On Dynamic Hedging of Single-Tranche Collateralized Debt Obligations

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\section*{Abstract}
This study deals with the dynamic hedging of single-tranche collateralized debt obligations (STCDOs). As a first step, we specify a top-down affine factor model in which a catastrophic risk component is incorporated in order to capture the dynamics of super-senior tranches. Next, we derive the model-based variance-minimizing strategy for the hedging of STCDOs with a dynamically rebalanced portfolio on the underlying index swap. We analyze the actual performance of the variance-minimizing and the model-free regression-based hedging on the iTraxx Europe data. Results of the in-sample hedging analysis indicate that the regression-based hedge outperforms the variance-minimizing hedge based on various criteria. In order to assess the two hedging strategies further, we run a simulation analysis where normal and extreme loss scenarios are generated via the method of importance sampling. Performing the hedging analysis on the set of simulated scenarios we find that, overall, the variance-minimizing strategy is more effective in terms of yielding less riskier hedging portfolios.

\textit{JEL Classification:} C51, G12  
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\section{1. Introduction}
Over the last decade standard models used in market practice for the risk management and hedging of credit derivatives have been strongly criticized for being inadequate. As the financial crisis of 2008 has highlighted once more there is a need for sound hedging methodologies for complex yet highly traded credit derivatives such as collateralized debt obligations (CDO).
A CDO is defined as a structured product backed by a portfolio of credit-risky assets. Synthetic CDOs are special types of CDOs in which the underlying credit-risky portfolio consists of single name credit default swaps (CDS). Synthetic single-tranche CDOs (STCDOs), on the other hand, make it possible to take an exposure on a specific segment of the underlying CDS portfolio. The gains process of a STCDO position is exposed to the risk arising from the credit events of any constituent. There is an exposure to changes in the spreads and the default loss payments. Hence, a reliable hedging methodology is expected to take both risk factors into account. So far, however, the market practice has been the use of sensitivity-based hedging approaches that solely focus on the spread risk. In this context, methodologies based on copula models (see, for example Li [2000]) are widely used. For such an approach, the first step is to price the CDO by using the copula model. As the second step, for each constituent the spread delta, which shows the relative change in the CDO price due to a change in the spread of the underlying CDS, is calculated by varying the spread of the CDS. Apparently, this type of a methodology is not able to provide protection against the default events in the underlying portfolio, as it focuses only on the spread risk. Moreover, consistent dynamic hedging may not be possible because of the static nature of copula models.

Several studies propose alternatives to copula-based hedging approaches. For instance, Laurent et al. [2011] study the hedging of STCDOs in a setting without spread risk. Although ignoring the spread risk leads to a complete market situation, which allows for the perfect replication of the STCDO with the underlying index swap, it may be unrealistic. Modeling the dynamics of defaults and credit spreads by a Markov chain with stochastic intensity, Arnsdorf & Halperin [2008] provide an interplay between the defaults and spreads. In particular, their model yields credit spread dynamics which jump at the arrival of defaults. Even though no theoretical result on the market completeness within the model is granted, they compute sensitivities and deltas for the hedging of tranches with the corresponding index. For the review of other studies on the sensitivity-based hedging of CDOs we refer to Cousin & Laurent [2008].

Frey & Backhaus [2010] study the dynamic hedging of STCDOs in a framework where spread risk is incorporated along with the default contagion. Reckoning with the incompleteness of the market arising from the presence of spread and default risk, this study utilizes a variance-minimizing strategy for the hedging of STCDOs with underlying CDSs (see, e.g., Schweizer [1999] for the review of variance-minimizing and other quadratic hedging approaches). Notably, Frey & Backhaus [2010] show that the variance minimization provides a model-based endogenous interpolation between the hedging against spread risk and default risk. For the hedging of a STCDO position with a dynamically rebalanced portfolio on the index, Filipović & Schmidt [2010] derive the variance-minimizing hedging strategy based on a general top-down framework. However, they do not provide any discussion on the application of this hedging strategy to a real market setting.
Cont & Kan [2011] is the unique study which compares actual performance of various hedging strategies for portfolio credit derivatives. Among other approaches this study also tests the performance of the variance-minimizing strategy for the hedging of STCDOs with the underlying index. It is shown that, while it may not hold true in general, in the case of extreme market conditions variance-minimizing hedge performs better than the other hedging strategies. Furthermore, model-free regression-based hedging is found to be strikingly efficient. Moreover, contrary to the existing literature (see, e.g., Bielecki et al. [2010]), Cont & Kan [2011] conclude that when tested with market data the bottom-up models do not necessarily outperform the top-down models.

In this paper, our focus is on the dynamic hedging of STCDOs with the underlying index swap. To this end, we first specify an affine model under the general top-down framework given in Filipović et al. [2011]. This is a two-factor affine model in which a catastrophic risk component is considered in order to capture the dynamics of senior tranches. This specification allows for the successful fit of the model to the market data simultaneously for all tranches (for a detailed discussion of the model see Eksi & Filipović [2013]). As the next task, we compute the variance-minimizing hedging strategy corresponding to our model specification. Then, we test the performance of the variance-minimizing hedge on the iTraxx Europe data. This analysis is complementary to Filipović & Schmidt [2010] in the sense that it carries the theoretical framework of that study into implementation. The novelty of this analysis is also due to the data set we use. Although this data set does not include any default event, it covers a sufficiently long time horizon which witnessed extreme market conditions such as the recent credit crisis.

For comparison purposes we also perform a regression-based hedging analysis. We find that both variance-minimizing and regression-based strategies are efficient in reducing the risk of the STCDO position. However, when compared to variance-minimizing hedging, the regression-based strategy is found to be more effective. This result is in accordance with the findings of Cont & Kan [2011].

In order to investigate further on the performance of both hedging strategies we run a simulation analysis. Specifically, along with the normal scenarios we generate extreme scenarios by employing the importance sampling technique. Performing the hedging analysis on the simulated scenario set, we find that the variance-minimizing strategy is more effective in terms of yielding less riskier hedging portfolios.

The remainder of this article is structured as follows: Section 2 gives the basics, cash flow structure and valuation formulas for STCDOs. Section 3 introduces the two-factor affine model. Following this, Section 4 deals with the computation of variance-minimizing and regression-based strategies, discusses the hedging algorithm and gives hedging performance criteria. Section 5 reveals the details of the simulation methodology. Section 6 is on the implementation, where the data set is described and the results for the in-sample hedging and the simulation study are given. In Section 7 the article concludes with a summary of the main findings.
2. Single-Tranche CDOs

We fix a stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})\) satisfying the usual conditions and where \(\mathbb{P}\) denotes the historical probability measure. We consider a CDO pool of credits with the overall outstanding notional normalized to 1. The aggregate loss process, representing the ratio of CDO-losses realized by time \(t\), is indicated by \(L_t\). It is assumed that

\[ L_t = \sum_{x \leq T} \Delta L_x \]

is \([0, 1]\)-valued, non-decreasing marked point process with absolutely continuous \(\mathbb{P}\)-compensator \(v(t, d\omega, dx)dt\).

We now define, for any \(T > 0\) and \(x \in [0, 1]\), the hypothetical \((T, x)\)-bond which pays \(1_{[\Delta L_x \leq x]}\) at maturity \(T\). In other words, this is a defaultable zero-recovery zero-coupon bond, which pays one if the realized loss fraction at \(T\) is less than or equal to \(x\), and zero otherwise. The \((T, x)\)-bond price at time \(t \leq T\) is denoted by \(P(t, T, x)\). It follows that the risk-free zero-coupon bond price \(P(t, T)\) equals \(P(t, T, 1)\). Throughout, we assume that the risk-free rate \(r\) is constant, so that \(P(t, T) = e^{-rt}\).

\((T, x)\)-bonds have the spanning property. That is, any European contingent claim \(F(L_T)\) with absolutely continuous payoff function \(F\) can be decomposed into the sum of \((T, x)\)-bond payoffs, \(F(L_T) = F(1) - \int_0^1 F'(x)1_{[\Delta L_x \leq x]}dx\). Hence the claim can be replicated by the static portfolio with the value process

\[ F(1)P(t, T) - \int_0^1 F'(x)P(t, T, x)dx. \]

This in particular allows for the pricing of a STCDO via \((T, x)\)-bonds in the following way. We recall that (see e.g. Embrechts et al. [2005, Chapter 9]) a STCDO issued at time 0 is specified by a sequence of coupon payment dates \(0 < T_1 < \cdots < T_n\), a tranche with attachment and detachment point \(x_1 < x_2\) in \([0, 1]\), and a coupon rate \(\kappa^{[x_1, x_2]}_{(0, n)}\). The attachment point \(x_1\) indicates the level at which losses in the underlying CDO pool begin to erode the notional of the tranche. At the detachment point \(x_2\) the full tranche is written down. Note that \((x_1, x_2) = (0, 1)\) corresponds to the entire index. The holder of a long position in such a STCDO

- receives \(\kappa^{[x_1, x_2]}_0 \times H^{(x_1, x_2)}(L_T)\) at \(T_i\), \(i = 1, 2, \ldots, n\) (coupon leg)
- pays \(-\Delta H^{(x_1, x_2)}(L_T) = H^{(x_1, x_2)}(L_{T_i}) - H^{(x_1, x_2)}(L_{T_{i-1}})\) at any time \(t \leq T_n\) where \(\Delta L_t \neq 0\) (protection leg)

where we define \(H^{(x_1, x_2)}(x) := \int_{x_1}^{x_2} 1_{[\Delta L_t \leq x]}dy = (x_2 - x)^+ - (x_1 - x)^+\). It follows from (1) that the value at time \(t \leq T_n\) of the coupon leg is given by \(\kappa^{[x_1, x_2]}_0 \times D^{(x_1, x_2)}(t)\), where we define the annuity factor

\[ D^{(x_1, x_2)}(t) = \sum_{j \leq T} \int_{x_1}^{x_2} P(t, T_j, x)dx, \]
and the time \( t \) value of the protection leg is

\[
V_p^{(x_1, x_2)}(t) = \int_{x_1}^{x_2} \left( P(t, T_n, x) - r \int_t^{T_n} P(t, u, x) du \right) dx.
\]

The par-coupon rate at \( t \) is then defined as the rate \( \kappa_t^{(x_1, x_2)} \) by which \( \kappa_0^{(x_1, x_2)} \) would need to be replaced for rendering the two legs equal in value. That is,

\[
\kappa_t^{(x_1, x_2)} = \frac{V_p^{(x_1, x_2)}(t)}{D^{(x_1, x_2)}(t)}.
\]

This in turn implies that the time \( t \) spot value, \( \Gamma_t^{(x_1, x_2)} \), of a long position in this STCDO equals

\[
\Gamma_t^{(x_1, x_2)} = \kappa_0^{(x_1, x_2)} - \kappa_t^{(x_1, x_2)} D^{(x_1, x_2)}(t).
\]  

(2)

The discounted gains process, \( G_t^{(x_1, x_2)} \), from holding a long position in the STCDO equals the sum of the accumulated discounted cash flows,

\[
A_t^{(x_1, x_2)} = \kappa_0^{(x_1, x_2)} \sum_{t < T_i} e^{-rT_i} H(L_{T_i}) + \int_0^t e^{-ru} dH(L_u),
\]

and the discounted spot value. That is,

\[
G_t^{(x_1, x_2)} = A_t^{(x_1, x_2)} + e^{-rt} \Gamma_t^{(x_1, x_2)}.
\]

As shown in Filipović & Schmidt [2010, Section 5.1] the discounted gains process satisfies the following dynamics

\[
dG_t^{(x_1, x_2)} = \int_{x_1}^{x_2} \left( e^{-rt} \left( \kappa_0^{(x_1, x_2)} - \kappa_t^{(x_1, x_2)} \right) D^{(x_1, x_2)}(t) \right) dx
\]

\[
+ r1_{[L \leq x]} dt + d\gamma(t, x) dx
\]

(3)

where \( \gamma(t, x) = r e^{-rt} \int_t^{T_n} P(t, u, x) du. \)

3. Model Specification

In this section we specify an affine factor model for the stochastic evolution of the \((T, x)\)-bond prices. Here, we consider an \( \mathbb{R}_2^2 \)-valued affine state process \((Y, Z)\) given by

\[
dY_t = \kappa_y (Z_t - Y_t) dt + \sigma_y \sqrt{Y_t} dW^{y}_t
\]

(4)

\[
dZ_t = \kappa_z (\theta_z - Z_t) dt + \sigma_z \sqrt{Z_t} dW^{z}_t
\]

(5)
for parameters \( \kappa_y \geq 0, \kappa_z \geq 0, \sigma_y, \sigma_z \geq 0 \), and where \( W = (W^y, W^z)^T \) is a two-dimensional \( \mathbb{P} \)-Brownian motion. The factor \( Z \) represents the stochastic long-run mean reversion level of factor \( Y \).

In the sequel, we also need the dynamics of the state process under a risk-neutral pricing measure \( \mathbb{Q} \sim \mathbb{P} \). To preserve the affine structure of the state process under \( \mathbb{Q} \) (see e.g. Cheridito et al. [2007] for details) we specify the market price of risk process \( \lambda_t = (\lambda^y_t, \lambda^z_t) \) in the following way

\[
\lambda_t^y = \frac{\lambda^y_t \sqrt{Y_t}}{\sigma^y}, \quad \lambda_t^z = \frac{\lambda^z_t \sqrt{Z_t}}{\sigma^z}.
\]

Then \( \tilde{W}_t = (\tilde{W}^y_t, \tilde{W}^z_t) = W_t + \int_0^t \lambda_s^\top dW_s \) becomes a Brownian motion under \( \mathbb{Q} \) with Radon-Nikodym density process

\[
\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t} = \exp \left( -\int_0^t \lambda_s dW_s - \frac{1}{2} \int_0^t \| \lambda_s \|^2 ds \right).
\]

The dynamics of the state process under \( \mathbb{Q} \) read

\[
dY_t = \left( \kappa_y + \lambda^y_t \right) \left( \frac{\kappa_y}{\kappa_y + \lambda^y_t} Z_t - Y_t \right) dt + \sigma_y \sqrt{Y_t} d\tilde{W}^y_t,
\]

\[
dZ_t = \left( \kappa_z + \lambda^z_t \right) \left( \frac{\kappa_z}{\kappa_z + \lambda^z_t} \theta_z - Z_t \right) dt + \sigma_z \sqrt{Z_t} d\tilde{W}^z_t.
\]

We assume no market price of default event risk. That means the \( \mathbb{Q} \)-compensator \( \nu^\mathbb{Q}(t, d\mathcal{X}) \) of the loss process \( L \) is equal to the \( \mathbb{P} \)-compensator:

\[
\nu^\mathbb{Q}(t, d\mathcal{X}) = \nu(t, d\mathcal{X}).
\]

The following theorem provides the affine specification for the \((T, x)\)-bond market.

**Theorem 3.1.** Let \( \alpha, \beta_y \) and \( \beta_z \) be some non-increasing and càdlàg functions with \( \alpha(x) = r \) and \( \beta_y(x) = \beta_z(x) = 0 \) for \( x \geq 1 \). Then, under the above assumptions, there exists a loss process \( L \) which is unique in law such that

\[
P(t, T, x) = 1_{\{t \leq x\}} e^{-\int_t^T (A(T,s) - B_y(T,s)Y_s - B_z(T,s)Z_s) ds}
\]

defines an arbitrage-free \((T, x)\)-bond market,\(^3\) where the functions \( A, B_y \) and \( B_z \) solve the Riccati

\(^3\)That is, \( e^{-\int_t^T P(t, T, x), 0 \leq t \leq T \), is a \( \mathbb{Q} \)-martingale, for any \( T > 0 \) and \( x \in [0, 1] \).
where \( \mu \) process \( L \), and dynamics with \( d \).

See Theorem 7.2 of Filipović et al. [2011].

**Proof.** See Theorem 7.2 of Filipović et al. [2011].

Moreover, the compensator of \( L \) is given by

\[
\nu(t, (0, x]) = \alpha(L_{-}) - \alpha(L_{-} + x) + (\beta_{y}(L_{-}) - \beta_{y}(L_{-} + x))Y + (\beta_{y}(L_{-}) - \beta_{y}(L_{-} + x))Z.
\]

**Proof.** See Theorem 7.2 of Filipović et al. [2011].

Plugging (6) into (3) yields the following corollary (see also Filipović & Schmidt [2010, Eqn. (20)]).

**Corollary 3.1.** The implied discounted gains process is a square-integrable \( \mathbb{Q} \)-martingale with dynamics

\[
dG^{(\xi, \kappa)} = e^{-\int_{0}^{t} \nu(t, d\xi)} \left[ B_{t}^{(\xi, \kappa)} d\tilde{W}_{t} + \int_{[0,1]} C_{t}^{(\xi, \kappa)}(\xi)(\mu(dt, d\xi) - \nu(t, d\xi)dt) \right]
\]

where \( \mu(dt, dx) \) denotes the integer-valued random measure associated to the jumps of the loss process \( L \), and

\[
B_{t}^{(\xi, \kappa)} = \int_{(\xi, \kappa)} \left\{ \int_{[0,1]} \left( \sum_{i \in T} P(t, T_{i}, x) \beta(t, T_{i}, x) \right) \right. \\
+ P(t, T_{n}, x) \beta(t, T_{n}, x) + r \int_{t}^{T_{n}} P(t, u, x) \beta(t, u, x)du \right\} dx \\
C_{t}^{(\xi, \kappa)}(\xi) = -\int_{(\xi, \kappa)} \left\{ \int_{T_{n}, x} \left( \sum_{i \in T} P(t, T_{i}, x) \right) \right. \\
+ P(t, T_{n}, x) \left( \sum_{\xi > x} \int_{t}^{T_{n}} P(t, u, x)du \right) \right\} dx
\]

with

\[
\beta(t, T, x) = \begin{pmatrix}
-B_{y}(T - t, x)\sigma_{y} \sqrt{Y_{t}}, & -B_{z}(T - t, x)\sigma_{z} \sqrt{Z_{t}}
\end{pmatrix}^{	op}.
\]
We emphasize that the functions $\alpha$, $\beta_y$ and $\beta_z$ in Theorem 3.1 are exogenous and determine the default event arrival intensity

$$\Lambda_t = \nu(t, (0, 1]) = \alpha(L_{-}) - r + \beta_y(L_{-})Y_t + \beta_z(L_{-})Z_t. \quad (7)$$

as well as the cumulative loss given default distribution function

$$F_L(t, x) = \frac{\nu(t, (0, x])}{\Lambda_t}.$$ 

We henceforth fix some nonnegative parameters $\gamma, a_0, b_0, c_0 \geq 0$, and set

$$\alpha(x) = \gamma\left(e^{-a_0(x^\wedge 1)} - e^{-a_0}\right) + r, \quad \beta_y(x) = e^{-b_0(x^\wedge 1)} - e^{-b_0}, \quad \beta_z(x) = c_0 1_{[0,1]}(x).$$

We obtain

$$\nu(t, (0, x]) = \gamma\left(e^{-a_0(L_{-}^\wedge 1)} - e^{-a_0}\right) + \left(e^{b_0(L_{-}^\wedge 1)} - e^{-b_0}\right)Y_t + c_0 1_{[0,1]}(L_{-})Z_t,$$

and hence the loss given default tail distribution becomes

$$1 - F_L(t, x) = \frac{\gamma\left(e^{-a_0(L_{-}^\wedge 1 + x)\wedge 1)} - e^{-a_0}\right) + \left(e^{-b_0(L_{-}^\wedge 1 + x)\wedge 1)} - e^{-b_0}\right)Y_t + c_0 1_{[0,1]}(L_{-} + x)Z_t}{\gamma\left(e^{-a_0(L_{-}^\wedge 1)} - e^{-a_0}\right) + \left(e^{-b_0(L_{-}^\wedge 1)} - e^{-b_0}\right)Y_t + c_0 1_{[0,1]}(L_{-})Z_t}.$$ 

Evidently, this is a weighted mixture of truncated exponential distributions and a point mass at $1 - L_{-}$. The latter models a catastrophic default event which extinguishes the entire CDO pool. This catastrophic risk component will be crucial for fitting the super senior tranche spread levels, (see also Chen et al. [2009] and Collin-Dufresne et al. [2012]). Notice that the loss given default distribution is not static. The mixture weights are stochastic. More specifically, $Y_t$ tunes the truncated exponential distribution part, while $Z_t$ drives the catastrophic component.

4. Hedging Analysis

Recall that the index corresponds to some weighted portfolio of single name CDSs. It is understood that the index can be replicated by holding the respective positions in the constituent CDSs. We thus consider the problem of hedging a long position in a STCDO with a dynamically rebalanced self-financing portfolio in the entire index and the money market account. In this section, we develop the hedging strategies, algorithms, and assessment criteria that are going to be implemented in the numerical part below.
Due to the presence of an infinite number of possible jump sizes of the loss process $L$, the market formed by the index and the money market account is incomplete. In particular, we cannot perfectly replicate a long position in a STCDO with trading in the index alone. Instead, following Cont & Kan [2011], we will consider variance-minimizing and regression-based hedging strategies.

A variance-minimizing hedge minimizes the $Q$-conditional variance of the hedging error. This method yields a self-financing hedging strategy, which coincides with the perfect replication in a complete market situation. A regression-based strategy is defined as the sequentially estimated regression coefficient of the daily discounted gains of the long position in the STCDO on the daily discounted gains of a short position in the index. While variance-minimizing hedging is well established theoretically, the regression-based strategy is found to be surprisingly effective, despite its lack of theoretical basis within the financial hedging context.

4.1. Variance-Minimizing Strategy

We recall from Corollary 3.1 that the discounted gains process $G^{x_{1},x_{2}}\mid t$ is a square-integrable $Q$-martingale. It follows that, for any time interval $[0, T]$ with $T < T_{n}$, the self-financing strategy $\phi = \phi^{VM}$, with

$$\phi^{VM}_{t} = \frac{d(G^{x_{1},x_{2}}\mid t, G^{0,1}\mid t)}{d(G^{0,1}\mid t)} = \frac{-B_{t}^{x_{1},x_{2}}B_{t}^{0,1} + \int_{0}^{t} C_{t}^{x_{1},x_{2}}(\xi)C_{t}^{0,1}(\xi)\nu(t, d\xi)}{(B_{t}^{0,1})^{2} + \int_{0}^{t} (C_{t}(0,1)(\xi))^{2}\nu(t, d\xi)},$$

along with the initial capital $c = G^{x_{1},x_{2}}\mid 0 = 0$ is the unique minimizer of the quadratic hedging error

$$\inf_{c, \phi} \mathbb{E}_{Q}\left[\left(G^{x_{1},x_{2}}\mid t - c + \int_{0}^{t} \phi_{s}dG^{0,1}_{s}\right)^{2}\right].$$

Here the infimum is taken over all $c \in \mathbb{R}$ and predictable processes $\phi$ with

$$\mathbb{E}_{Q}\left[\int_{0}^{T} \phi_{s}^{2}d(G^{0,1}\mid t)\right] < \infty.$$

See e.g. Filipović & Schmidt [2010, Theorem 5.1]. The strategy $\phi^{VM}$ is called the variance-minimizing strategy. Note that it does not depend on the hedging time horizon $T$.

4.2. Regression-Based Strategy

In regression-based hedging, the idea is to regress the daily gains of the STCDO on the daily gains of the index. Here let us denote the discounted daily gains of a long position in the STCDO by

$$\delta G^{x_{1},x_{2}}_{k} = G^{x_{1},x_{2}}_{k} - G^{x_{1},x_{2}}_{k-1}, \quad k \geq 1$$
where $0 = t_0 < t_1 < \ldots$ denote the calendar days.

Then, we assume that the discounted daily gains of the STCDO and the index are related by

$$\delta G_{t_k}^{(x_1,x_2)} = a + b \delta G_{t_k}^{(0,1)} + \epsilon_k, \quad k \geq 1$$  \hspace{1cm} (8)

where $\epsilon_k$ are i.i.d. mean-zero normal noise terms. Given this statistical model, the idea is to find the daily estimates of the parameters $a$ and $b$. This is achieved via method of linear least squares in which we find estimates $\hat{a}$ and $\hat{b}$ which minimize the squared error of the observed data up to the current day. Then, each day we set the hedge ratio $\phi_{RB}^t = -\hat{b}$, that is, in each day the hedge ratio is computed via estimating the parameter $b$ in above regression model by using the available data up to that day. From the very well acknowledged formula for the least squares estimate, we have

$$\phi_{RB}^t = -\frac{\sum_{t_k \leq t} (\delta G_{t_k}^{(0,1)} - \overline{\delta G_{t_k}^{(0,1)}})(\delta G_{t_k}^{(x_1,x_2)} - \overline{\delta G_{t_k}^{(x_1,x_2)}})}{\sum_{t_k \leq t} (\delta G_{t_k}^{(0,1)} - \overline{\delta G_{t_k}^{(0,1)}})^2} = -\hat{\rho}_t \overline{\sigma_{t_k}^{(x_1,x_2)}}$$  \hspace{1cm} (9)

denote the empirical average and standard deviation of the daily gains of the STCDO, and the empirical correlation between daily gains of the STCDO and the index by time $t$, respectively.

From (8) it follows that the empirical standard deviation of the error terms $\epsilon_k$ by time $t$, given by

$$\overline{\sigma_t}^2 = \overline{\sigma_t^{(x_1,x_2)}} \sqrt{1 - \hat{\rho}_t^2},$$  \hspace{1cm} (10)

serves as a measure for the goodness of the regression-based hedge. In other words, the smaller the standard deviation of the discounted daily gains of the STCDO, and the larger the absolute value of the correlation between the discounted daily gains of the STCDO and the index, the better the performance of regression-based hedge.
4.3. Hedging Algorithm

We first specify a hedging time horizon \([0, T]\), and calendar days \(0 = t_0 < t_1 < \cdots t_K = T\) with \(\delta t := t_k - t_{k-1} \equiv 1/360\).

We denote by \(CP = \{T_1, \ldots, T_n\}\) the set of coupon payment dates and assume that any coupon payment date \(T_i\) lies on the grid of calendar days, that is \(T_i \in \{t_1, \ldots, t_K\}\) for all \(i\).

Assume a time series of the loss process \(L_t\), and the spot value processes \(\Gamma(t_1, t_2)\) \(\delta t\) and \(\Gamma(0, 1)\) \(\delta t\), \(k = 0, \ldots, K\), are given. Consider a hedging strategy denoted by \(\phi_t\). The resulting nominal value process \(V_t\) of the self-financing hedging portfolio with zero initial capital, \(V_{t_0} = 0\), is given by the recursive formula

\[
V_{t_k} = V_{t_{k-1}} e^{\delta t_k} + PL_{t_k}^{(0,1)}
\]

where

\[
PL_{t_k}^{(0,1)} = \phi_{t_{k-1}} \left( \Gamma_{t_k}^{(1,0)} - \Gamma_{t_{k-1}}^{(1,0)} e^{\delta t_k} + \sum_{i \in CP} H^{(0,1)}(L_{t_{k-1}}) - H^{(0,1)}(L_{t_k}) \right)
\]

indicates the nominal daily profit and loss on \((t_{k-1}, t_k]\). The discounted gains process \(G_{t_k}^{(0,1)}\) of holding the index is then given by \(G_{t_k}^{(0,1)} = e^{-\delta t_k} V_{t_k}^{(0,1)}\) where \(V_{t_k}^{(0,1)}\) denotes the nominal value process for the strategy \(\phi_t \equiv 1\). And following the same way one can obtain \(G_{t_k}^{(x_1, x_2)}\).

The hedging algorithm consists of computation and comparison of the nominal value process of the hedging portfolio and the gains process of the tranche position for each day of the hedging period.

4.4. Assessment of Hedging Performance

In order to conclude which hedging strategy outperforms the other, we need some criteria. Regarding this, Cont & Kan [2011] suggest two measures. One of them is the relative hedging error and for a fixed final date \(K\) it is given by the absolute value of the ratio of average daily profit of the hedge position, \(V_T + e^{rT} G_{T_k}^{(x_1, x_2)}\) to the average daily profit of un-hedged position, \(e^{rK} G_{t_k}^{(x_1, x_2)}\). The other criterion is the reduction in volatility and measures the reduction in the dispersion of the profit distribution with respect to the unhedged position. Formally, it is defined as the ratio of the volatility of daily profits from the hedged position to the volatility of the daily profits from the unhedged tranche position. According to these criteria, a hedging strategy performs better as long as the related relative hedging error and reduction in volatility values are smaller.

The first criterion that we considered for the evaluation of the performance of a hedging strategy is the reduction in volatility measure. As the second criterion, assuming the total outstanding value to be equal to 100, we use the normalized total portfolio profit and loss (P&L) series given by

\[
100 \times \left( \frac{V_T + e^{rT} G_{T_k}^{(x_1, x_2)}}{1} - \frac{x_2 - x_1}{11} \right)
\]
We conclude that a hedging strategy performs better as the normalized P&L series stays closer to zero.

The reason why we use normalized total portfolio P&L series instead of a relative hedging error is twofold. First, when the value of the discounted gains gets too small, the relative hedging error gets too large and may yield misleading results. The other reason is that the relative hedging error strongly depends on the final date \( K \). That is, while for some date \( K \) in the sample period the relative hedging error can get very small and implies a successful hedge, for some other day the result may be the opposite.

5. Simulation Analysis

In the simulation analysis our objective is to elaborate more on the performance of the two hedging strategies in a more general framework, where scenarios with nonzero losses are permitted. Recall that in the current modeling setup we deal with three stochastic processes, namely the factors \( Y \), \( Z \) and the loss process \( L \). We use Euler discretization to approximate the discrete time evolution of the factors \( Y \) and \( Z \) in Equation (4)-(5) on the equidistant time grid \( 0 = t_0 < t_1 < \ldots < t_K = T \). For \( k = 0, 1, \ldots, K - 1 \) we have

\[
\begin{align*}
Y_{t_{k+1}} &= Y_{t_k} + \kappa_Y (Z_{t_k} - Y_{t_k}) \delta t + \sigma_Y \sqrt{Y_{t_k}} \delta W_{t_k}^y, \quad Y_0 = \gamma \in \mathbb{R}^+ \quad \text{(11)} \\
Z_{t_{k+1}} &= Z_{t_k} + \kappa_Z (\theta_z - Z_{t_k}) \delta t + \sigma_z \sqrt{Z_{t_k}} \delta W_{t_k}^z, \quad Z_0 = \zeta \in \mathbb{R}^+ \quad \text{(12)}
\end{align*}
\]

where \( \delta W_{t_{k+1}}^y, \delta W_{t_{k+1}}^z = W_{t_{k+1}}^y - W_{t_k}^y, W_{t_{k+1}}^z - W_{t_k}^z \) are i.i.d. \( N(0, \sqrt{\delta t}) \). Using this fact to simulate trajectories of length \( K \) for each of the processes \( Y \) and \( Z \), we first generate \( K \) numbers from \( N(0, \sqrt{\delta t}) \). Inserting \( \delta W_{t_{k+1}}^y, \delta W_{t_{k+1}}^z \) s and the estimated parameters in (11) and (12) then yields a trajectory of length \( K \) for the processes \( Y \) and \( Z \), respectively. Euler discretization methodology may give negative numbers for \( Y_{t_k} \) and \( Z_{t_k} \). To avoid the negative values, whenever realized we change the negative numbers by \( 10^{-8} \).

As the next step, to simulate the loss process \( L \) we use the simulated factors \( Y, Z \), set of parameter estimates and another parameter \( \Psi \) which we interpret as the importance sampling parameter. The reason why we need the parameter \( \Psi \) is as follows. There does not occur any default during the sample period we use. Hence the parameter set coming from the in-sample analysis is not able to generate a remarkable number of jumps. Moreover, the Monte Carlo simulation is known to fail in generating rare events unless the number of simulated scenarios is very large. Nevertheless, a frequently used technique in stress scenario generation is importance sampling (see, e.g. Boyle et al. [1997][Section 2.7] and references therein). In this context, it is possible to manipulate the number of jumps via amplifying the jump intensity given in (7) with the importance sampling parameter \( \Psi \). However, one should take care of the necessary measure change for the adjustment of probabilities assigned to each scenario. This is necessary in particular for
computing the empirical distribution function of the losses. We define the cumulative intensity process \( \hat{\Lambda}_t = \int_0^t \Lambda_s ds \) and sketch the algorithm for simulating a loss trajectory of length \( K \) as follows.

1. Initiate the jump time \( \tau = 0 \), the number of jumps \( N = 0 \), the loss process \( L_{\tau} = 0 \), and the cumulative arrival intensity \( \hat{\Lambda}_{\tau} = 0 \).

2. Generate a number \( U \) from exponential distribution with parameter 1.

3. While \( k < K \) and \( \hat{\Lambda}_{\tau} - \hat{\Lambda}_k < U \) calculate \( \hat{\Lambda}_{k+1} \) via
   \[
   \hat{\Lambda}_{k+1} = \hat{\Lambda}_k + \Psi(\alpha(L_k) + \beta_1(L_k) Y_k + \beta_2(L_k) Z_k - r) \delta t
   \]
   set \( L_{k+1} = L_k, \ k \mapsto k + 1 \)

4. If \( \hat{\Lambda}_k - \hat{\Lambda}_r \geq U \), i.e., when a jump occurs generate a number \( s \) from the standard uniform distribution. Compute the jump size via
   \[
   \Delta L_k = F_{L}^{-1}(L_k, Y_k, Z_k, s)
   \]
   where \( F_L \) is the cumulative loss given default distribution given by
   \[
   F_L(L_k, Y_k, Z_k, x) = \frac{\alpha(L_k) + \beta_1(L_k) Y_k + \beta_2(L_k) Z_k - \alpha(L_k + x)}{\alpha(L_k) + \beta_1(L_k) Y_k + \beta_2(L_k) Z_k - r} - \frac{\beta_1(L_k + x) Y_k + \beta_2(L_k + x) Z_k}{\alpha(L_k) + \beta_1(L_k) Y_k + \beta_2(L_k) Z_k - r}
   \]
   Update the loss path, jump time and number of jumps
   \[
   L_{k+1} = L_k + \Delta L_k, \ \tau = \tau_k, \ N = N + 1
   \]

5. If \( k < K \) return to 2, else stop.

Employing the methodology described above, we simulate 2000 scenarios, 1000 of which are the normal scenarios and generated via taking importance sampling parameter \( \Psi = 1 \). On the other hand, in order to simulate 1000 stress scenarios we take \( \Psi = 100 \). We set the probability estimate of each of the 1000 normal scenarios equal to \( q(i) = 1/1000, \ i = 1, 2, ..., 1000 \), while for the stress scenarios we adjust the probability estimate of each scenario in the following way.

Let \( \tau_n \) denotes the \( n^{th} \) jump time of the process \( L \). Changing the jump intensity of the process \( L \) from \( \Lambda_t \) to \( \Psi \Lambda_t \) and leaving the jump size distribution unchanged is tantamount to an equivalent
change of measure where the measure $\mathbb{P}^\Psi \sim \mathbb{P}$ is characterized by

$$\frac{d\mathbb{P}^\Psi}{d\mathbb{P}}|_{\mathcal{F}_t} = M_t$$

with the Radon-Nikodym derivative $M_t$ given by (see, e.g. Brémaud [1981][Chapter VIII, T10])

$$M_t = \left( \prod_{n \geq 1} \Psi_{1_{[\tau_n \leq t]}} \right) e^{\int_0^t (1 - \Psi) \Lambda_s ds} = \left( \prod_{n \geq 1} \Psi_{1_{[\tau_n \leq t]}} \right) e^{\hat{\Lambda}(1 - \Psi)}.$$  

(13)

Now suppose we generate the $i^{th}$ stress scenario from the distribution $\mathbb{P}^\Psi$. For this particular scenario we denote the total number of jumps realized in $[0,t_K]$ by $N_i$. Then, according to (13) we define the corresponding weight of the scenario $i$ under $\mathbb{P}$ by

$$w(i) = e^{(\Psi - 1) \sum_{j=1}^{x_{j_i}} \alpha(L_y_j) + \beta(L_y_j) Y_j + \beta(L_y_j) Z_j - r \delta_t}.$$ 

Due to the law of large numbers we have $\frac{\sum_{i=1}^{1000} w(i)}{1000} \approx 1$. Nevertheless, we normalize the weight $w(i)$ in an exact way and obtain the following estimate for the $\mathbb{P}$-probability of scenario $i$

$$p(i) = \frac{w(i)}{\sum_{i=1}^{1000} w(i)}.$$ 

Finally, to aggregate the 2000 scenarios we give equal weight to normal and stress scenarios, and set probabilities $\bar{q}(i) = q(i)/2$ and $\bar{p}(i) = p(i)/2$, so that $\sum_{i=1}^{1000} \bar{q}(i) + \bar{p}(i) = 1$, as it should be.

6. Implementation and Results

6.1. Data

The raw data comprises daily observations of iTraxx Europe from 30 August 2006 to 3 August 2010. The stripped data, which has been sourced from Bank Austria\footnote{We thank Peter Schaller for providing the data.}, is the zero-coupon spreads across maturities and tranches, that is

$$R(t, \tau, j) = -\frac{1}{\tau} \log D(t, \tau, j) - r$$

where the zero-coupon discount curve $D(t, \tau, j)$ is given by

$$\tau \mapsto D(t, \tau, j) = \frac{1}{x_j - x_{j-1}} \int_{x_{j-1}}^{x_j} P(t, t + \tau, x) dx$$
for four different times to maturity $\tau := T - t = 3, 5, 7, 10$ and six tranches $j = 1, \ldots, 6$ with standard attachment and detachment points $0\%, 3\%, 6\%, 9\%, 12\%, 22\%, 100\%$. This corresponds to 972 observation days in each of which we have a $4 \times 6$ observation matrix.

We illustrate the time series of the 5-year zero-coupon spreads in Figure 1a. Naturally, the market conditions are reflected in the data set. The tranche data exhibits a relatively stable pattern from the beginning of the data period to July 2007, where the global financial markets are started to be affected from the credit crisis. In March 2008, we observe a spike which stems from the panic due to the possibility of the collapse of Bear Stearns. Furthermore, a drastic upward movement is observed starting from September 2008. This time period corresponds to the breakdown of the credit market due to events such as the bankruptcy of Lehman Brothers.

![Figure 1a: iTraxx Europe 5-year zero-coupon spread data for all tranches.](image)

![Figure 1b: iTraxx Europe index spread data for all maturities.](image)

Figure 1: iTraxx Europe data from 30 August 2006 to 3 August 2010

In Figure 1b we provide index spreads across four maturities. Figure 1a and Figure 1b together show that the tranche data and the index data have the same up and downward trends during the time period considered. Another feature of the data set we use is that there is no default event during the sample period. This implies, $L_{tk} = 0, k = 1, \ldots, 972$.

In addition to the given loss path $L_k$, what is essentially needed in the hedging analysis is the parameter estimates and filtered series for the two-factor affine model. In this context, we appeal to the results in Eksi & Filipović [2013, Section 4.2] where a quasi-maximum likelihood based on the Kalman filter is used to estimate the model parameters and filter out the unobservable factors simultaneously. Table 1 gives the related parameter estimates and Figure 2 provides the plots of filtered factor series.
Table 1: Parameter estimates of the two-factor affine model in the period 30 August 2006-3 August 2010.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta_y$</td>
<td>0.0055</td>
</tr>
<tr>
<td>$\kappa_y$</td>
<td>0.52</td>
</tr>
<tr>
<td>$\kappa_z$</td>
<td>0.22</td>
</tr>
<tr>
<td>$\sigma_y$</td>
<td>0.38</td>
</tr>
<tr>
<td>$\sigma_z$</td>
<td>0.25</td>
</tr>
<tr>
<td>$\lambda_y$</td>
<td>-0.66</td>
</tr>
<tr>
<td>$\lambda_z$</td>
<td>-0.30</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>98.78</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>0.32</td>
</tr>
<tr>
<td>$b_0$</td>
<td>26.40</td>
</tr>
<tr>
<td>$c_0$</td>
<td>0.09</td>
</tr>
</tbody>
</table>

Figure 2: Filtered series of the factor $Y$ and $Z$ for the period 30 Aug 2006-3 Aug 2010.

Given the parameter estimates and the filtered factor series ($Y_t, Z_t$) we utilize formula (2) to construct series for the spot value of 5-year index and 5-year STCDO tranches. Then, we study the hedging of STCDOs with the index. The next section gives results in detail.

6.2. In-Sample Hedging Results

As given in equation (10), the empirical standard deviation, $\hat{\sigma}_{t}^{x_1,x_2}$ and the empirical linear correlation, $\hat{\rho}_{t}$ are the main determinants of the performance of regression-based hedge. Furthermore, it is revealed in (9) that the relative standard deviation $\frac{\hat{\sigma}_{t}^{x_1,x_2}}{\hat{\sigma}_{t}^{x_1}}$ is one of the determinants of regression-based hedging strategy. In this context, as a supplementary result we depicted the empirical variance and relative variance series and the empirical correlation series in Figure 3 and in Figure 4, respectively. It is observed from Figure 3 that for all tranches the empirical variance series are increasing until June 2009 and slowly decrease afterwards. On the other hand, Figure 4 shows that for all tranches but 12 – 22% and 22 – 100% two significant downward shifts occur successively in July 2007 and in March 2008. For the 12 – 22% tranche the correlation series stays relatively constant while the series for the 22 – 100% tranche exhibits a sharp increase in June 2007.
For the whole sample period we perform the hedging analysis given in Section 4.3 for all attachment and detachment points. Taking $3 - 6\%$, $9 - 12\%$ and $22 - 100\%$ tranches as representatives for equity, mezzanine and senior tranches, respectively, Figure 5 depicts the series for the gains process and nominal spot value of the STCDOs and hedging portfolio values both for the regression-based and variance-minimizing strategies.
Figure 5: Hedging results for the sample period: gains process, nominal spot value and hedging portfolio value for equity, mezzanine and senior tranches.

(a) Equity tranche  
(b) Mezzanine tranche  
(c) Senior tranche

The normalized total portfolio P&L and variance-minimizing and regression-based hedging strategies are plotted in Figure 6. For all tranches the regression-based strategy, denoted by the dashed line, is observed to be more stable during the hedging period. It is revealed in plots 6a, 6c and 6e that both for the variance-minimizing and regression-based hedges, the change in $\phi$ at the beginning of the crisis around July 2007 exhibits different patterns for each tranche. In particular, for equity and mezzanine tranches $\phi$ is decreasing in absolute value, indicating a reduction in insurance demand and for the senior tranche there is the opposite behavior. For the regression-based hedge, equation (9) is helpful in understanding why the behavior of $\phi$ differs across tranches. As it can be observed in Figure 3b, for the senior tranches the relative variance is almost constant until July 2007 and then shows a sharp increase. On the other hand, in Figure 4
it is revealed that there is a large increase in the empirical correlation value for the senior tranche. Therefore, one can say that the different behaviors of the relative variance and correlation account for the different movements of the regression-based hedging strategy among tranches.

Figure 6: Hedging results for the sample period: the hedging strategy $\phi$ and normalized total portfolio P&L for equity, mezzanine and senior tranches.
Regarding the performance comparison of the two hedging strategies the P&L criterion does not lead to any conclusion, as in some parts of the period the P&L series for the regression-based strategy is closer to zero and in the rest this holds true for the variance-minimizing hedge. However, if we focus on the last day of the sample period in Figure 6, it is seen that for equity and senior tranches the terminal P&L values for the regression-based hedge are closer to zero, implying a better performance, whereas for the mezzanine tranche the variance-minimizing strategy performs better.

The performance of the regression-based strategy for the mentioned tranches can be explained via equation (10), Figure 3a and Figure 4, where very high correlation between the tranche gains process and the index is observed. However, we want to point out that the presence of defaults in the data set may deteriorate the linear correlation between the index and tranches. This in turn may cause the regression-based strategy to perform worse. We leave this discussion to the next section where the hedging analysis will be done under more general scenarios with non-zero losses.

In Figure 7 we provide the reduction in volatility for the variance-minimizing and regression-based strategies. According to the reduction in volatility criterion, the regression-based hedge performs better than the variance-minimizing hedge for all tranches.

Figure 7: Reduction in volatility for variance-minimizing and regression-based hedge.

We now compare our in-sample hedging results with those in Cont & Kan [2011]. Note that the data and the sample periods of the current study and Cont & Kan [2011] differ. To provide a more appropriate basis for a comparison, we first redo the in-sample hedging analysis for the period covered in Cont & Kan [2011], that is, the period 25 March-25 September 2008. According to Figure 8, the regression-based hedge is observed to be more efficient in terms of the reduction in volatility criterion. This agrees with the results in Cont & Kan [2011, Figure 12]
We compare the variance-minimizing and regression-based strategies under the criterion of relative hedging error. In particular, we compute the relative hedging error values at dates 16 September and 25 September 2008. The results are depicted in Figure 9, where bar graphs 9a and 9b together show that the relative hedging error criterion depends very much on the final date of the hedging period. When we compare Figure 9 to Cont & Kan [2011, Figure 13], we see that the results are very much in the same direction.

Figure 9: Relative hedging error at 16 September and 25 September 2008.

(a) Relative hedging error at 25 September 2008

(b) Relative hedging error at 16 September 2008
6.3. Simulation Results

Using the method given in Section 5 and the set of parameter estimates given in Table 1 we simulate 2000 trajectories for the time horizon of 250 days. Among the generated trajectories 1000 of them correspond to normal scenarios and the other 1000 represent stress scenarios. Next, we investigate the performance of the variance-minimizing and the regression-based hedging strategies on the set of simulated scenarios. Moreover, we perform a conditional simulation analysis in which 2000 loss scenarios are generated conditional on the original filtered factor trajectories.

We take the set of all simulated scenarios and focus on the last date, \( T = 250 \), of the simulation period. The empirical cumulative loss distribution function at the final date is given in Figure 10. According to this figure the simulation procedure is successful in the sense that it is able to produce loss scenarios ranging between 0% and 10%.

Figure 10: Empirical cumulative loss distribution at \( T = 250 \) for a total number of 2000 scenarios.

We plot the date \( T \) cumulative distribution of the total hedging portfolio P&L distribution for variance-minimizing and regression-based strategies in Figure 11. This figure implies that in most of the simulated trajectories both variance-minimizing and regression-based hedging strategies yield normalized total portfolio P&L values which are close to zero. In other words, both strategies are successful on average. However, for all tranches, the regression-based strategy, which is indicated by the dashed line, is observed to produce more extreme losses as the density corresponding to the P&L values coming from the regression-based strategy has a longer left tail. Hence, we can say that on average the variance-minimizing strategy is more successful than the regression-based hedge for scenarios which permit non-zero losses.
To investigate further on the relative performances of hedging strategies, sitting at time $t = 0$, we estimate the risk of regression-based and variance-minimizing hedging portfolios for each time horizon ranging between 1 to 250 days. In particular, we compute the value at risk (VaR) and expected shortfall at the confidence levels 99% and 99.9%, respectively. Results for the 99% confidence level are illustrated in Figure 12. For both confidence levels, for all tranches and for every time horizon VaR and expected shortfall corresponding to the regression-based strategy are observed to lie above the VaR and expected shortfall of the variance-minimizing hedge. This means that the regression-based strategy yields riskier hedging portfolios than the ones generated by the variance-minimizing strategy.
Figure 12: VaR and Expected shortfall (ES) for the 99% confidence level and various time horizons.

(a) VaR, equity tranche
(b) Expected shortfall, equity tranche
(c) VaR, mezzanine tranche
(d) Expected shortfall, mezzanine tranche
(e) VaR, senior tranche
(f) Expected shortfall, senior tranche

As the second criterion, for each scenario we compute the reduction in volatility for all tranches. Following this, we compute the descriptive statistics for reduction in volatility by taking the Radon-Nikodym densities of the stress scenarios into account. Table 2 and Table 3 illustrate the results for regression-based and variance-minimizing hedges, respectively. For all
tranches, the mean values of reduction in volatility for the regression-based hedge are higher than the mean values for the variance-minimizing hedge. This suggests that, according to the reduction in volatility criterion, for all tranches the variance-minimizing hedge performs better on average. On the other hand, the regression-based hedge yields more right-skewed distribution for reduction in volatility, implying that most of the values lie to the left of the mean. Moreover, coefficient of variation (CV) values imply that for all tranches the regression-based hedge yields more dispersed reduction in volatility values.

Table 2: Descriptive Statistics of reduction in volatility for the regression-based hedge.

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Median</th>
<th>Std</th>
<th>CV</th>
<th>Max</th>
<th>Min</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity</td>
<td>74.09</td>
<td>47.97</td>
<td>140.56</td>
<td>1.89</td>
<td>3531.4</td>
<td>6.96</td>
</tr>
<tr>
<td>Mezzanine</td>
<td>57.62</td>
<td>27.13</td>
<td>103.57</td>
<td>1.79</td>
<td>2766.7</td>
<td>5.21</td>
</tr>
<tr>
<td>Senior</td>
<td>72.10</td>
<td>42.09</td>
<td>112.75</td>
<td>1.56</td>
<td>3761.5</td>
<td>5.43</td>
</tr>
</tbody>
</table>

Table 3: Descriptive Statistics of reduction in volatility for the variance-minimizing hedge

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Median</th>
<th>Std</th>
<th>CV</th>
<th>Max</th>
<th>Min</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equity</td>
<td>64.95</td>
<td>55.72</td>
<td>43.56</td>
<td>0.67</td>
<td>601.6</td>
<td>3.36</td>
</tr>
<tr>
<td>Mezzanine</td>
<td>43.28</td>
<td>36.01</td>
<td>34.33</td>
<td>0.79</td>
<td>2223.6</td>
<td>10.26</td>
</tr>
<tr>
<td>Senior</td>
<td>50.35</td>
<td>34.40</td>
<td>62.19</td>
<td>1.23</td>
<td>3261.6</td>
<td>15.98</td>
</tr>
</tbody>
</table>

We also obtain the density function of the reduction in volatility for both hedging strategies. While doing this, we first compute the related frequencies of the possible values for reduction in volatility. The important point in this step is to adjust the frequencies coming from the stress scenarios, that is, scenarios which are generated by taking the importance sampling parameter $\Psi = 100$. This is readily possible since we have the related Radon-Nikodym densities. After obtaining the adjusted frequencies, we use the Kernel smoothing technique in order to get the density function of the reduction in volatility for regression-based and variance-minimizing strategies. For each tranche we plot the related density in Figure 13, where a logarithmic scale is used in the horizontal axis for illustrative purposes.
We now present the results of the conditional simulation analysis. We fix the filtered factor series given in Figure 2 and conditional on these trajectories we simulate 2000 loss scenarios again with the importance sampling parameter values $\Psi = 1$ and $\Psi = 100$. Conditional distribution of the simulated loss process at time $T$ is depicted in Figure 14. One striking result is that, when compared with the loss distribution function given in Figure 10, conditional loss distribution in Figure 14 gives higher probability to losses greater than 10%. Moreover, simulation results suggest that for normal scenarios, $\Psi = 1$, in 815 of 1000 simulated loss trajectories, there occurred a jump, that is, a default. These findings suggest that an actual default event in the iTraxx was very much likely to occur. In such a scenario, the regression-based hedge is expected to result in a larger loss than the variance-minimizing strategy. This is because the latter slightly underperforms the former in times of no default, but is likely to provide better protection against
losses in case of a default.

Figure 14: Empirical conditional cumulative loss distribution function at \( T = 250 \) for a total number of 2000 scenarios.

7. Conclusion

In this paper we have dealt with the hedging of STCDOs with the underlying index default swap. We first proposed a two-factor affine model in which a catastrophic risk component is considered as a tool for explaining the dynamics of the senior tranches. Next, we computed the variance-minimizing hedging strategy based on the affine model. We analyzed the actual performance of the variance-minimizing and model-free regression-based hedging strategies on the iTraxx Europe data. We also ran a simulation analysis, in which the objective was to test the performance of hedging strategies under more general loss scenarios.

Our findings suggest that within the data period, both hedging strategies are effective in reducing the risk of a STCDO. According to the relative hedging error and reduction in volatility criteria given in Cont & Kan [2011], the regression-based strategy is observed to be more successful than the variance-minimizing hedge within the data period. This result is consistent with the findings of Cont & Kan [2011]. However, results of our simulation study show that under more general scenarios the regression-based strategy yields riskier hedging portfolios and hence the variance-minimizing strategy is likely to provide better protection against losses in periods with defaults.

References


