_-, u-\alpha)$. Observe that $\ell_0(x,u,\alpha)$ is a convex function of both u and α because $0 < p^* < 1$. Therefore, the function $\ell_1(x,u) := \inf_\alpha \ell_0(x,u,\alpha)$ is convex in u since the infimum is taken over a convex set*[3]. *Hence, at each step of* (16), *one only needs to solve a convex minimization problem and this can be done very efficiently.*

5 Convergence to the continuous time model

In this section, we prove that the numerical scheme converges to the unique viscosity[4] solution of the continuous time equation

$$\begin{aligned} H(t,x,u,\mathrm{D}\varphi(t,x,u), \mathrm{D}^2\varphi(t,x,u)) = 0, \quad (t,x,u) \in [0,T) \times \mathbb{R}_+ \times I \\ v(T,x,u) = U^{-1}(u) + g(x), \quad (x,u) \in \mathbb{R}_+ \times I. \end{aligned} \tag{17}$$

Here,

$$H(t,x,u,\mathrm{D}\varphi, \mathrm{D}^2\varphi) := \varphi_t + \frac{1}{2}\sigma^2 x^2 \varphi_{xx} + \inf_{a \in \mathbb{R}} \left\{ \frac{1}{2} a^2 \varphi_{uu} + a(\sigma x \varphi_{xu} - \frac{\mu}{\sigma}\varphi_u) \right\}.$$

The gradient $\mathrm{D}\varphi$ and Hessian $\mathrm{D}^2\varphi$ are taken with respect to (t,x,u) and (x,u), respectively. To simplify the presentation, we ignore state-constraints by taking $I = \mathbb{R}$.

The proof follows the framework of Barles and Souganidis; we prove the numerical scheme is *monotone*, *consistent*, and *stable*. The final step of the proof requires that the limit equation satisfies comparison, i.e. if u_1, u_2 are viscosity sub and super-solutions, resp., and $u_1(T,\cdot,\cdot) \le u_2(T,\cdot,\cdot)$, then $u_1 \le u_2$ everywhere. Comparison can be established in the unconstrained case under fairly general hypotheses (see Crandall et al.) in certain classes of functions. However, we will not attempt to

[3] see Boyd and Vandenberghe.

[4] For the definition of a viscosity solution, the reader is referred to Crandall et al..

prove a general comparison result here. For the purposes of this paper, we therefore set a standing assumption that there is comparison in a suitable class of functions. This class of functions is typically determined by analyzing the value function of the stochastic control problem from which (17) arises. The value function itself should be a member of this class and hence it is the unique viscosity solution in that class of functions.

Assumption 5.1. *There exists a unique (possibly discontinuous) viscosity solution of* (17) *in an appropriate class of functions.*

The convergence result of Barles and Souganidis exploits the stability of viscosity solutions under uniform limits. Informally, this says a locally bounded family of viscosity solutions to a convergent sequence of equations converges to a viscosity solution of the limit equation (see Barles and Perthame). The monotonicity property of the numerical scheme is akin to the ellipticity criterion which forms the linchpin of viscosity theory. In financial applications, monotonicity is also a desirable property since it guarantees the numerical scheme respects no-arbitrage inequalities at each computational step.

Lemma 5.2. *The numerical scheme* (11) *is monotone: if* v_1, v_2 *are two solutions of* (11) *and* $v_1(T,\cdot,\cdot) \leq v_2(T,\cdot,\cdot)$*, then* $v_1 \leq v_2$*.*

Proof. Suppose v_1, v_2 are two solutions of (11) and that $v_1(T,x) \leq v_2(T,x)$, for all $x \geq 0$. Then it is immediate that $v_1(T-h,x) \leq v_2(T-h,x)$ for all $x \geq 0$ simply by construction. Iterating finitely many times until completion yields $v_1(t,x) \leq v_2(t,x)$ for all $t \leq T$. □

We now define the *relaxed semi-limits*:

$$v^*(t,x,u) := \lim_{h\downarrow 0} \ \sup\left\{ v^h(t',x',u') \ : \ t' \in \mathcal{P}_{n(h)}, \ |t'-t| + |x'-x| + |u-u'| \leq h \right\},$$
$$v_*(t,x,u) := \lim_{h\downarrow 0} \ \inf\left\{ v^h(t',x',u') \ : \ t' \in \mathcal{P}_{n(h)}, \ |t'-t| + |x'-x| + |u-u'| \leq h \right\}.$$

In order to proceed, it needs to be demonstrated that these limits are finite. To this end, we show that v^h is *locally uniformly bounded*: for any compact set $\mathcal{D} \subset \mathbb{R}_+ \times \mathbb{R}$, there exists $h_0 > 0$ and a constant $M > 0$ such that

$$\sup_{0<h<h_0} \{ |v^h(t,x,u)| \ : \ (t,x,u) \in [0,T] \times \mathcal{D} \cap \mathbf{dom}\ v^h \} \leq M.$$

Indeed, it suffices to construct local upper and lower bounds for v^h which are independent of h.

Lemma 5.3. *The upper relaxed semi-limit* v^* *is finite.*

Proof. Let $(x,u) \in \mathbb{R}_+ \times \mathbb{R}$ and a mesh refinement $h = \frac{T}{n} > 0$ be given. Take $v^h(T,x,u) = U^{-1}(u) + g(x)$. For any $t \in \mathcal{P}_n$ observe that

$$\begin{aligned} v^h(t,x,u) &= \inf_{\alpha\in\mathbb{R}} \{ p^* v^h(t+h, x_+, u+\alpha) + (1-p^*) v^h(t+h, x_-, u-\alpha) \} \\ &\leq p^* v^h(t+h, x_+, u) + (1-p^*) v^h(t+h, x_-, u) \\ &\leq U^{-1}(u) + \sum_{k=0}^{n} \binom{n}{k} (p^*)^{n-k} (1-p^*)^k g(x(1_+)^{n-k}(1_-)^k) \\ &= U^{-1}(u) + \mathbb{E}_{p^*}\left[g(X_T) \middle| X_t = x \right]. \end{aligned}$$

As $h \to 0$, the term in the expectation converges locally uniformly in (t,x) to the (Black-Scholes) price of the claim g.

□

We now prove that the scheme produces approximations which are locally uniformly bounded from below. The estimates will depend on the precise form of the utility function (e.g. smoothness properties, domains of definition, effective domains, etc.) and each case can be tackled individually. When the wealth process is bounded from below, then the value function is as well and there is nothing to prove. This is again the case when the inverse of the utility function is bounded from below. We demonstrate the technique of the proof for the most well-known utility function which does not fall into the trivial class of examples, the exponential utility. The proof in other non-trivial cases can be modeled on the one that follows.

Lemma 5.4. *The lower relaxed semi-limit v_* is finite.*

Proof. By monotonicity, it is clear that $v^h \geq v_0^h$, where v_0^h is the solution to (11) for $g \equiv 0$. Thus, it suffices to show that v_0^h is locally uniformly bounded from below. Let us assume that $U(c) := -e^{-\gamma c}$ for some $\gamma > 0$. Then $F(u) := U^{-1}(u) = -\frac{1}{\gamma}\log(-u)$.

Set

$$\ell_0(u) = \inf_{\alpha\in\mathbb{R}} \{p^* F(u+\alpha) + (1-p^*)F(u-\alpha)\}.$$

The infimum[5] is achieved at

$$\alpha^* = \alpha^*(u) := (2p^*-1)u.$$

Therefore, utilizing convexity and homotheticity of F, we see

$$\begin{aligned}
\ell_0(u) &= p^* F(u+\alpha^*(u)) + (1-p^*)F(u-\alpha^*(u)) \\
&\geq F(u + (2p^*-1)\alpha^*(u)) \\
&= F([1+(2p^*-1)^2]u) \\
&= F(u) - \frac{1}{\gamma}\log(1+[2p^*-1]^2).
\end{aligned}$$

Note that p^* is explicit and $\beta := 1 + (2p^*-1)^2 = 1 + \left(\frac{\mu}{\sigma}\right)^2 h$.

For the next step, we write

$$\ell_1(u) = \inf_{\alpha} \{p^* \ell_0(u+\alpha) + (1-p^*)\ell_0(u-\alpha)\}.$$

Based on the previous discussion, it follows that

$$\begin{aligned}
\ell_1(u) &\geq \inf_{\alpha} \{p^* F(u+\alpha) + (1-p^*)F(u-\alpha)\} - \frac{1}{\gamma}\log(\beta) \\
&\geq F(u) - \frac{2}{\gamma}\log(\beta).
\end{aligned}$$

Repeating this procedure $N = \lceil T/h \rceil$ times, reveals that

$$\ell_N(u) \geq F(u) - \frac{N+1}{\gamma}\log(\beta)$$

Since $N \in O(h^{-1})$ and $\beta - 1 \in O(h)$, we observe that as h tends to 0 (equivalently, N tends to infinity), $\frac{N+1}{\gamma}\log(\beta)$ is finite. Therefore since F is also locally bounded, we see $\ell_N(u)$ is locally

[5] In general, the observation on the order of the optimizer in *Step 1.* of the proof in Lemma 5.6 is used here. In this case, it is easily recovered through the explicit form of the utility function.

uniformly bounded from below in both N and u, provided h is sufficiently small. Finally, by construction $v_0^h(T-nh,u) \geq \ell_{n-1}(u) \geq \ell_0(u) + O(1)$. This completes the proof.

□

Lemma 5.5. *The scheme* (11) *is stable, in the sense that* v^h *are viscosity solutions of* (11) *and they are locally uniformly bounded.*

Proof. We have already shown that $(v^h)_{0<h<h_0}$ are locally uniformly bounded and that the relaxed semi-limits exist. Finally, the viscosity property of v^h follows immediately from the fact that they are classical solutions to the dynamic programming equation (in that they satisfy (11) pointwise).

□

Lemma 5.6. *The scheme* (11) *is consistent: for any smooth test function* $\varphi \in C^\infty([0,T]\times\mathbb{R}_+\times\mathbb{R})$ *and any point* $(t,x,u) \in [0,T) \times \mathbb{R}_+ \times \mathbb{R}$*, there holds*

$$\lim_{h\to 0^+} \frac{\varphi(t,x,u) - \varphi^h(t,x,u)}{h} = H(t,x,u,D\varphi(t,x,u),D^2\varphi(t,x,u)).$$

Proof. Since the optimization is on $\mathbb{R}$, we will assume that φ is strictly convex in u. This ensures that the numerical algorithm produces a unique minimizer $\alpha^* \in \mathbb{R}$ for (16).
Step 1. Fix $t, x > 0$. We will show that the minimizer, in (16), $\alpha^* \in O(h^{1/2})$. Define

$$F(x_1,x_2,\alpha) := p^*\varphi(t,x_1,u+\alpha) + (1-p^*)\varphi(t,x_2,u-\alpha).$$

Given any $x \in \mathbb{R}_+$, there exists $\hat{\alpha} := \hat{\alpha}(x,x)$ for which $\alpha \mapsto F(x,x,\alpha)$ is minimized. By strict convexity of φ in u, we obtain $F_{\alpha\alpha}(x,x,\hat{\alpha}) > 0$ and therefore by the implicit function theorem, there exists an open ball $B_r(x,x) \subset \mathbb{R}_+\times\mathbb{R}_+$ and a smooth function $\hat{\alpha} : B_r(x,x) \to \mathbb{R}$ such that for all $(x_1,x_2) \in B_r(x,x)$, there holds $F_\alpha(x_1,x_2,\hat{\alpha}(x_1,x_2)) = 0$. That is, $\hat{\alpha}(x_1,x_2)$ is the minimizer of $\alpha \mapsto F(x_1,x_2,\alpha)$ and it is smooth in x_1, x_2. Therefore, by Taylor expansion

$$\alpha^* = \hat{\alpha}(x_+,x_-) = \hat{\alpha}(x,x) + O(h^{1/2}) \quad \text{as } h \to 0.$$

Hence, it suffices to show that $\hat{\alpha}(x,x) \in O(h^{1/2})$.

Without loss of generality, we need only consider $F(\alpha) := F(x,x,\alpha)$ and set $\alpha^* = \hat{\alpha}(x,x)$. By convexity, it is clear that for any $\alpha > 0$ we have

$$\varphi_u(t,x,u+\alpha) \geq \varphi_u(t,x,u-\alpha).$$

On the other hand,

$$p^*\varphi_u(t,x,u+\alpha^*) = (1-p^*)\varphi_u(t,x,u-\alpha^*). \tag{18}$$

We need to distinguish three cases. For $h > 0$ sufficiently small we have: i) if $\varphi_u(t,x,u) > 0$, then $F_\alpha(0) < 0$ and thus $\alpha^* > 0$; ii) if $\varphi_u(t,x,u) < 0$, then $F_\alpha(0) > 0$ and thus $\alpha^* > 0$; iii) If $\varphi_u(t,x,u) = 0$, then $F_\alpha(0) = 0$ and $\alpha^* = 0$ and there is nothing to prove.

Suppose $\alpha^* > 0$ (the case $\alpha^* < 0$ can be handled similarly). We apply a Taylor expansion to obtain

$$\varphi_u(t,x,u+\alpha^*) = \varphi_u(t,x,u-\alpha^*) + 2\alpha^*\varphi_{uu}(t,x,\tilde{u}),$$

for some $\tilde{u} \in [u-\alpha^*, u+\alpha^*]$. Since $\varphi_{uu} > 0$ on the compact set $[u-\alpha^*, u+\alpha^*]$, there is $\epsilon > 0$ such that $\varphi_{uu} > \epsilon$.

Combining the observations yields

$$(1-2p^*)\varphi_u(t,x,u-\alpha^*) > 2p^*\alpha^*\epsilon$$

which using the fact $1-2p^* = \frac{\mu}{\sigma}h^{1/2}$ and $\varphi_u(t,x,u) \geq \varphi_u(t,x,u-\alpha^*)$ simplifies to

$$\frac{\mu}{\sigma}h^{1/2}\varphi_u(t,x,u) \geq (1-\frac{\mu}{\sigma}h^{1/2})\alpha^*\epsilon > 0.$$

Therefore, we may conclude that $\alpha^* \in O(h^{1/2})$, as $h \to 0$.
Step 2. Define

$$\varphi^h(t,x,u) := \inf_{\alpha,z}\left\{\max\left\{\begin{array}{c}\varphi\left(t+h,x_+,u+\alpha\right) - z\left(x_+ - x\right)\\ \\ \varphi\left(t+h,x_-,u-\alpha\right) - z\left(x_- - x\right)\end{array}\right\}\right\}.$$

Write $\alpha^* = \hat{\alpha}(x_-,x_+) = a(x_-,x_+)h^{1/2}$. Next, perform a Taylor expansion of $\varphi(t+h,x_\pm,u \pm a(x_-,x_+)h^{1/2})$ around (t,x,u) to obtain

$$\begin{aligned}\varphi(t+h,x_\pm,u{\pm}ah^{1/2}) &- z(x_\pm - x) - \varphi(t,x,u)\\ &= h^{1/2}(\pm\sigma x\varphi_x \pm a\varphi_u \mp zx\sigma)(t,x,u)\\ &+ h\left(\varphi_t + \mu x\varphi_x - zx\mu + \frac{1}{2}\sigma^2x^2\varphi_{xx} + a\sigma x\varphi_{xu} + \frac{1}{2}a^2\varphi_{uu}\right)(t,x,u) + O(h^{3/2}),\end{aligned}$$

where we crucially use the fact that $a(x_-,x_+) \in O(1)$ as $h \to 0$ and the smoothness of φ to conclude that the higher order terms are in $O(h^{3/2})$ as $h \to 0$.
Step 3. Recall from (15) that the optimal z^* is given by

$$z^* := z^*(a) = \frac{\varphi(t+h,x_+,u+ah^{1/2}) - \varphi(t+h,x_-,u-ah^{1/2})}{x_+ - x_-}.$$

By Taylor expansion, we obtain

$$\begin{aligned}\varphi(t+h,x_\pm,u{\pm}ah^{1/2}) =&\\ &\varphi(t+h,x,u) + x(\mu h \pm \sigma h^{1/2})\varphi_x(t+h,x,u) \pm ah^{1/2}\varphi_u(t+h,x,u)\\ &+\frac{1}{2}[x(\mu h \pm \sigma h^{1/2})]^2\varphi_{xx}(t+h,x,u) + \frac{1}{2}a^2h\varphi_{uu}(t+h,x,u)\\ &+[x(\mu h \pm \sigma h^{1/2})(\pm ah^{1/2})]\varphi_{xu}(x,u) + O(h^{3/2}).\end{aligned}$$

Note that there are no order $h^{1/2}$ terms in z^*. More precisely,

$$\begin{aligned}z^* &= \frac{\sigma x v_x(t+h,x,u) + av_u(t+h,x,u)}{x\sigma} + O(h)\\ &= \frac{\sigma x v_x(t,x,u) + av_u(t,x,u)}{x\sigma} + O(h).\end{aligned}$$

Hence,

$$\begin{aligned}\varphi(t+h,&x_\pm,u \pm ah^{1/2}) - z(x_\pm - x) - \varphi(t,x,u)\\ &= h\left(\varphi_t - a\frac{\mu}{\sigma}\varphi_u + \frac{1}{2}\sigma^2x^2\varphi_{xx} + a\sigma x\varphi_{xu} + \frac{1}{2}a^2\varphi_{uu}\right)(t,x,u) + O(h^{3/2}).\end{aligned}$$

Using the first order condition (18), we obtain that

$$a = \frac{-\sigma x \varphi_{xu}(t,x,u) + \frac{\mu}{\sigma}\varphi_u(t,x,u)}{\varphi_{uu}(t,x,u)} + O(h).$$

Which agrees with the optimizer in the continuous time equation

$$\varphi_t + \frac{1}{2}\sigma^2 x^2 \varphi_{xx} + \inf_{a\in\mathbb{R}}\left\{\frac{1}{2}a^2\varphi_{uu} + a(\sigma x\varphi_{xu} - \frac{\mu}{\sigma}\varphi_u)\right\}$$

at the leading order. This completes the proof. □

We have all of the results necessary to conclude convergence. However, for the convenience of the reader we sum up the conclusion of the argument with the following theorem.

Theorem 5.7. *Let v^h be the solution to* (11) *with time step discretization $h > 0$ and let v be a viscosity solution of* (17). *Then $v^h \to v$ locally uniformly in* $[0,T] \times \mathbb{R}_+ \times \mathbb{R}$.

Proof. By definition, the relaxed semi-limits of v^h satisfy $v_* \le v^*$ on $[0,T] \times \mathbb{R}_+ \times \mathbb{R}$. Together with Lemma 5.6, it follows that v_* and v^* are viscosity super and sub-solutions, respectively of (17). Finally, Assumption 5.1 implies that $v^* \le v_*$ and therefore $v_* = v^*$. Provided that v_*, v^* belong to the same class of functions as v, then $v_* = v^* = v$. As a consequence, it immediately follows that v^h converges locally uniformly to v.

□

6 Application to Quantile Hedging

6.1 The formulation of the quantile hedging problem for complete markets

The idea of quantile hedging is formulated in Föllmer and Leukert as follows: assume that the discounted price process of the underlying is a semi-martingale $X = (X_t)_{t\in[0,T]}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. Let $\mathcal{M}_\mathbb{P}$ denote the set of equivalent martingale measures. Absence of arbitrage means $\mathcal{M}_\mathbb{P} \neq \emptyset$. In a complete market, there exists a unique equivalent martingale measure $\mathbb{P}^* \in \mathcal{M}_\mathbb{P}$. Now, consider a contingent claim given by an $\mathcal{F}_T$-measurable, non-negative random variable H such that $H \in L^1(\Omega, \mathbb{P}^*)$. Completeness implies the existence of a predictable process ξ^H, providing a perfect hedge for H, i.e.,

$$\mathbb{E}^*[H|F_t] = H_0 + \int_0^t \xi^H_s dX_s, \quad \forall t \in [0,T], \quad \mathbb{P}\text{-a.s.}$$

where $\mathbb{E}^*$ denotes expectation with respect to $\mathbb{P}^*$. Thus, the claim can be replicated by the self-financing trading strategy (H_0, ξ^H). There is an inherent assumption that the investor allocates the required initial capital $H_0 = \mathbb{E}^*[H]$. However, the investor may be unwilling or unable to invest the initial capital H_0. In this case, the solution to the quantile hedging problem provides the best hedge the investor can achieve with a given smaller amount $\widetilde{V_0} < H_0$ of initial capital. The probability that the hedge ie successful is taken as the optimality criterion. More precisely, the solution constructs an admissible strategy (V_0, ξ^*), with $V_0 \le \tilde{V}_0$ such that the corresponding value process V^* satisfies

$$\mathbb{P}\left[V^*_T = V_0 + \int_0^T \xi^*_s dX_s \ge H\right] = \max \mathbb{P}\left[V_T = \hat{V}_0 + \int_0^T \xi_s dX_s \ge H\right],$$

where the maximum is taken over the set of all admissible portfolio strategies $(\hat{V}_0, \xi)$ satisfying

$$\hat{V}_0 \leq \widetilde{V_0}.$$

The probability $\mathbb{P}[V_T \geq H]$ is termed as the success probability.

The solution using our method

In view of the original problem,

$$V(t,x) := \inf\left\{y \in \mathbb{R}_+ \ : \ \exists Z \in \mathcal{A}^t, \ \mathbb{E}\left[U\left(Y_T^{t,y,Z} - g(X_T^{t,x})\right)\right] \geq u^*\right\},$$

we take $U(x) = \mathbf{1}_{[0,\infty)}(x)$ to obtain

$$V(t,x) = \inf\left\{y \in \mathbb{R}_+ \ : \ \exists Z \in \mathcal{A}^t, \ \mathbb{P}\left[Y_T^{t,y,Z} \geq g(X_T^{t,x})\right] \geq u^*\right\},$$

where u^* is the success probability. Finally this is reformulated into (9) and we take $U^{-1}(u) \equiv 0$.

6.2 Quantile Hedging in a Market with Frictions

A general definition of financial market frictions is provided by DeGennaro and Robotti as anything that interferes with trades that rational individuals make (or would make in the absence of market frictions). The sources of financial market frictions are diverse and widespread but still can be classified into (although not a completely exhaustive list) transactions costs, portfolio constraints, taxes and regulations, differential borrowing and lending rates, asset indivisibility, and agency and information problems. In this section we focus on the portfolio constraints type of financial market frictions. The effects of leverage (or portfolio) constraints on optimal hedging of stock and bond options in discrete time was first studied in Naik and Uppal. Broadie et al. extended the work of Naik and Uppal to continuous-time and we use their results as a benchmark for our model. As a result, we can model the borrowing constraint by requiring that the amount borrowed cannot exceed C_b times total wealth, i.e.,

$$(Z_t X_t - Y_t) \leq C_b Y_t \ \Leftrightarrow \ Z_t \leq (1 + C_b)\frac{Y_t}{X_t}, \qquad 0 \leq t \leq T. \tag{19}$$

Similarly, we require the short selling amount to be less than C_s times total wealth by imposing the short selling constraint:

$$-Z_t X_t \leq C_s Y_t \ \Leftrightarrow \ Z_t \geq -C_s \frac{Y_t}{X_t}, \qquad 0 \leq t \leq T. \tag{20}$$

Dynamic programming with portfolio constraints

We now discuss dynamic programming in the context of portfolio constraints. Under the restrictions (19) and (20), the dynamic programming equation in (11) can be stated as

$$v(t,x,u) = \inf_{z,\alpha}\{\max\{F(\alpha,z), G(\alpha,z)\}\},$$

where the infimum is taken over

$$z \in \left[\frac{-C_s v(t,x,u)}{x}, \frac{(1+C_b)v(t,x,u)}{x}\right].$$

As before, we define

$$\begin{aligned} F(\alpha, z) &:= v\,(t+h, x_+, u+\alpha) - z(x_+ - x), \\ G(\alpha, z) &:= v\,(t+h, x_-, u-\alpha) - z(x_- - x). \end{aligned}$$

The presence of the value function in the portfolio constraints gives the optimization task an implicit nature. In order to solve the problem, we first fix α, which we assume is the minimizer. Then, we calculate $z^*(\alpha)$ (compare (15)), i.e.,

$$z^*(\alpha) = \frac{v\,(t+h, x_+, u+\alpha) - v\,(t+h, x_-, u-\alpha)}{2x\sigma\sqrt{h}}.$$

We differentiate the following cases:

(i) $$\frac{-C_s v(t,x,u)}{x} < z^*(\alpha) < \frac{(1+C_b)v(t,x,u)}{x};$$

(ii) $$z^*(\alpha) \geq \frac{(1+C_b)v(t,x,u)}{x};$$

(iii) $$z^*(\alpha) \leq \frac{-C_s v(t,x,u)}{x}.$$

Case (i): In this case, the optimal $z^*(\alpha)$ from the unconstrained problem already satisfies the portfolio constraints. Therefore, we set $z^{c,*}(\alpha) := z^*(\alpha)$. Next, we define the function

$$H_1(\alpha) := F(\alpha, z^{c,*}) = G(\alpha, z^{c,*}). \tag{21}$$

In view of the discussion in Section 4.1, this yields

$$H_1(\alpha) = \frac{v\,(t+h, x_+, u+\alpha)\,(x_- - x) + v\,(t+h, x_-, u-\alpha)\,(x_+ - x)}{2x\sigma\sqrt{h}}.$$

Case (ii): Setting

$$z^{c,*}(\alpha) := \frac{(1+C_b)v(t,x,u)}{x},$$

implies

$$\begin{aligned} v(t,x,u) &= \max\left\{F(\alpha, z^{c,*}), G(\alpha, z^{c,*})\right\} \\ &= \max\left\{ \begin{array}{l} v\,(t+h, x_+, u+\alpha) - (1+C_b)v(t,x,u)(\mu h + \sigma\sqrt{h}) \\ v\,(t+h, x_-, u-\alpha) - (1+C_b)v(t,x,u)(\mu h - \sigma\sqrt{h}) \end{array} \right\}. \end{aligned}$$

Since $z^{*,c}(\alpha) \leq z^*(\alpha)$, the proof of Lemma 4.2 implies that $F(\alpha, z^{c,*})$ is the maximum. Therefore, we define

$$H_2(\alpha) := F(\alpha, z^{c,*}) = \frac{v\,(t+h, x_+, u+\alpha)}{1 + (1+C_b)(\mu h + \sigma\sqrt{h})}.$$

Case (iii): Set

$$z^{c,*}(\alpha) := \frac{-C_s v(t,x,u)}{x}.$$

Then,

$$\begin{aligned} v(t,x,u) &= \max\{F(\alpha, z^{c,*}), G(\alpha, z^{c,*})\} \\ &= \max\left\{ \begin{array}{l} v(t+h, x_+, u+\alpha) + C_s v(t,x,u)(\mu h + \sigma\sqrt{h}), \\ v(t+h, x_-, u-\alpha) + C_s v(t,x,u)(\mu h - \sigma\sqrt{h}) \end{array} \right\}. \end{aligned}$$

Since $z^{c,*}(\alpha) \geq z^*(\alpha)$, the proof of Lemma 4.2 indicates that $G(\alpha, z^{c,*}(\alpha))$ is the maximum. So we define

$$H_3(\alpha) := G(\alpha, z^{c,*}(\alpha)) = \frac{v(t+h, x_-, u-\alpha)}{1 - C_s(\mu h - \sigma\sqrt{h})}.$$

Now we are ready to formulate the computational procedure. Define an indexing map

$$\zeta(z) := \begin{cases} 1, & \text{if } \frac{-C_s v(t,x,u)}{x} < z < \frac{(1+C_b)v(t,x,u)}{x}, \\ 2, & \text{if } z \geq \frac{(1+C_b)v(t,x,u)}{x}, \\ 3, & \text{if } z \leq \frac{-C_s v(t,x,u)}{x}. \end{cases}$$

In view of the case-by-case anaylsis, the value function can be computed by evaluating

$$v(t,x,u) = \min_{\alpha}\{H_{\zeta(z^*(\alpha))}(\alpha)\}.$$

7 Numerical Results

In this section, we present the numerical results of our model for European Vanilla and barrier call and put options. Moreover, we investigate the quantile hedging costs under portfolio constraints.

As a benchmark, we utilize the closed form solutions for Vanilla call and put options derived in Föllmer and Leukert. It is important to note that they provide an analytical solution for the case of a call option when $\mu < \sigma^2$ and only a semi-closed solution exists for $\mu > \sigma^2$. Hence, in order to use the analytical solution we take volatility: $\sigma = 0.3$; mean return rate: $\mu = 0.08$ as in Föllmer and Leukert for European call options.We also take the initial stock price: $S_0 = 100$, strikes $K = 90, 100, 110$; maturities: $T = 1, 3, 6, 9, 12$ months; number of time steps: $N = 100$; time discretization: $h = 0.001$, and for simplicity interest rate is taken to be zero. We consider a shortfall probability up to 10% as a reasonable level to accept. Figures 1, 2, and 3 show the absolute percentage errors for in-the-money (ITM), at-the-money (ATM), and out-of-the-money (OTM) call options computed with our method. We see that as the shortfall probability changes from 0 to 10%, the absolute percentage error always remains less than 1% for almost all parameter values. The only exception is the 1-month OTM option (not shown in the figure). However as the number of time steps increases it is possible to produce higher accuracy and very small percentage errors. Similar results are obtained for put options.

The fourth column in Tables 1-3, shows how much percentage gain an investor can make by accepting a certain shortfall probability. For instance, for a 6-month ATM call option it is possible to pay 5.5% less by accepting a 1% shortfall probability and the discount increases up to 39.5% if one accepts a 10% shortfall probability. We observe that percentage gains increase with time to maturity for ITM call options and decrease for ATM and OTM call options. There is also numerical evidence in Tables 1-3 that an inverse relationship exists between the moneyness of a call option and profit from the quantile hedging strategy. The above relations hold in the reverse direction for put options.

If strict borrowing constraints are imposed on the wealth process, then percentage gains from the quantile hedging approach become substantial for the short term ITM, ATM, OTM call options.

For instance, if an investor is allowed to borrow at most two times more than his current wealth, i.e. when $C_b = 2$, then by accepting a 1% shortfall risk for an ATM call option, he can pay 23.7% less than the Black-Scholes price. This is almost four times more profitable than in the unconstrained case. However, as the shortfall probability increases, the effect of portfolio constraints on the relative profitability of the quantile hedging strategy diminishes. If we take a 10% shortfall size in the previous example, the constrained case is only at most 1.5 times more profitable than the unconstrained case. One can also observe from Figures 4-6 that as time to maturity increases, the effect of borrowing constraints become less significant for all options independent of the moneyness of the option.

Relaxing the portfolio constraints on the wealth process, i.e. increasing C_b, we observe convergence to the unconstrained case, as expected. For example, taking $C_b = 1000$ yields exactly the same results as the unconstrained case. As a second benchmark, we use the explicit formulae given in Broadie et al. for the minimum super-replication prices under portfolio constraints in continuous time. Their methodology is to first create a dominating claim which has suitably increasing payoffs with respect to the original claim. They show that the price of the original claim with portfolio constraints is equal to the price of the dominating claim without constraints. In particular, the dominating claim for a standard call option with payoff $b(X) = (X - K)^+$ is given by

$$\hat{b}(X) = \begin{cases} X - K, & \text{if } X \geq \frac{Ku}{u-1}, \\ \frac{K}{u-1}\left(\frac{(u-1)X}{Ku}\right)^u, & \text{if } X < \frac{Ku}{u-1}, \end{cases} \tag{22}$$

where u represents the borrowing constraint (corresponding to $1 + C_b$ in our context). Let y^b denote the minimum super-replication call price with borrowing constraints calculated using (22) on a binomial tree with 100 time steps. Let y_0^b denote the minimal initial capital required under borrowing constraints when the shortfall probability is zero. We compute y_0^b with our method again in 100 time steps. A list of y^b and y_0^b values are presented in Table 4 for different call options. A comparison of y_0^b and y^b values in Table 4 indicates that our method provides a very close approximation to the continuous time model.

Quantile hedging costs for exotic options can also be computed. Consider an up-and-out barrier call option with the parameters as in Table 5. Typically, barrier options are constructed for the purpose of decrease the initial cost of a similar option without a barrier. Quantile hedging strategies can further decrease the initial cost of the option. The results in Table 5 show that percentage gains increase with time to maturity regardless of the moneyness of the option. We also observe an inverse relationship between the moneyness of the call option and the profit from the quantile hedging strategy.

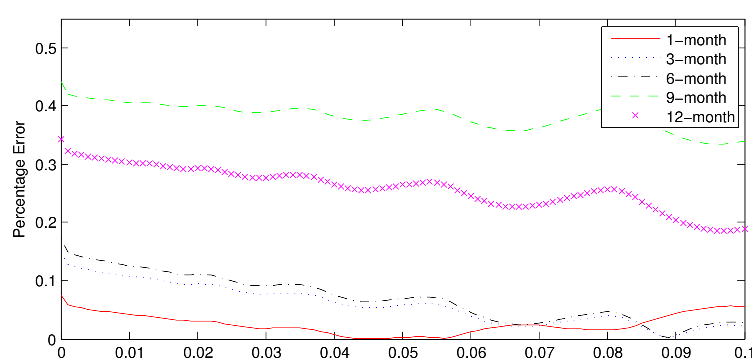

Figure 1: Call Option with $S = 100$, $K = 90$, $\sigma = 0.3$, $\mu = 0.08$

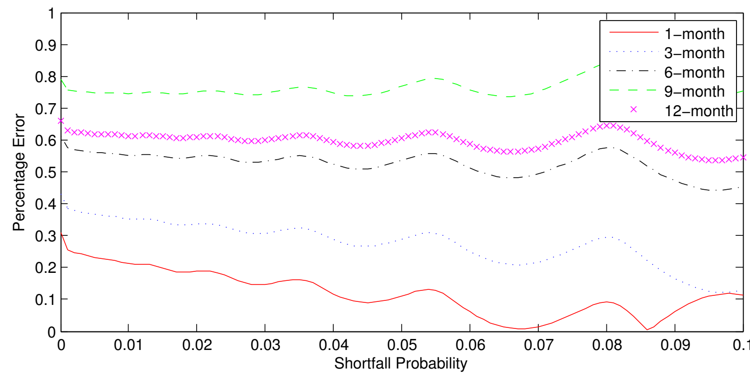

Figure 2: Call Option with $S = 100$, $K = 100$, $\sigma = 0.3$, $\mu = 0.08$

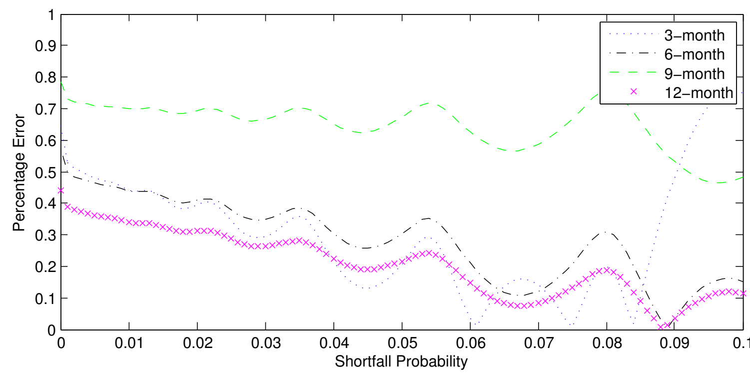

Figure 3: Call Option with $S = 100$, $K = 110$, $\sigma = 0.3$, $\mu = 0.08$

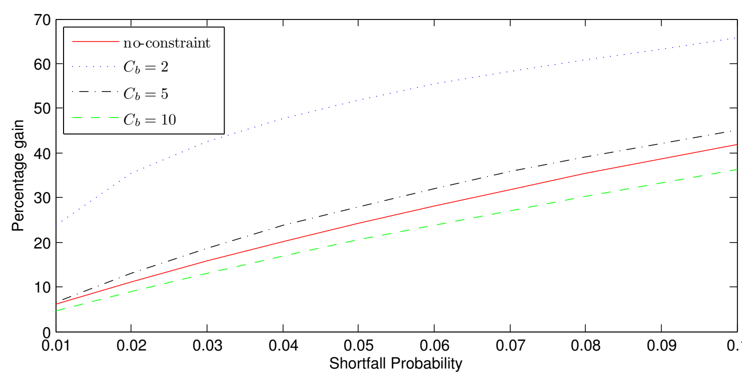

Figure 4: Call Option with $S = 100$, $K = 90$, $\sigma = 0.3$, $\mu = 0.08$

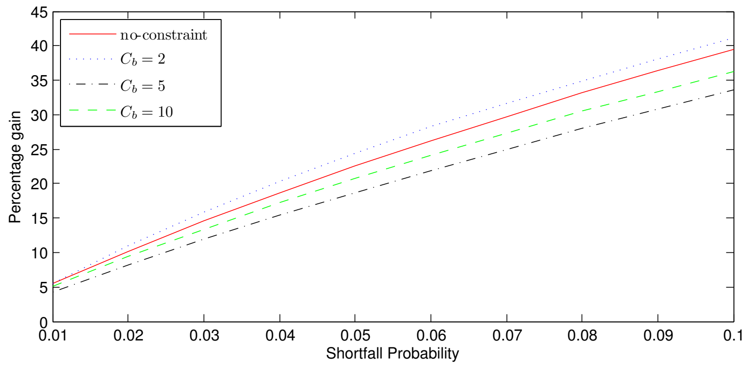

Figure 5: Call Option with $S = 100$, $K = 100$, $\sigma = 0.3$, $\mu = 0.08$

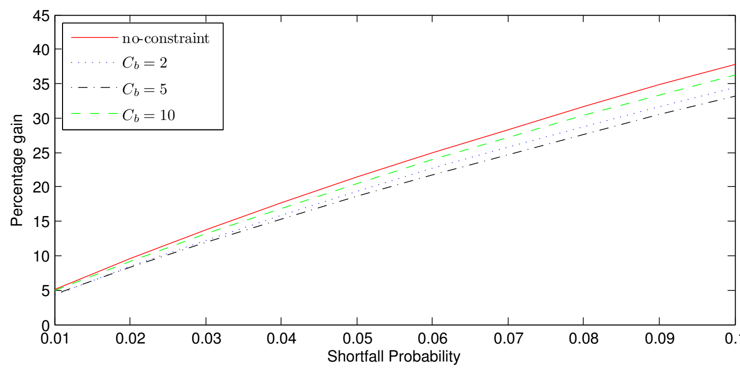

Figure 6: Call Option with $S = 100$, $K = 110$, $\sigma = 0.3$, $\mu = 0.08$

T	Shortfall Prob	Initial Cost	% Gain	C_b 2	% Gain	C_b 5	% Gain	C_b 10	% Gain	C_b 1000
0.083	0	10.44		18.70		12.45		11.02		10.44
0.083	0.01	10.15	2.8	16.89	9.7	12.13	2.6	10.73	2.6	10.15
0.083	0.02	9.89	5.3	15.65	16.3	11.82	5.1	10.47	5.0	9.89
0.083	0.03	9.64	7.6	14.78	21.0	11.53	7.4	10.23	7.2	9.64
0.083	0.04	9.40	9.9	14.10	24.6	11.25	9.7	9.99	9.4	9.40
0.083	0.05	9.17	12.1	13.53	27.7	10.98	11.8	9.76	11.5	9.17
0.083	0.06	8.95	14.2	13.03	30.3	10.72	13.9	9.54	13.5	8.95
0.083	0.07	8.74	16.3	12.59	32.7	10.47	15.9	9.32	15.4	8.74
0.083	0.08	8.53	18.3	12.18	34.9	10.23	17.8	9.11	17.3	8.53
0.083	0.09	8.32	20.3	11.79	37.0	9.99	19.7	8.91	19.2	8.32
0.083	0.10	8.12	22.2	11.44	38.8	9.76	21.6	8.71	21.0	8.12
0.5	0	13.97		20.72		15.76		14.52		13.97
0.5	0.01	13.44	3.7	19.99	3.5	15.24	3.3	14.00	3.6	13.44
0.5	0.02	12.98	7.1	19.26	7.0	14.77	6.3	13.54	6.8	12.98
0.5	0.03	12.55	10.2	18.58	10.4	14.34	9.0	13.10	9.8	12.55
0.5	0.04	12.13	13.1	17.92	13.5	13.92	11.7	12.69	12.6	12.13
0.5	0.05	11.74	16.0	17.32	16.4	13.52	14.2	12.29	15.3	11.74
0.5	0.06	11.36	18.7	16.75	19.1	13.13	16.7	11.91	18.0	11.36
0.5	0.07	10.99	21.3	16.20	21.8	12.76	19.0	11.54	20.5	10.99
0.5	0.08	10.62	23.9	15.67	24.4	12.39	21.4	11.18	23.0	10.62
0.5	0.09	10.28	26.4	15.19	26.7	12.03	23.7	10.84	25.4	10.28
0.5	0.10	9.94	28.8	14.71	29.0	11.69	25.8	10.50	27.7	9.94
1	0	16.95		22.80		18.46		17.40		16.95
1	0.01	16.30	3.9	22.04	3.4	17.81	3.5	16.74	3.8	16.30
1	0.02	15.71	7.3	21.34	6.4	17.22	6.7	16.16	7.1	15.71
1	0.03	15.17	10.6	20.66	9.4	16.67	9.7	15.61	10.3	15.17
1	0.04	14.64	13.6	20.01	12.2	16.15	12.5	15.09	13.3	14.64
1	0.05	14.14	16.6	19.41	14.9	15.65	15.2	14.58	16.2	14.14
1	0.06	13.66	19.5	18.82	17.5	15.16	17.9	14.10	19.0	13.66
1	0.07	13.19	22.2	18.24	20.0	14.70	20.4	13.63	21.6	13.19
1	0.08	12.73	24.9	17.67	22.5	14.23	22.9	13.17	24.3	12.73
1	0.09	12.29	27.5	17.12	24.9	13.80	25.3	12.74	26.8	12.29
1	0.10	11.86	30.0	16.59	27.3	13.37	27.6	12.31	29.3	11.86

Table 1: Quantile Hedging costs for a call option under borrowing constraint with parameters $S_0 = 100$, $K = 90$, $\mu = 0.08$, $\sigma = 0.3$.

T	Shortfall Prob	Initial Cost	% Gain	C_b 2	% Gain	C_b 5	% Gain	C_b 10	% Gain	C_b 1000
0.083	0	3.44		15.15		7.47		4.92		3.44
0.083	0.01	3.23	6.1	11.55	23.7	6.98	6.6	4.68	4.7	3.23
0.083	0.02	3.06	11.2	9.76	35.5	6.50	13.1	4.47	9.0	3.06
0.083	0.03	2.90	15.9	8.69	42.6	6.08	18.7	4.28	13.0	2.90
0.083	0.04	2.75	20.2	7.93	47.7	5.70	23.7	4.09	16.9	2.75
0.083	0.05	2.61	24.3	7.31	51.7	5.39	27.9	3.91	20.5	2.61
0.083	0.06	2.47	28.2	6.76	55.4	5.09	31.9	3.74	23.8	2.47
0.083	0.07	2.35	31.8	6.31	58.3	4.80	35.8	3.59	27.1	2.35
0.083	0.08	2.22	35.4	5.93	60.8	4.56	39.0	3.43	30.2	2.22
0.083	0.09	2.11	38.7	5.55	63.3	4.32	42.1	3.28	33.3	2.11
0.083	0.10	2.00	41.9	5.20	65.7	4.10	45.2	3.13	36.3	2.00
0.5	0	8.40		16.87		10.77		9.13		8.40
0.5	0.01	7.93	5.5	15.94	5.5	10.30	4.3	8.67	5.1	7.93
0.5	0.02	7.54	10.2	15.02	11.0	9.88	8.2	8.27	9.4	7.54
0.5	0.03	7.17	14.6	14.19	15.9	9.49	11.9	7.91	13.4	7.17
0.5	0.04	6.83	18.7	13.43	20.4	9.12	15.4	7.57	17.2	6.83
0.5	0.05	6.50	22.6	12.75	24.4	8.75	18.7	7.24	20.7	6.50
0.5	0.06	6.19	26.2	12.09	28.3	8.41	21.9	6.93	24.1	6.19
0.5	0.07	5.90	29.7	11.53	31.7	8.08	25.0	6.64	27.3	5.90
0.5	0.08	5.61	33.2	10.98	34.9	7.76	28.0	6.35	30.5	5.61
0.5	0.09	5.34	36.4	10.44	38.1	7.45	30.8	6.08	33.4	5.34
0.5	0.10	5.08	39.5	9.91	41.2	7.15	33.6	5.82	36.3	5.08
1	0	11.84		18.85		13.69		12.35		11.84
1	0.01	11.24	5.1	18.04	4.3	13.09	4.4	11.74	4.9	11.24
1	0.02	10.71	9.6	17.27	8.4	12.55	8.3	11.21	9.2	10.71
1	0.03	10.22	13.8	16.54	12.2	12.06	11.9	10.72	13.2	10.22
1	0.04	9.75	17.7	15.87	15.8	11.60	15.3	10.25	16.9	9.75
1	0.05	9.31	21.4	15.21	19.3	11.15	18.6	9.81	20.5	9.31
1	0.06	8.89	24.9	14.56	22.7	10.72	21.7	9.39	23.9	8.89
1	0.07	8.49	28.3	13.99	25.8	10.31	24.7	8.99	27.2	8.49
1	0.08	8.09	31.7	13.44	28.7	9.91	27.6	8.59	30.4	8.09
1	0.09	7.72	34.8	12.89	31.6	9.52	30.5	8.23	33.4	7.72
1	0.10	7.36	37.8	12.36	34.4	9.14	33.2	7.86	36.3	7.36

Table 2: Quantile Hedging costs for a call option under borrowing constraint with parameters $S_0 = 100$, $K = 100$, $\mu = 0.08$, $\sigma = 0.3$.

T	Shortfall Prob	Initial Cost	% Gain	C_b 2	% Gain	C_b 5	% Gain	C_b 10	% Gain	C_b 1000
0.083	0	0.61		12.52		4.64		2.01		0.61
0.083	0.01	0.48	21.0	6.73	46.2	3.55	23.6	1.72	14.4	0.48
0.083	0.02	0.39	36.1	4.58	63.4	2.73	41.2	1.44	28.4	0.39
0.083	0.03	0.32	48.3	3.48	72.2	2.16	53.4	1.22	39.4	0.32
0.083	0.04	0.25	58.6	2.68	78.6	1.77	61.9	1.00	50.1	0.25
0.083	0.05	0.20	67.4	2.17	82.7	1.43	69.3	0.83	58.8	0.20
0.083	0.06	0.16	74.6	1.66	86.7	1.12	75.8	0.69	65.6	0.16
0.083	0.07	0.12	80.7	1.33	89.4	0.93	80.1	0.56	71.9	0.12
0.083	0.08	0.08	86.4	1.08	91.4	0.74	84.1	0.44	78.3	0.08
0.083	0.09	0.06	90.0	0.81	93.5	0.55	88.1	0.33	83.6	0.06
0.083	0.10	0.04	93.3	0.58	95.3	0.39	91.5	0.22	88.8	0.04
0.5	0	4.72		13.97		7.30		5.47		4.72
0.5	0.01	4.32	8.5	12.74	8.8	6.84	6.3	5.07	7.3	4.32
0.5	0.02	3.98	15.6	11.51	17.6	6.43	11.9	4.73	13.4	3.98
0.5	0.03	3.69	21.9	10.50	24.8	6.05	17.1	4.44	18.9	3.69
0.5	0.04	3.41	27.7	9.60	31.3	5.68	22.2	4.16	23.9	3.41
0.5	0.05	3.16	33.1	8.88	36.5	5.32	27.0	3.91	28.5	3.16
0.5	0.06	2.92	38.0	8.16	41.6	5.01	31.4	3.66	33.0	2.92
0.5	0.07	2.70	42.7	7.51	46.3	4.71	35.5	3.44	37.1	2.70
0.5	0.08	2.49	47.3	7.00	49.9	4.42	39.4	3.22	41.2	2.49
0.5	0.09	2.30	51.3	6.49	53.5	4.15	43.2	3.00	45.1	2.30
0.5	0.10	2.11	55.2	5.97	57.3	3.87	46.9	2.80	48.7	2.11
1	0	8.11		15.76		10.03		8.64		8.11
1	0.01	7.55	6.9	14.86	5.7	9.47	5.6	8.09	6.5	7.55
1	0.02	7.07	12.8	14.01	11.1	8.99	10.4	7.61	12.0	7.07
1	0.03	6.64	18.1	13.18	16.4	8.54	14.9	7.18	17.0	6.64
1	0.04	6.23	23.1	12.45	21.0	8.11	19.1	6.77	21.7	6.23
1	0.05	5.85	27.8	11.77	25.3	7.71	23.1	6.39	26.1	5.85
1	0.06	5.50	32.2	11.08	29.7	7.33	26.9	6.03	30.2	5.50
1	0.07	5.16	36.4	10.44	33.8	6.96	30.6	5.70	34.1	5.16
1	0.08	4.83	40.5	9.88	37.3	6.62	34.0	5.37	37.9	4.83
1	0.09	4.53	44.1	9.35	40.7	6.28	37.4	5.07	41.4	4.53
1	0.10	4.23	47.8	8.82	44.0	5.94	40.7	4.77	44.8	4.23

Table 3: Quantile Hedging costs for a call option under borrowing constraint with parameters $S_0 = 100$, $K = 110$, $\mu = 0.08$, $\sigma = 0.3$.

S	K	T	min initial cost	C_b		
				2	5	10
100	90	0.083	y^b	18.702	12.453	11.029
			y_0^b	18.699	12.449	11.024
100	90	0.5	y^b	20.729	15.772	14.541
			y_0^b	20.721	15.759	14.522
100	90	1	y^b	22.815	18.489	17.437
			y_0^b	22.805	18.462	17.397
100	100	0.083	y^b	15.148	7.477	4.923
			y_0^b	15.146	7.473	4.916
100	100	0.5	y^b	16.886	10.805	9.172
			y_0^b	16.869	10.770	9.134
100	100	1	y^b	18.872	13.717	12.430
			y_0^b	18.846	13.693	12.345
100	110	0.083	y^b	12.519	4.649	2.016
			y_0^b	12.518	4.642	2.006
100	110	0.5	y^b	13.981	7.329	5.517
			y_0^b	13.974	7.297	5.469
100	110	1	y^b	15.778	10.094	8.679
			y_0^b	15.755	10.028	8.644

Table 4: Quantile Hedging cost comparison to continuous-time minimal super-replication cost of call option under borrowing constraint with parameters $\mu = 0.08$, $\sigma = 0.3$.

T	Shortfall Prob	initial cost for $K = 100$	% Gain	initial cost for $K = 90$	% Gain
0.0833	0	3.39		10.37	
0.0833	0.01	3.19	5.8	10.09	2.7
0.0833	0.02	3.02	10.9	9.83	5.2
0.0833	0.03	2.86	15.6	9.59	7.5
0.0833	0.04	2.72	19.9	9.35	9.8
0.0833	0.05	2.58	24.0	9.12	12.0
0.0833	0.06	2.45	27.9	8.91	14.1
0.0833	0.07	2.32	31.6	8.69	16.2
0.0833	0.08	2.20	35.1	8.48	18.2
0.0833	0.09	2.09	38.4	8.28	20.1
0.0833	0.1	1.98	41.7	8.08	22.1
0.5	0	2.95		6.80	
0.5	0.01	2.73	7.5	6.50	4.4
0.5	0.02	2.53	14.1	6.22	8.5
0.5	0.03	2.34	20.6	5.95	12.5
0.5	0.04	2.16	26.7	5.68	16.4
0.5	0.05	2.00	32.1	5.44	20.0
0.5	0.06	1.85	37.3	5.20	23.6
0.5	0.07	1.69	42.5	4.96	27.1
0.5	0.08	1.54	47.7	4.72	30.6
0.5	0.09	1.41	52.1	4.50	33.8
0.5	0.1	1.30	56.1	4.29	36.9
1	0	1.67		3.93	
1	0.01	1.46	12.8	3.64	7.5
1	0.02	1.28	23.5	3.37	14.2
1	0.03	1.11	33.7	3.12	20.6
1	0.04	0.95	43.5	2.87	27.0
1	0.05	0.82	51.0	2.66	32.4
1	0.06	0.70	58.1	2.45	37.7
1	0.07	0.58	65.1	2.24	43.0
1	0.08	0.47	72.0	2.04	48.1
1	0.09	0.37	77.8	1.85	53.0
1	0.1	0.31	81.7	1.69	57.0

Table 5: Quantile Hedging costs for an up-and-out barrier call option with parameters $S_0 = 100$, $B = 130$, $\mu = 0.08$, $\sigma = 0.3$.

8 Other applications

We now illustrate how to adapt the algorithm to a few, seemingly different, financial applications.

8.1 Application to Utility Indifference Pricing

Utility based pricing is a theoretically appealing pricing methodology in incomplete markets. It was first introduced by Hodges and Neuberger and has been extensively studied in recent finance literature. Indifference pricing provides a link between pricing a derivative product and maximizing utility of wealth (see Musiela and Zariphopoulou). This is important, not only for sell side (e.g. investment banks), but also for the buy side (e.g. wealth managers). The inclusion of risk aversion and wealth dependence in utility indifference pricing makes it an economically natural and justifiable method in incomplete markets. Carassus and Rásonyi define the seller's indifference price as "...the minimal amount a seller should add to his or her initial wealth so as to reach an optimal expected utility when delivering the claim which is greater than or equal to the one he or she would have obtained trading in the basic assets only."

Let p^s denote the seller's indifference price. The problem with λ units of a claim can be formulated in our setting as follows:

$$\max_{Z\in\mathcal{A}^t} \mathbb{E}\left[U\left(Y_N^{t,y+\lambda p^s,Z} - \lambda g(X_T^{t,x})\right)\right] = \max_{Z\in\mathcal{A}^t} \mathbb{E}\left[U\left(Y_T^{t,y,Z}\right)\right] \tag{23}$$

Similarly the buyer's price p^b can be obtained from

$$\max_{Z\in\mathcal{A}^t} \mathbb{E}\left[U\left(Y_N^{t,y-\lambda p^b,Z} + \lambda g(X_T^{t,x})\right)\right] = \max_{Z\in\mathcal{A}^t} \mathbb{E}\left[U\left(Y_T^{t,y,Z}\right)\right] \tag{24}$$

In the original problem (5), we start with a strictly increasing and concave utility function U, but the threshold u^* is also determined by the utility function

$$u^*(t,y) = \sup_{Z\in\mathcal{A}^t} \mathbb{E}\left[U\left(Y_T^{t,y,Z}\right)\right]. \tag{25}$$

8.2 Application to Good-Deal Bounds

Cochrane and Saá-Requejo describe the notion of a *good deal* as an investment opportunity with a *too* high Sharpe ratio. Their idea is to rule out good deals by putting a bound, called a *good-deal bound*, on the Sharpe ratio. In a paper published at the same time, Bernardo and Ledoit use the *gain-loss ratio* instead of the Sharpe ratio. The gain-loss ratio, defined as the quantity $\frac{\mathbb{E}[y^+]}{\mathbb{E}[y^-]}$, where y^+ and y^- are the positive and negative parts of y, is a measure of the attractiveness of an investment opportunity. More specifically, it is ratio between an investment's expected positive excess payout and its expected negative excess payout. Imposing a bound on the gain-loss ratio yields tighter price bounds than no-arbitrage price bounds. Similar to the gain-loss ratio is the *sufficiently attractive expected gain* (SAGE) proposed by Pinar et al.. A sequence of portfolio holdings is said to yield a *SAGE at level* $\lambda > 1$ whenever

$$\mathbb{E}[y^+] - \lambda\mathbb{E}[y^-] > 0,$$

where λ is the loss aversion parameter. In our context, we define

$$U(y) = y^+ - \lambda y^-,$$

and we take $u^* = 0$ in the general problem (5). We formulate the *seller's price* as

$$\begin{aligned}\mathbf{S}^\lambda := \inf\Big\{\ y\in\mathbb{R}\ :\ & \exists Z\in\mathcal{A}^t \text{ such that} \\ & \mathbb{E}\left[\left(Y_T^{t,y,Z} - g(X_T^{t,x})\right)^+ - \lambda\left(Y_T^{t,y,Z} - g(X_T^{t,x})\right)^-\right] \geq 0\Big\},\end{aligned}$$

The above problem can be equivalently reformulated as

$$\begin{aligned}\mathbf{S}^\lambda := \inf\Big\{\ y\in\mathbb{R}\ :\ & \alpha \text{ is } \mathcal{F}_t^s\text{-adapted, } Z\in\mathcal{A}^t, \text{ and} \\ & Y_T^{t,y,Z} \geq U^{-1}(u_T^{t,u^*,\alpha}) + g(X_T^{t,x}) \quad \mathbb{P}\text{-a.s. }\Big\},\end{aligned}$$

where

$$U^{-1}(u) = \begin{cases} u, & \text{if } u \geq 0, \\ \frac{u}{\lambda}, & \text{if } u < 0, \end{cases}$$

with the terminal condition

$$v(T, x, u) = \begin{cases} u + g(x), & \text{if } u \geq 0, \\ \frac{u}{\lambda} + g(x), & \text{if } u < 0. \end{cases}$$

Similarly, the *buyer's price* is defined as

$$\mathbf{B}^{\lambda} := \sup\Big\{ \, y \in \mathbb{R} \; : \exists \tilde{Z} \in \mathcal{A}^t \text{ such that} \\ \mathbb{E}\left[\left(-Y_N^{t,y,\tilde{Z}} + g(X_T^{t,x})\right)^+ - \lambda\left(-Y_N^{t,y,\tilde{Z}} + g(X_T^{t,x})\right)^-\right] \geq 0 \Big\}.$$

8.3 Application to Expected Shortfall

Value at risk (VaR) is defined as the worst expected loss over a given time period at a given confidence level under normal market conditions. VaR is a popular risk measure for fund managers, corporate treasurers, banks and other financial institutions. Its widespread use can be attributed to mandatory regulations endorsed by the Basel Committee on Banking Supervision. However, by definition VaR considers only probability of loss and not the size of the loss. Artzner et al. introduce an alternative risk measure called the Expected Shortfall (ES) which takes into account both the size and the probability of shortfall. They propose four axioms that should be satisfied by a coherent risk measure and show that ES satisfies three of the four axioms. Sub-additivity is an important feature for a reasonable risk measure. Whereas ES is sub-additive, VaR is not. A violation of the sub-additivity axiom means that portfolio diversification can increase risk, in stark contrast to the general concept of risk.

In our setting, we fix a bound on the expected shortfall size

$$\mathbb{E}\left[\max\left(0, g(X_T^{t,x}) - Y_T^{t,y,Z}\right)\right],$$

and we try to minimize the required initial capital. In the general problem (5), we take

$$U(x) = -\max(0, -x).$$

Let us remark that, for this choice of utility function it is easily demonstrated that computed solutions are *not* locally uniformly bounded from below (c.f. Lemma 5.4). This can be easily remedied by truncation, i.e. imposing state constraints on wealth. Then

$$V(t,x) := \inf\left\{ \, y \in \mathbb{R} \; : \; \exists Z \in \mathcal{A}^t \text{ such that } \mathbb{E}\left[\max\left(0, g(X_T^{t,x}) - Y_T^{t,y,Z}\right)\right] \leq -u^* \right\},$$

In terms of the equivalent formulation of the original problem

$$v(t,x,u) := \inf\Big\{ \, y \in \mathbb{R} \; : \alpha \text{ is } \mathcal{F}_t^s\text{-adapted, } Z \in \mathcal{A}^t, \text{ and} \\ Y_T^{t,y,Z} \geq U^{-1}(u_T^{t,u^*,\alpha}) + g(X_T^{t,x}) \; \mathbb{P}\text{-a.s.} \Big\},$$

where

$$U^{-1}(u) = \begin{cases} \infty, & \text{if } u \geq 0, \\ u, & \text{if } u < 0, \end{cases}$$

with the terminal condition

$$V(T,x,u) = \begin{cases} \infty, & \text{if } u \geq 0, \\ u + g(x), & \text{if } u < 0. \end{cases}$$

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