Optimal Liquidity Provision in Limit Order Markets

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Abstract

A small investor provides liquidity at the best bid and ask prices of a limit order market. For small spreads and frequent orders of other market participants, we explicitly determine the investor’s optimal policy and welfare. In doing so, we allow for general dynamics of the mid price, the spread, and the order flow, as well as for arbitrary preferences of the liquidity provider under consideration.

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1 Introduction

Trades on financial markets are instigated by various motives. For example, mutual funds rebalance their portfolios, derivative positions are hedged, and margin calls may necessitate the liquidation of large asset positions. Such trades require counterparties who provide the necessary liquidity to the market. Traditionally, this market making role was played by designated “specialists”, who agreed on contractual terms to match incoming orders in exchange for earning the spread between their bid and ask prices. As stock markets have become automated, this quasi-monopolistic setup has given way to limit order markets on many trading venues. Here, electronic limit order books collect all incoming orders, and automatically pair matching buy and sell trades. Such limit order markets allow virtually all market participants to engage in systematic liquidity provision, which has become a popular algorithmic trading strategy for hedge funds.

The present study analyzes optimal strategies for liquidity provision and their performance. In contrast to most previous work on market making, we do not consider a single large monopolistic specialist (e.g., [9, 2, 14, 3, 11]) who optimally sets the bid-ask spread. Instead, as in [21, 13],...
we focus on a small liquidity provider, who chooses how much liquidity to provide by placing limit buy and sell orders at exogenously given bid and ask prices, respectively. For tractability, we assume that the mid-price of the risky asset follows a martingale and consider the practically relevant limiting regime of small spreads and frequent orders of other market participants. Thereby, we obtain explicit formulas in a general setting, allowing for arbitrary dynamics of the mid price, the spread, and the order flow, as well as for general preferences of the liquidity provider under consideration.

The optimal policy is determined by an upper and lower boundary for the monetary position in the risky asset, to which the investor trades whenever an exogenous order of another market participant arrives. Hence, these target positions determine the amount of liquidity the investor posts in the limit order book. Kühn and Stroh [21] characterize these boundaries by the solution of a free boundary problem for a log-investor with unit risk aversion, who only keeps long positions in a market with constant order flow and bid-ask prices following geometric Brownian motion with positive drift. In the present study we show in a general setting with a martingale mid price that – in the limit for small spreads and frequent orders of other market participants – the upper and lower target positions are given explicitly by

\[ \beta_t^u = \frac{2 \varepsilon_t \alpha_t^2}{\text{ARA}(x_0) \sigma_t^2}, \quad \beta_t^l = -\frac{2 \varepsilon_t \alpha_t^1}{\text{ARA}(x_0) \sigma_t^2}. \]

In these formulas, \(2 \varepsilon_t\) is the width of the relative bid-ask spread, \(\alpha_t^1\) and \(\alpha_t^2\) are the arrival rates of market sell and buy orders of other market participants, \(\sigma_t\) is the volatility of the risky asset returns, and \(\text{ARA}(x_0)\) is the absolute risk aversion of the investor at her initial position \(x_0\). To wit, the optimal amount of liquidity provided is inversely proportional to the inventory risk caused by the asset’s local variance, scaled by the investor’s risk aversion. Conversely, liquidity provision is proportional to the compensation per trade (i.e., the spread \(2 \varepsilon_t\)), and the arrival rates \(\alpha_t^1\) resp. \(\alpha_t^2\). The product of these two terms plays the role of the risky asset’s expected returns in the usual Merton position, in that it describes the investor’s average revenues per unit time, that are traded off against her risk aversion and the variance of the asset returns to determine the optimal target position. Here, however, revenues are derived by netting other traders’ buy and sell orders, unlike for the classical Merton problem, where they are generated by participating in trends of the risky asset. Note that the above policy is myopic, in that it only depends on the local dynamics of the model; future variations are not taken into account at the leading order.

The performance of the above strategy can also be quantified. At the leading order, its certainty equivalent is given by

\[ x_0 + \frac{\text{ARA}(x_0)}{2} E\left[ \int_0^T \left( \beta_t^u A_t^1 + \beta_t^l A_t^2 \right) \sigma_t^2 dt \right], \]

where \(\omega \in A_t^1\) if the investor’s last trade before time \(t\) was a purchase so that her position is close to the upper boundary \(\beta_t^u\). Likewise, \(\omega \in A_t^2\) if the investor’s position is close to the lower boundary \(\beta_t^l\) after her last trade was a sale of the risky asset. Hence, the certainty equivalent of providing liquidity in the limit order market is given by the average (with respect to states and business time \(\sigma_t^2 dt\)) of future squared target positions, rescaled by risk aversion. If all model parameters are

\[ ^1 \text{Related results for models with small trading costs have recently been determined by [27, 23, 28, 20, 19]. These correspond to optimal trading strategies for liquidity takers, whose demand is matched by liquidity providers such as the ones considered here.} \]

\[ ^2 \text{This is in direct analogy to the results for models with proportional transaction costs [19, Equation (3.4)]; since the mid price follows a martingale in our model, the marginal pricing measure coincides with the physical probability here.} \]
constant, the above formula simplifies to
\[ x_0 + \frac{(2\varepsilon \alpha^1)(2\varepsilon \alpha^2)}{2\text{ARA}(x_0)\sigma^2} T. \]

In this case, liquidity provision is therefore equivalent at the leading order to an annuity proportional to the “drift rates” \(2\varepsilon \alpha^i\) of the investor’s revenues from purchases resp. sales, divided by two times the investor’s risk aversion, times the risky asset’s variance. In the symmetric case \(\alpha^1 = \alpha^2 = \alpha\), this is in direct analogy to the corresponding result for the classical Merton problem in the Black-Scholes model, where the equivalent annuity is given by the squared Sharpe ratio divided by two times the investor’s risk aversion. For a given total order flow \(\alpha^1 + \alpha^2\), asymmetries \(\alpha^1 \neq \alpha^2\) reduce liquidity providers’ certainty equivalents, since they reduce the opportunities to earn the spread with little inventory risk by netting successive buy and sell trades.

A key assumption of our model is that market orders of other market participants do not move the current best bid or ask prices. This represents the most benign setting for liquidity providers, since it allows them to earn the full spread between alternating buy and sell trades, only subject to the risk of intermediate price changes. This changes substantially if market prices systematically rise resp. fall for purchases resp. sales of other market participants, as acknowledged in the voluminous literature on price impact (e.g., [4, 1, 25]). These effects can stem, e.g., from adverse selection, as informed traders prey on the liquidity providers [10], or from large incoming orders that eat into the order book [25]. Our model can be extended to account for price impact of incoming orders equal to a fraction \(\kappa \in [0, 1)\) of the half-spread \(\varepsilon_t\). This extension is still tractable; indeed, the above formula (1.1) for the leading-order optimal position limits generalizes to
\[ \beta_t = \frac{2\varepsilon_t(1 - \kappa)\alpha^1_t - \kappa^2 \alpha^1_t}{\text{ARA}(x_0)\sigma^2_t}, \quad \beta_t = -\frac{2\varepsilon_t(1 - \kappa)\alpha^2_t - \kappa^2 \alpha^2_t}{\text{ARA}(x_0)\sigma^2_t}. \]

For a symmetric order flow \((\alpha^1_t = \alpha^2_t = \alpha_t)\), these formulas reduce to
\[ \beta_t = \frac{2\varepsilon_t(1 - \kappa)\alpha_t}{\text{ARA}(x_0)\sigma^2_t}, \quad \beta_t = -\frac{2\varepsilon_t(1 - \kappa)\alpha_t}{\text{ARA}(x_0)\sigma^2_t}. \]

That is, liquidity provision is simply reduced by a factor of \(1 - \kappa\) in this case. Here, the intuition is that, for \(\kappa \approx 1\), price impact almost neutralizes the proportional transaction cost \(\varepsilon_t\) the liquidity provider earns per trade. Hence, market making becomes unprofitable in this case as dwindling earnings are outweighed by inventory risk.

Even with price impact, the leading-order optimal certainty equivalent is still given by (1.2), if one replaces the trading boundaries accordingly. Hence, liquidity providers’ profits shrink as they reduce their position limits due to price impact.

The remainder of the article is organized as follows. Our model is introduced in Section 2. Subsequently, the main results of the paper are presented in Section 3, and proved in Section 4. Finally, Section 5 extends the model to allow for price impact of incoming orders.

2 Model

2.1 Limit Order Market

We consider a financial market with one safe asset, normalized to one, and one risky asset, which can be traded either by market orders or by limit orders. Market orders are executed immediately,
but purchases at time \( t \) are settled at a higher ask price \((1 + \varepsilon_t)S_t\), whereas sales only earn a lower bid price \((1 - \varepsilon_t)S_t\). In contrast, limit orders can be put into the order book with an arbitrary exercise price, but are only executed once a matching order of another market participant arrives. Handling the complexity of limit orders with arbitrary exercise prices is a daunting task. To obtain a tractable model, we therefore follow [21] in assuming that limit buy or sell orders can only be placed at the current best bid or ask price, respectively. This can be justified as follows for small investors, whose orders do not move market prices. In this case, placing (and constantly updating) limit buy orders at a “marginally” higher price than the current best-bid price \((1 - \varepsilon_t)S_t\) guarantees execution as soon as the next sell order of another trader arrives. Limit buy orders with a higher exercise price are executed at the same time but at a higher cost, whereas exercise prices below the current best bid are only executed later. This argument implicitly assumes that the incoming orders of other market participants are liquidity-driven and small, so that they do not move market prices (we show how to relax this assumption in Section 5). Moreover, the investor under consideration is even smaller, in that her orders also don’t influence market prices and are executed immediately against any incoming order of another market participant. These assumptions greatly reduce the complexity of the problem. Yet, the model still retains the key tradeoff between making profits by providing liquidity, and the inventory risk caused by the positions built up along the way.

Let us now formalize trading in this limit order market. All stochastic quantities are defined on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfying the usual conditions. Strategies are described by quadruples \( \mathcal{G} = (M^B_t, M^S_t, L^B_t, L^S_t)_{t \in [0,T]} \) of predictable processes. Here, the nondecreasing processes \( M^B_t \) and \( M^S_t \) represent the investor’s cumulated market buy and sell orders until time \( t \), respectively. \( L^B_t \) (resp. \( L^S_t \)) specifies the size of the limit buy order with limit price \((1 - \varepsilon_t)S_t\) (resp. the limit sell order with limit price \((1 + \varepsilon_t)S_t\)) in the book at time \( t \), i.e., the amount that is bought or sold if an exogenous market sell or buy order arrives at time \( t \). Fix an initial position of \( x_0 \) units in the safe and \( x = 0 \) units in the risky asset. The risky position \( \varphi_t \) changes by market orders, and when incoming limit buy or sell orders are executed at the jump times of some counting processes \( N^B_t \) or \( N^S_t \), respectively. At the jump times of \( N^B_t \), the sell order of another market participant arrives so that the risky position of the liquidity provider is increased according to the number of corresponding limit orders in the book, and analogously for incoming buy orders at the jump times of \( N^S_t \):

\[
\varphi_t = M^B_t - M^S_t + \int_0^t L^B_s \, dN^1_s - \int_0^t L^S_s \, dN^2_s. \tag{2.1}
\]

For self-financing strategies, the corresponding safe position \( \varphi^0_t \) then evolves as follows:

\[
\varphi^0_t = x_0 - \int_0^t (1 + \varepsilon_s)S_s \, dM^B_s + \int_0^t (1 - \varepsilon_s)S_s \, dM^S_s - \int_0^t L^B_s(1 - \varepsilon_s)S_s \, dN^1_s + \int_0^t L^S_s(1 + \varepsilon_s)S_s \, dN^2_s. \tag{2.2}
\]

\footnote{That is, \( \varepsilon_t \) is the halfwidth of the bid-ask spread.}
\footnote{This is made explicit in the model of Guilbaud and Pham [13], where the liquidity provider can gain priority of execution by improving the spread by one tick. We abstract from this issue for the sake of tractability.}
\footnote{Partial execution is studied by Guilbaud and Pham [12].}
\footnote{The assumption that \( L^B_t \) and \( L^S_t \) can be arbitrary predictable processes is justified because the submission and deletion of limit orders is typically free.}
\footnote{Note that the integrators \( M^B_t \) and \( M^S_t \) are processes of finite variation possessing left and right limits, but are in general neither left- nor right-continuous. See [21] for the precise definition of the corresponding integrals.}
To wit, market orders are executed at the less favorable side of the bid-ask spread, whereas limit orders are matched against other traders’ orders at the more favorable side. Any pair $(\varphi_t^0, \varphi_t^1)_{t \in [0,T]}$ that can be written as (2.1-2.2) for some strategy $\mathcal{G}$ is called a self-financing portfolio process.

Let now specify the primitives of our model. We work in a general Itô process setting; in particular, no Markovian structure is required. The mid price follows

$$\frac{dS_t}{S_t} = \sigma_t dW_t,$$

for a Brownian motion $W_t$ and a volatility process $\sigma_t$. Assuming the mid-price of the risky asset to be a martingale allows to disentangle the effects of liquidity provision from pure investment due to trends in the risky asset; on a technical level, it is also needed to obtain both long and short positions even in the limit for small spreads. This assumption is reasonable since “market making is typically not directional, in the sense that it does not profit from security prices going up or down” [13]. Moreover, as in the optimal execution literature (e.g., [4, 1, 25]), it is also justified by the time scales under consideration: we are not dealing with long-term investment here, but much rather focusing on high-frequency liquidity provision strategies which are typically liquidated and evaluated at the end of a trading day [24]. Models for high-frequency strategies designed to profit from the predictability of short-term drifts are studied in [7, 12].

The arrival times of sell and buy orders by other market participants are modeled by counting processes $N^1_t$ and $N^2_t$ with absolutely continuous jump intensities $\alpha^1_t$ and $\alpha^2_t$, respectively. We assume that $N^1_t$ and $N^2_t$ a.s. never jump at the same time. In contrast to most of the previous literature, we do not restrict ourselves to Poisson processes with independent and identically distributed inter-arrival times. Instead, we allow for general arrival rates, thereby recapturing uncertainty about future levels and also empirical observations such as the U-shaped distribution of order flow over the trading day.

We are interested in limiting results for a small spread $\varepsilon_t$. Therefore, we parametrize it as

$$\varepsilon_t = \varepsilon \mathcal{E}_t,$$

for a small parameter $\varepsilon$ and an Itô process $\mathcal{E}_t$. Unlike for models with proportional transaction costs (e.g., [29, 17]), where it is natural to assume that all other model parameters remain constant as the spread tends to zero, the width of the spread is inextricably linked to the arrival rates of exogenous market orders here. Indeed, market orders naturally occur more frequently for more liquid markets with smaller spreads. Hence, we rescale the arrival rates accordingly:

$$\alpha^1_t = \lambda^1_t \varepsilon^{-\vartheta}, \quad \alpha^2_t = \lambda^2_t \varepsilon^{-\vartheta},$$

for some $\vartheta \in (0,1)$.

Here, $\vartheta > 0$ ensures that orders arrive continuously as the bid-ask spread vanishes for $\varepsilon \to 0$. $\vartheta < 1$ is needed to ensure that the profits from liquidity provision vanish as $\varepsilon \to 0$. Higher arrival rates necessitate extensions of the model such as a price impact of incoming orders; see Section 5 for more details. In our optimal policy and the corresponding utility, the exponent $\vartheta$ only appears in the rates of the asymptotic expansions; the leading-order terms are fully determined by the arrival rates $\alpha^1_t, \alpha^2_t$.

The processes $\lambda^1_t, \lambda^2_t, \sigma_t$, and $\mathcal{E}_t$ satisfy the following technical assumptions:

**Assumption 2.1.** $\lambda^1_t, \lambda^2_t, \sigma_t$, and $\mathcal{E}_t$ are positive continuous semimartingales that are bounded and bounded away from zero. Their predictable finite variation parts and the quadratic variation processes of their local martingale parts are absolutely continuous with a bounded continuous rate.

That is, $\alpha^i_t$ are predictable processes and $N^i_t - \int_0^t \alpha^i_s \, ds$ are local martingales for $i = 1, 2$. 

5
Note that we allow for any stochastic dependence of the processes $\lambda^i_t$ and $E^i_t$. In the market microstructure literature (cf., e.g., [3]), plausible distributions of trading times as functions of the current bid-ask prices are derived.

### 2.2 Preferences

The investor’s preferences are described by a general von Neumann-Morgenstern utility function $U : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following mild regularity conditions:

**Assumption 2.2.**

(i) $U$ is strictly concave, strictly increasing, and twice continuously differentiable.

(ii) The corresponding absolute risk aversion is bounded and bounded away from zero:

$$c_1 < \text{ARA}(x) := -\frac{U''(x)}{U'(x)} < c_2, \quad \forall x \in \mathbb{R},$$  \hspace{1cm} (2.3)

for some constants $c_1, c_2 > 0$.

**Remark 2.3.** Since $U'(x) = U'(0) \exp(\int_0^x U''(y)/U'(y) \, dy)$, Condition (2.3) implies that

$$U'(x), |U''(x)| \leq C \exp(-c_2x), \quad \forall x \leq 0 \quad \text{and} \quad U'(x), |U''(x)| \leq C \exp(-c_1x), \quad \forall x > 0,$$  \hspace{1cm} (2.4)

for some constant $C > 0$.

The arch-example satisfying these assumptions is of course the exponential utility $U(x) = -\exp(-\alpha x)$ with constant absolute risk aversion $\alpha$. Analogues of our results can also be obtained for utilities defined on the positive half line, such as power utilities with constant relative risk aversion. Here, we focus on utilities whose absolute risk aversion is uniformly bounded, because these naturally lead to bounded monetary investments in the risky asset, in line with the “risk budgets” often allocated in practice:

**Definition 2.4.** A family of self-financing portfolio processes $(\varphi^0_t, \varphi^+_t)_{\varepsilon \in (0,1)}$ in the limit order market is called admissible if the monetary position $\varphi^+_t S_t$ held in the risky asset is uniformly bounded and converges to zero pointwise for $\varepsilon \rightarrow 0$.

Here, the pointwise convergence to zero is naturally satisfied for “almost” optimal strategies, because the martingale assumption on the mid price of the risky asset implies that no investment is optimal with a zero spread. This notion of admissibility also makes sense economically, as high-frequency liquidity providers typically try to keep their inventory close to zero [24].

### 3 Main Results

The main results of the present study are a trading policy that is optimal at the leading order $\varepsilon^2(1-\vartheta)$ for small spreads $\varepsilon_t = \varepsilon \varepsilon_t$, and an explicit formula for the utility that can be obtained by applying it. To this end, define the monetary trading boundaries

$$\beta^-_t = \frac{2\varepsilon_t \alpha^2}{\text{ARA}(x_0)\sigma^2_t}, \quad \beta^+_t = -\frac{2\varepsilon_t \alpha^1}{\text{ARA}(x_0)\sigma^1_t}$$

and consider the strategy that keeps the risky position $\varphi^+_t S_t$ in the interval $[\beta^+_t, \beta^-_t]$ by means of market orders, while constantly updating the corresponding limit orders so as to trade to $\beta^+_t$.
resp. \( \bar{\beta}_t \) whenever the buy resp. sell order of another market participant allows to sell or buy at favorable prices, respectively. Formally, this means that the process \((\beta^e_t)_{t \in [0,T]}\) is defined as the unique solution to the Skorokhod problem

\[
d\beta^e_{t+} = \beta^e_t \sigma_t dW_t + (\beta_t - \beta^e_t) dN^1_t + (\beta - \beta^e_t) dN^2_t + d\Psi_t, \quad \beta^e_0 = 0, \tag{3.1}
\]

where \( \Psi_t \) is the minimal finite variation process that keeps the solution in \([\beta^e_t, \bar{\beta}^e_t]^{10} \) This corresponds to the strategy

\[
M^B_t := \int_0^t \frac{1}{S_s} d\Psi^+_s, \quad M^S_t := \int_0^t \frac{1}{S_s} d\Psi^-_s, \quad L^B_t := \frac{\bar{\beta}_t - \beta^e_t}{S_t}, \quad L^S_t := \frac{\beta^e_t - \beta_t}{S_t}, \tag{3.2}
\]

with admissible portfolio process

\[
d\varphi^e_{t+} = \frac{1}{S_t} d\Psi^+_t - \frac{1}{S_t} d\Psi^-_t + \frac{\bar{\beta}_t - \beta^e_t}{S_t} dN^1_t + \frac{\beta - \beta^e_t}{S_t} dN^2_t, \quad \varphi_0 = 0,
\]

\[
d\varphi_0^e = -(1 + \varepsilon_t) d\Psi^+_t + (1 - \varepsilon_t) d\Psi^-_t + (1 - \varepsilon_t)(\beta^e_t - \beta_t) dN^1_t + (1 + \varepsilon_t)(\beta^e_t - \beta_t) dN^2_t, \quad \varphi_0 = x_0,
\]

and liquidation wealth process \( \hat{X}_t^e := \varphi^e_t + \varphi_t 1_{(\varphi_t \geq 0)}(1 - \varepsilon_t)S_t + \varphi_t 1_{(\varphi_t < 0)}(1 + \varepsilon_t)S_t \). The following is the main result of the present paper:

**Theorem 3.1.** Suppose Assumptions 2.1 and 2.2 hold. Then, the above policy is optimal at the leading order \( \varepsilon^{2(1-\vartheta)} \), in that:

\[
E[U(\hat{X}_T^e)] = U \left( x_0 + \frac{\varepsilon^{2(1-\vartheta)}}{\text{ARA}(x_0)} E \left[ \int_0^T \left( \frac{2\varepsilon_t^2(\lambda_t^2)^2}{\sigma_t^2} - 1 \right) A_t^1 + \frac{2\varepsilon_t^2(\lambda_t^2)^2}{\sigma_t^2} - 1 \right] A_t^2 \right) + o(\varepsilon^{2(1-\vartheta)}), \quad \varepsilon \to 0,
\]

for the corresponding liquidation wealth processes \((\hat{X}_t^\varepsilon)_{\varepsilon \in (0,1)}, \) whereas

\[
E[U(X_T^e)] \leq U \left( x_0 + \frac{\varepsilon^{2(1-\vartheta)}}{\text{ARA}(x_0)} E \left[ \int_0^T \left( \frac{2\varepsilon_t^2(\lambda_t^2)^2}{\sigma_t^2} - 1 \right) A_t^1 + \frac{2\varepsilon_t^2(\lambda_t^2)^2}{\sigma_t^2} - 1 \right] A_t^2 \right) + o(\varepsilon^{2(1-\vartheta)}), \quad \varepsilon \to 0,
\]

for any competing family \((X_t^\varepsilon)_{\varepsilon \in (0,1)}\) of admissible liquidation wealth processes.

Here, \( \omega \in A^1 \) resp. \( \omega \in A^2 \) means that the investor’s last trade before time \( t \) was a purchase or sale, respectively, i.e. \( A^i = \{ \omega \mid (\omega, t) \in A^i \}, \quad i = 1,2 \) for the predictable sets

\[
A^1 = \{ (\omega, t) \mid \sup \{ s \in (0, t) \mid \Delta N^1_s > 0 \} > \sup \{ s \in (0, t) \mid \Delta N^2_s > 0 \} \},
A^2 = \{ (\omega, t) \mid \sup \{ s \in (0, t) \mid \Delta N^2_s > 0 \} \geq \sup \{ s \in (0, t) \mid \Delta N^1_s > 0 \} \}. \tag{3.3}
\]

(By convention, before the first jump of \((N^1, N^2)\) all time points belong to \( A^2 \).

If the model parameters \( \lambda_t, \sigma_t, \varepsilon_t \) are all constant, then the above formula reduces to

\[
E[U(\hat{X}_T^e)] = U \left( x_0 + \frac{2\lambda^1 \lambda^2}{\text{ARA}(x_0)\sigma^2} T \varepsilon^{2(1-\vartheta)} \right) + o(\varepsilon^{2(1-\vartheta)}), \quad \varepsilon \to 0.
\]

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10That is, \( \bar{\beta}_t \leq \beta^e_t \leq \bar{\beta}_t \) and \( \Psi_t \) is a continuous process of finite variation such that \( \int_0^t 1_{(\beta^e_t = \bar{\beta}_t)} d\Psi_s \) is increasing, \( \int_0^t 1_{(\beta^e_t = \bar{\beta}_t)} d\Psi_s = 0 \) is decreasing, and \( \int_0^t 1_{(\beta^e_t < \bar{\beta}_t < \bar{\beta}_t)} d\Psi_s \) vanishes. The solution is constructed explicitly in (4.1).
4 Proofs

This section contains the proof of our main result, Theorem 3.1. We proceed as follows: first, it is shown that as the spread \( \varepsilon_t = \varepsilon E_t \) tends to zero and jumps to the trading boundaries \( \beta_t, \overline{\beta}_t \) become more and more frequent for our policy \( \beta^\varepsilon_t \), almost all time is eventually spent near \( \beta_t, \overline{\beta}_t \). Motivated by this result, we then construct a frictionless “shadow market”, which is at least as favorable as the original limit order market, and for which the optimal policy is precisely given by the approximation of \( \beta^\varepsilon_t \) which oscillates between \( \beta_t \) and \( \overline{\beta}_t \). In a third step, we then show that the utility obtained from applying our original policy \( \beta^\varepsilon_t \) matches the one for the optimizer in the more favorable frictionless shadow market at the leading order for small spreads, so that our candidate \( \beta^\varepsilon_t \) is indeed optimal at the leading order.

4.1 An Approximation Result

As described above, we start by showing that our policy \( \beta^\varepsilon_t \) spends almost all time near the boundaries \( \beta_t, \overline{\beta}_t \) as the spread \( \varepsilon_t = \varepsilon E_t \) collapses to zero and orders of other market participants become more and more frequent:

**Lemma 4.1.** On the stochastic interval \( ] \inf \{ t > 0 \mid \Delta N^1_t > 0 \text{ or } \Delta N^2_t > 0 \}, T [ \), the process

\[
\left( \beta^\varepsilon_t - \beta_t 1_{A_t^1} - \overline{\beta}_t 1_{A_t^2} \right) \varepsilon^{\theta - 1}
\]

converges to 0 uniformly in probability for \( \varepsilon \to 0 \).

**Proof.** The solution of the Skorokhod problem (3.1) can be constructed explicitly. Indeed, let \( (\tau^\varepsilon_i)_{i \in \mathbb{N}} \) be the jump times of \( N^1_t \), i.e. the jumps of \( \beta^\varepsilon_t \) to the upper boundary \( \overline{\beta}_t \). (To ease notation we suitably extend the model beyond \( T \).) From \( \tau^\varepsilon_i \) up to the next jump time of \( (N^1_t, N^2_t) \), the solution is then given by

\[
\beta^\varepsilon_t = \exp \left( \int_{\tau^\varepsilon_i}^t \sigma_u dW_u - \frac{1}{2} \int_{\tau^\varepsilon_i}^t \sigma_u^2 du - \sup \left\{ \int_{\tau^\varepsilon_i}^s \sigma_u dW_u - \frac{1}{2} \int_{\tau^\varepsilon_i}^s \sigma_u^2 du - \ln (\overline{\beta}_s) \mid s \in [\tau^\varepsilon_i, t] \right\} \right) \tag{4.1}
\]

(analogously after jump times of \( N^2_t \), and \( \Psi = \Psi_{\tau^\varepsilon_i} + \beta^\varepsilon_t - \overline{\beta}_t \tau^\varepsilon_i - \int_{\tau^\varepsilon_i}^t \beta^\varepsilon_u du dW_u \).

Fix any \( \overline{\varepsilon} > 0 \). Note that, after a limit sell order execution, \( \beta^\varepsilon_t \) jumps to \( \beta_t < 0 \) and then cannot enter the region \([0, (1 - \overline{\varepsilon})\overline{\beta}_t]\) before the next limit buy order execution. As a result, we can estimate the excursions away from the upper trading boundary \( \overline{\beta}_t \) as follows:

\[
P \left( \exists t \in (\tau^\varepsilon_i, T] \text{ s.t. } \frac{\beta^\varepsilon_t}{\beta_t} \in [0, 1 - \overline{\varepsilon}) \right) \leq P \left( M_{1, \varepsilon} \cup \bigcup_{i=1}^{\left\lfloor 2T \lambda_{\text{max}}^{1} \varepsilon^{-\theta} \right\rfloor} M_{2, \varepsilon, i} \cup \bigcup_{i=1}^{\left\lfloor 2T \lambda_{\text{max}}^{1} \varepsilon^{-\theta} \right\rfloor} M_{3, \varepsilon, i} \right), \tag{4.2}
\]

where

\[
M_{1, \varepsilon} := \{ \omega \in \Omega \mid N^1_t(\omega) > \left\lfloor 2T \lambda_{\text{max}}^{1} \varepsilon^{-\theta} \right\rfloor \},
\]

\[
M_{2, \varepsilon, i} := \{ \omega \in \Omega \mid \tau^\varepsilon_{i+1}(\omega) - \tau^\varepsilon_i(\omega) > \varepsilon^{\theta/2} \},
\]

\[
M_{3, \varepsilon, i} := \{ \omega \in \Omega \mid \frac{\beta^\varepsilon_t(\omega)}{\beta_t(\omega)} \in [0, 1 - \overline{\varepsilon}) \text{ for some } t \in (\tau^\varepsilon_i(\omega), \tau^\varepsilon_{i+1}(\omega) \wedge (\tau^\varepsilon_i(\omega) + \varepsilon^{\theta/2})] \},
\]

with \( \lambda_{\text{max}}^{1} \varepsilon^{-\theta} \) being an upper bound for the jump intensity of the counting process \( N^1_t \). In plain English, there are either many jumps to the upper boundary \( \overline{\beta}_t \), and/or there is a long-time
excursion away from $\overline{\beta}_t$, and/or there is a short excursion that nevertheless takes the risky position $\beta^c_t$ sufficiently far way from the boundary $\overline{\beta}_t$. In the sequel, we show that the probability for these events is smaller than $\bar{\varepsilon}$ for sufficiently small spreads $\varepsilon$. Observe that after the first jump of $(N^1_t, N^2_t)$ we have $0 < \beta^c_t \leq \overline{\beta}_t$ on $A^1_t$ and $\beta^c_t \leq \beta^c_t < 0$ on $A^2_t$. As $\overline{\beta}_t = \frac{2\lambda^c_t \varepsilon^{1-\theta}}{\sigma^c_t}$ and the process $\frac{2\lambda^c_t \varepsilon^{1-\theta}}{\sigma^c_t}$ is bounded, the estimate for (4.2) in turn yields $|\beta^c_t - \overline{\beta}_t|\varepsilon^{\theta-1}A^1_t \to 0$ uniformly in probability. By applying the same arguments to $\beta^c_t$ on $A^2_t$ we obtain the assertion.

Let us now derive the required estimates for (4.2). First, recall that the time-changed process $u \mapsto N^1_u$ with $\Gamma_u := \inf\{v \geq 0 \mid \int_0^v \lambda^c_t \varepsilon^{-\theta} ds = u\}$ is a standard Poisson process (cf. [3, Theorem 16]) and $\Gamma^1_{\max} \varepsilon^{-\theta} T \geq T$. As a result:

$$P(M^1_{i,\varepsilon}) \leq P(N^1_{\Gamma^1_{\max} \varepsilon^{-\theta} T} > [2T\lambda^1_{\max} \varepsilon^{-\theta}]) \leq \frac{\bar{\varepsilon}}{3}, \quad \text{for } \varepsilon \text{ small enough,}$$

by the law of large numbers.

Next, since the jump intensity of $N^1_t$ is bounded from below by some $\lambda_{\min} \varepsilon^{-\theta} > 0$, we obtain

$$P\left(\frac{\tau^\varepsilon_{i+1} - \tau^\varepsilon_i}{\lambda^1_{\min} \varepsilon^{-\theta}} > \frac{x}{\lambda^1_{\min} \varepsilon^{-\theta}}\right) \leq \exp(-x), \quad x \in \mathbb{R}_+, \ i \in \mathbb{N}.$$

Choosing $x = 2 \ln([2T\lambda^1_{\max} \varepsilon^{-\theta}])$, this estimate yields

$$P\left(\frac{\tau^\varepsilon_{i+1} - \tau^\varepsilon_i}{\lambda^1_{\min} \varepsilon^{-\theta}} > \frac{e^{\theta} 2 \ln([2T\lambda^1_{\max} \varepsilon^{-\theta}])}{\lambda^1_{\min} \varepsilon^{-\theta}}\right) \text{ for some } i = 1, \ldots, [2T\lambda^1_{\max} \varepsilon^{-\theta}] \leq \frac{1}{[2T\lambda^1_{\max} \varepsilon^{-\theta}]}.$$

As $e^{\theta} 2 \ln([2T\lambda^1_{\max} \varepsilon^{-\theta}]) \leq \varepsilon^{\theta/2}$ for $\varepsilon$ small enough, this in turn gives

$$P\left(\bigcup_{i=1}^{[2T\lambda^1_{\max} \varepsilon^{-\theta}]} M_{2,\varepsilon, i}\right) \leq \frac{\bar{\varepsilon}}{3}, \quad \text{for } \varepsilon \text{ small enough.}$$

Finally, by (4.1) and Assumption 2.1 we have that

$$\frac{\beta^c_t}{\overline{\beta}_t} = \exp\left(M_t^{i,\varepsilon} + A_t^{i,\varepsilon} - \sup \left\{M_s^{i,\varepsilon} + A_s^{i,\varepsilon} \mid s \in [\tau^\varepsilon_t, t]\right\}\right), \quad i \in \mathbb{N},$$

for some continuous martingale $M_t^{i,\varepsilon}$ with respect to the filtration $(\mathcal{F}^{\tau^\varepsilon_t + u})_{u \geq 0}$ and a continuous finite variation process $A_t^{i,\varepsilon}$. In addition, both $\langle M_t^{i,\varepsilon}, M_t^{i,\varepsilon} \rangle$ and $A_t^{i,\varepsilon}$ have a bounded drift rate with respect to $\text{Id}_u(\omega) = u$, uniformly in $i$ and $\varepsilon$. Let $K \in \mathbb{R}_+$ be such a bound and define the stopping times $\Gamma_u := \inf\{v \geq 0 \mid \langle M_t^{i,\varepsilon}, M_t^{i,\varepsilon} \rangle_v = u\}$. We have $\Gamma_{K^{\varepsilon^{\theta/2}}} \geq e^{\theta/2}$ and the process $u \mapsto \tilde{W}_u := \Gamma_u^{i,\varepsilon}$ is a standard Brownian motion with respect to $(\mathcal{F}^{\tau^\varepsilon_t + \Gamma_u})_{u \geq 0}$ by the Dambis-Dubins-Schwarz theorem. Together with $e^{\theta/2} \leq \frac{1}{K} \ln\left(\frac{1-e^{\theta/2}}{1-\varepsilon}\right)$ for $\varepsilon$ small enough and $A_t^{i,\varepsilon} - \sup \left\{A_s^{i,\varepsilon} \mid s \in [0, t]\right\} \geq -K^{e^{\theta/2}}$ for all $t \in [0, e^{\theta/2}]$, we arrive at

$$P\left(\bigcup_{i=1}^{[2T\lambda^1_{\max} \varepsilon^{-\theta}]} M_{3,\varepsilon, i}\right) \leq [2T\lambda^1_{\max} \varepsilon^{-\theta}] P(\exists s, t \in [0, K^{e^{\theta/2}}] s.t. \exp(\tilde{W}_t - \tilde{W}_s - K^{e^{\theta/2}}) < 1 - \varepsilon)$$

$$\leq [2T\lambda^1_{\max} \varepsilon^{-\theta}] P(\exists s, t \in [0, K^{e^{\theta/2}}] s.t. \exp(\tilde{W}_t - \tilde{W}_s) < 1 - \varepsilon)$$

$$\leq \frac{32T\lambda^1_{\max} \sqrt{K^{e^{\theta/2}}}}{\sqrt{2\pi \varepsilon}} \exp\left(\frac{-\varepsilon^2}{32K^{e^{\theta/2}}}\right)$$

$$\leq \frac{\bar{\varepsilon}}{3}, \quad \text{for } \varepsilon \text{ small enough.}$$

(4.5)
Here, the second to last inequality is the consequence of the following estimate for excursions of a standard Brownian motion $W_t$: for any $\bar{\varepsilon} > 0$ and $t_0 > 0$ we have

$$P \left( \exists s, t \in [0, t_0] \text{ s.t. } \exp(W_t - W_s) < 1 - \frac{\bar{\varepsilon}}{2} \right) \leq P \left( \max_{0 \leq t \leq t_0} |W_t| > -\frac{1}{2} \ln \left( 1 - \frac{\bar{\varepsilon}}{2} \right) \right) \leq 4P \left( W_{t_0} > \frac{1}{2} \ln \left( 1 - \frac{\bar{\varepsilon}}{2} \right) \right) \leq 4P \left( W_1 > \frac{\bar{\varepsilon}}{4\sqrt{t_0}} \right) \leq 16 \frac{\sqrt{t_0}}{\sqrt{2\pi\bar{\varepsilon}}} \exp \left( \frac{-\bar{\varepsilon}^2}{32t_0} \right).$$

Piecing together (4.3), (4.4) and (4.5), it follows that the probability in (4.2) is indeed bounded by $\bar{\varepsilon}$ for sufficiently small $\varepsilon$. This completes the proof. \hfill \Box

### 4.2 An Auxiliary Frictionless Shadow Market

Similarly as for markets with proportional transaction costs \cite{18} and for limit order markets \cite{21}, we reduce the original optimization problem to a frictionless version, by replacing the mid-price $S_t$ with a suitable “shadow price” $\tilde{S}_t$. The latter is potentially more favorable for trading but nevertheless leads to an equivalent optimal strategy and utility. The key difference to \cite{21} is that we focus on asymptotic results for small spreads here. Hence, it suffices to determine “approximate” shadow prices, which are at least as favorable for all spreads, but for which the corresponding optimizers in the limit order market are “almost” optimal only for small spreads. This simplifies the construction significantly, and thereby allows to treat the general framework considered here.

Indeed, the approximation result established in Lemma \ref{lem:shadow} suggests that it suffices to look for a frictionless shadow market where the optimal policy oscillates between the upper and lower boundaries $\beta^1_t, \beta^2_t$ at the jump times of the counting processes $N^1_t, N^2_t$. To this end, it turns out that one can simply let the shadow price jump to the bid resp. ask price whenever a limit buy resp. sell order is executed, and then let it evolve as the bid resp. ask price until the next jump time. To make this precise, let $\tilde{S}_0 = (1 + \varepsilon_0)S_0$ and define

$$\frac{d\tilde{S}_t}{\tilde{S}_{t-}} = \sigma_t dW_t + A^1_t \left( \frac{1 + \varepsilon_t}{1 - \varepsilon_t} - 1 \right) dN^2_t - \frac{\sigma_t}{1 - \varepsilon_t} d(W, \varepsilon)_t + \frac{\sigma_t}{1 + \varepsilon_t} d(W, \varepsilon)_t,$$

$$=: d\tilde{R}_t,$$

with $A^1$ and $A^2$ from (3.3), where the terms $\frac{\sigma_t}{1 - \varepsilon_t} d(W, \varepsilon)_t$ and $\frac{\sigma_t}{1 + \varepsilon_t} d(W, \varepsilon)_t$ ensure that $\tilde{S}_t = (1 - \varepsilon_t)S_t$ on $A^1_{t+}$ and $\tilde{S}_t = (1 + \varepsilon_t)S_t$ on $A^2_{t+}$ even for time-varying $\varepsilon_t$. (The correction terms can easily be derived by applying the integration by parts formula.) Then:

$$(1 - \varepsilon_t)S_t \leq \tilde{S}_t \leq (1 + \varepsilon_t)S_t, \quad \tilde{S}_t = (1 - \varepsilon_t)S_t \text{ on } \Delta N^1_t > 0, \quad \tilde{S}_t = (1 + \varepsilon_t)S_t \text{ on } \Delta N^2_t > 0. \quad (4.7)$$

That is, the frictionless price process $\tilde{S}_t$ evolves in the bid-ask spread, and therefore always leads to at least as favorable trading prices for market orders. When more favorable trading prices are
available due to the execution of limit orders, \( \tilde{S}_t \) jumps to match these. Hence, trading \( \tilde{S}_t \) is at least as profitable as the original limit order market.

The key step now is to determine the optimal policy for \( \tilde{S}_t \). In the corresponding frictionless market, portfolios can be equivalently parametrized directly in terms of monetary positions \( \tilde{\eta}_t = \varphi_t \tilde{S}_t \) held in the risky asset, with associated wealth process

\[
\tilde{X}_t^\eta = x_0 + \int_0^t \tilde{\eta}_s d\tilde{R}_s.
\]

As before, a family of policies \((\tilde{\eta}_t^\varepsilon)_{\varepsilon>0}\) is called admissible if it is uniformly bounded and converges to zero pointwise for \( \varepsilon \to 0 \). Let us compute the expected utility obtained by applying an arbitrary admissible policy \( \tilde{\eta}_t^\varepsilon \). By Assumption \[2.3\] the integrals with respect to \( \varepsilon \)- and \( \langle W, \varepsilon \rangle_t \) in \[4.6\] are dominated for \( \varepsilon \to 0 \). Indeed, the continuous martingale parts are dominated by \( \sigma_t dW_t \) and the drift terms are dominated by the drifts of the integrals with respect to the counting processes \( N_t^j \), which are of order \( 2e^{1-\vartheta} \varepsilon \lambda_i^j \). Hence, these terms can be safely neglected in the sequel.

Itô’s formula as in \[16\] Theorem I.4.57 and \[16\] Theorem II.1.8 yield:

\[
U(\tilde{X}_T^\eta) = U(x_0) + \int_0^T U'(\tilde{X}_t^\eta) \tilde{\eta}_t dW_t + \frac{1}{2} \int_0^T U''(\tilde{X}_t^\eta)(\tilde{\eta}_t^\varepsilon)^2 dt + \left( U(\tilde{X}_t^\eta) + \eta_s \sigma \right) \ast (\mu R - \nu R) + \left( U(\tilde{X}_t^\eta) - U(\tilde{X}_t^\eta) \right) \ast \nu R.
\]

The integrals with respect to the Brownian motion \( W_t \) and the compensated random measure \( \mu R - \nu R \) are true martingales. To see this, first consider the Brownian integral. By \[2.4\] we have

\[
U'(\tilde{X}_t^\eta) \leq C \exp(-\alpha \tilde{X}_t^\eta) 1_{\{\tilde{X}_t^\eta < 0\}} + U'(0) 1_{\{\tilde{X}_t^\eta \geq 0\}}
\]

for some constants \( C, \alpha > 0 \). Therefore and due to the boundedness of \( \tilde{\eta}_t^\varepsilon \), it suffices to show that

\[
E \left[ \int_0^T \exp(-2\alpha \tilde{X}_t^\eta) dt \right] < \infty.
\]

By the Doleans-Dade exponential formula \[16\] Theorem I.4.61 and \[16\] Theorem II.1.8, we have

\[
\exp(-2\alpha \tilde{X}_t^\eta) = \exp(-2\alpha x_0) \exp\left(-2\alpha \int_0^t \tilde{\eta}_s^\varepsilon dW_s + (\exp(-2\alpha \tilde{\eta}_s^\varepsilon x) - 1) \ast (\mu R - \nu R) \right)_t
\]

\[
\times \exp\left( \int_0^t (2\alpha^2 (\tilde{\eta}_s^\varepsilon)^2 \sigma_s^2) ds + (\exp(-2\alpha \tilde{\eta}_s^\varepsilon x) - 1) \ast \nu R \right).
\]

For all \( t \in [0, T] \), the ordinary exponential in the above representation is uniformly bounded by a single constant. This is because \( \sigma_t^2 \) as well as \( (\tilde{\eta}_t^\varepsilon)^2 \) are both uniformly bounded and, since the intensities \( \varepsilon^{-\vartheta} \lambda_i^1, \varepsilon^{-\vartheta} \lambda_i^2 \) are bounded for any \( \varepsilon > 0 \), the same holds for the jump part (for sufficiently small \( \varepsilon \)):

\[
(\exp(-2\alpha \tilde{\eta}_s^\varepsilon x) - 1) \ast \nu R_t = \int_0^t (\exp(-2\alpha \tilde{\eta}_s^\varepsilon x/(1 - \varepsilon_s)) - 1) 1_{A^1} \varepsilon^{-\vartheta} \lambda_i^2 ds + \int_0^t (\exp(2\alpha \tilde{\eta}_s^\varepsilon x/(1 + \varepsilon_s)) - 1) 1_{A^2} \varepsilon^{-\vartheta} \lambda_i^1 ds.
\]

\[4.9\] now follows since the stochastic exponential in \[4.10\] is not only a local martingale, but also a supermartingale with decreasing expectation because it is positive for sufficiently small \( \varepsilon \).
The argument for the integral with respect to the compensated random measure $\mu \tilde{R} - \nu \tilde{R}$ in (4.8) is similar. By the mean value theorem, (2.4), and [16, Theorem II.1.33] it suffices to show

$$E \left[ \exp(-2\alpha \tilde{X}_T^\eta) \right] < \infty.$$ 

But this follows verbatim as for the Brownian integral above, again using that the jumps of $\tilde{R}_t$ as well as the corresponding jump intensities are all uniformly bounded. In summary, (4.8) therefore gives\(^\text{11}\)

$$E[U(\tilde{X}_T^\eta)] - U(x_0)
\leq E \left[ \int_0^T \left( \frac{1}{2} U''(\tilde{X}^\eta_t)(\tilde{\eta}_t)^2 \sigma_t^2 + U'(\tilde{X}^\eta_t)\tilde{\eta}_t \frac{2\epsilon_t}{1-\epsilon_t} e^{-\theta \lambda_t^1} \right) 1_{A_t^1} + \frac{1}{2} U''(\tilde{X}^\eta_t)(\tilde{\eta}_t)^2 \sigma_t^2 - U'(\tilde{X}^\eta_t)\tilde{\eta}_t \frac{2\epsilon_t}{1+\epsilon_t} e^{-\theta \lambda_t^1} \right) 1_{A^2_t} \right] \right],$$

(4.11)

where for the inequality we have used the concavity of $U$ and inserted the definition of $\tilde{R}_t$. By (2.4), (4.9) also shows that the random variables in (4.11) are integrable. Moreover, for any admissible family of policies $\tilde{\eta}_t$,

$$\int_0^TU''(\tilde{X}^\eta_t) dt \quad \text{and} \quad \int_0^TU''(\tilde{X}^\eta_t) dt \quad \text{are uniformly integrable for } \epsilon \in (0, \epsilon_0),$$

(4.12)

where $\epsilon_0 > 0$ is a sufficiently small constant. Indeed, it follows from the proof of (4.9) that the bound therein holds uniformly in $\epsilon \in (0, \epsilon_0)$. Then, using Jensen’s inequality, we observe that $(\int_0^TU''(\tilde{X}^\eta_t) dt)^2$, $(\int_0^TU''(\tilde{X}^\eta_t) dt)^2$ are uniformly bounded in expectation, which in turn yields (4.12).

For fixed wealth $\tilde{X}^\eta_t$, the integrand in the upper bound of (4.11) is a quadratic function in the policy $\tilde{\eta}_t$. Plugging in the pointwise maximizer $\frac{2\epsilon^{1-\theta} \epsilon_t \lambda_t^2}{\text{ARA}(\tilde{X}^\eta_t)^2 (1-\epsilon_t)} 1_{A^1_t} - \frac{2\epsilon^{1-\theta} \epsilon_t \lambda_t^1}{\text{ARA}(\tilde{X}^\eta_t)^2 (1+\epsilon_t)} 1_{A^2_t}$, which is of order $O(\epsilon^{1-\theta})$ (uniformly in $\omega, t$) by (2.3), therefore yields the following upper bound\(^\text{12}\)

$$E[U(\tilde{X}_T^\eta)] - U(x_0)
\leq \int_0^T E \left[ \right]$$

$$+ o(\epsilon^{2(1-\theta)}).$$

Here, we used $2\epsilon_t/(1+\epsilon_t) = 2\epsilon_t + O(\epsilon^2)$ and that, by (4.12), the remainder is uniformly bounded in expectation.

For the admissible family of feedback policies

$$\tilde{\eta}_t^\epsilon = \frac{2\epsilon^{1-\theta} \epsilon_t \lambda_t^2}{\text{ARA}(\tilde{X}^\eta_t)^2} 1_{A^1_t} - \frac{2\epsilon^{1-\theta} \epsilon_t \lambda_t^1}{\text{ARA}(\tilde{X}^\eta_t)^2} 1_{A^2_t},$$

(4.13)

\(^\text{11}\)If the integrals with respect to $\epsilon_t$ and $\langle W, \epsilon_t \rangle_t$ are taken into account explicitly, these only lead to an additional higher-order term that can be bounded by a constant times $\epsilon_t^\eta$ for small $\epsilon$.

\(^\text{12}\)If the integrals with respect to $\epsilon_t$ and $\langle W, \epsilon_t \rangle_t$ are taken into account explicitly, this does not change the pointwise optimizer and the corresponding upper bound at the leading order.
this inequality becomes an equality at the leading order $\epsilon^{2(1-\theta)}$, namely:

$$E[U(\tilde{X}_T^{\tilde{\eta}^*})] - U(x_0)$$

$$= E\left[\int_0^T \left( \frac{1}{2} U''(\tilde{X}_{t^-}^{\tilde{\eta}^*})(\tilde{\eta}_{t^-}^*)^2 \sigma_t^2 + U'(\tilde{X}_{t^-}^{\tilde{\eta}^*}) \tilde{\eta}_{t^-}^* \frac{2\epsilon_t}{1 - \epsilon_t} \epsilon^{-\theta} \lambda_t^2 \right) dt \right] + o(\epsilon^{2(1-\theta)})$$

$$= \int_0^T E \left[ \left( \frac{U'(\tilde{X}_{t^-}^{\tilde{\eta}^*})^2 2\epsilon^{(1-\theta)} \sigma_t^2 (\lambda_t^2)^2}{1} \right) \right] + o(\epsilon^{2(1-\theta)})$$

$$= \int_0^T E \left[ \frac{U'(\tilde{X}_{t^-}^{\tilde{\eta}^*})^2}{1} + o(\epsilon^{2(1-\theta)}) \right]$$

Here, the first equality follows from the mean value theorem because the differential remainder is bounded by $C|U'(\tilde{X}_{t^-}^{\tilde{\eta}^*} + \xi) - U'(\tilde{X}_{t^-}^{\tilde{\eta}^*})|\epsilon^{2(1-\eta)}$ for some constant $C > 0$, not depending on $\epsilon$ as $\tilde{\eta}^*/\epsilon^{1-\theta}$ is bounded for $\epsilon \to 0$ by (2.3), and some bounded random variable $\xi$ which tends to 0 pointwise for $\epsilon \to 0$. With (4.12) it follows that the term is uniformly integrable, so that the remainder is indeed of order $o(\epsilon^{2(1-\eta)})$.

As a result:

$$E[U(\tilde{X}_T^{\tilde{\eta}^*})] - E[U(\tilde{X}_T^{\tilde{\eta}^*})] \leq \epsilon^{2(1-\theta)} M \int_0^T E \left[ \left( \frac{U'(\tilde{X}_{t^-}^{\tilde{\eta}^*})^2}{U''(\tilde{X}_{t^-}^{\tilde{\eta}^*})} - U'(\tilde{X}_{t^-}^{\tilde{\eta}^*})^2 \right) \right] dt + o(\epsilon^{2(1-\theta)}),$$

where the constant $M$ is a uniform bound for $2\epsilon^{(1-\theta)} (\lambda_t^2)^2/\sigma_t^2$ and $2\epsilon^{(1-\theta)} (\lambda_t^2)^2/\sigma_t^2$.

Now, since any family $\tilde{\eta}_t^*$ of admissible policies is uniformly bounded and converges to zero pointwise, the dominated convergence theorem for stochastic integrals [26, Theorem IV.32] shows that $\tilde{X}_T^{\tilde{\eta}_t^*} \to x_0$ and in turn $U'(\tilde{X}_T^{\tilde{\eta}_t^*})^2/U''(\tilde{X}_T^{\tilde{\eta}_t^*}) \to U'(x_0)^2/U''(x_0)$ in probability as $\epsilon \to 0$. As above, by (4.12) we have uniform integrability, so that this convergence in fact holds in $L^1$. Hence, (4.15) and the dominated convergence theorem for Lebesgue integrals yield

$$E[U(\tilde{X}_T^{\tilde{\eta}^*})] \leq E[U(\tilde{X}_T^{\tilde{\eta}^*})] + o(\epsilon^{2(1-\theta)}),$$

that is, the family $(\tilde{\eta}_t^*)_{\epsilon>0}$ is approximately optimal at the leading order $\epsilon^{2(1-\theta)}$.

Together with (4.14), the same argument also yields that the corresponding leading-order optimal utility is given by

$$E[U(\tilde{X}_T^{\tilde{\eta}^*})] = U(x_0) - \frac{U'(x_0)^2}{2U''(x_0)} E \left[ \int_0^T (\tilde{\eta}_{t^-}^*)^2 d\langle \tilde{R} \rangle_t \right] + o(\epsilon^{2(1-\theta)})$$

$$= U \left( x_0 + \frac{\epsilon^{2(1-\theta)}}{A_R(x_0)} E \left[ \int_0^T \left( \frac{2\epsilon^2 (\lambda_t^2)^2}{\sigma_t^2} + \frac{2\epsilon^2 (\lambda_t^2)^2}{\sigma_t^2} \right) \right] \right) + o(\epsilon^{2(1-\theta)})$$

where the second equality follows from Taylor’s theorem and the definition of $\tilde{\eta}^*_t$.

If all the model parameters $\lambda_t, \sigma_t, \epsilon_t$ are constant, the integrals in this formula can be computed explicitly. Indeed, since $P[A_t] = 1 - P[A_t^c] = \lambda^1/(\lambda^1 + \lambda^2)$, it then follows that

$$E[U(\tilde{X}_T^{\tilde{\eta}^*})] = U \left( x_0 + \frac{2\lambda^1 \lambda^2}{A_R(x_0)\sigma^2} \epsilon^{2(1-\theta)} T \right) + o(\epsilon^{2(1-\theta)}).$$
4.3 Proof of the Main Result

We now complete the proof of our main result. To this end, we use that the policy \( \beta^\varepsilon_t \) proposed in Section 3 is uniformly close to the almost optimal policy \( \eta^\varepsilon \) in the shadow market with price process \( \widetilde S_t \) by Lemma 4.1. Since trading in the frictionless shadow market is at least as favorable as in the original limit order market, and the policy \( \beta^\varepsilon_t \) trades at the same prices in both markets, this in turn yields the leading-order optimality of \( \beta^\varepsilon_t \).

**Proof of Theorem 3.1.** Let

\[
\eta^\varepsilon_t := \beta^\varepsilon_t \left( (1 - \varepsilon_t)1_{\{\beta^\varepsilon > 0\}} + (1 + \varepsilon_t)1_{\{\beta^\varepsilon < 0\}} \right),
\]

where \( \beta^\varepsilon_t \) is the solution of (3.1), which is of order \( O(\varepsilon^{1-\theta}) \) uniformly in \( \omega, t \). Note that \( \eta^\varepsilon_t \) is the risky position of the policy \( \beta^\varepsilon_t \) if the risky asset is valued at the shadow price \( \widetilde S_t \) instead of the mid price \( S_t \).

**Step 1:** We want to compare the \( \widetilde S_t \)-wealth of \( \eta^\varepsilon_t \) with the wealth of the approximate optimizer \( \widetilde \eta^\varepsilon_t \) in the \( \widetilde S_t \)-market defined in (4.13).

By (4.12), in the expansion (4.11), one can replace \( U'(\widetilde X_t^\varepsilon) \) and \( U''(\widetilde X_t^\varepsilon) \) by \( U'(x_0) \) and \( U''(x_0) \), respectively, leading to a remainder of order \( o(\varepsilon^{2(1-\theta)}) \), for any admissible family of policies \( \widetilde \eta^\varepsilon_t \) with the property that \( \widetilde \eta^\varepsilon_t / \varepsilon^{1-\theta} \) is uniformly bounded. Applied to \( \eta^\varepsilon_t \) and \( \widetilde \eta^\varepsilon_t \), this yields

\[
E[U(\widetilde X_T^\varepsilon)] - E[U(X_T^\varepsilon)] = E \left[ \int_0^T \left( U''(x_0)(\eta^\varepsilon_t)^2 \sigma_t^2 + U'(x_0)\eta^\varepsilon_t \frac{2\varepsilon_t}{1-\varepsilon_t} \varepsilon^{-\theta} \lambda_t^2 \right) \, dt \right] + O(\varepsilon^{2(1-\theta)})
\]

\[
\quad + E \left[ \int_0^T \frac{1}{2} U''(x_0) \sigma_t^2 \left( (\eta^\varepsilon_t - \beta 1_A_1^t) - \beta 1_A_2^t \right)^2 + (\widetilde \eta^\varepsilon_t - \beta 1_A_1^t - \beta 1_A_2^t)^2 - (\eta^\varepsilon_t - \beta 1_A_1^t - \beta 1_A_2^t)^2 \right] \, dt \right]
\]

(4.16)

By Lemma 4.1 and since \( \eta^\varepsilon_t - \beta^\varepsilon_t = O(\varepsilon^{2-\theta}) \), we have \( (\eta^\varepsilon_t - \beta 1_A_1^t - \beta 1_A_2^t) / \varepsilon^{1-\theta} \rightarrow 0 \) after the first jump of \( (N_1^t, N_2^t) \), uniformly in probability. The same holds for \( \widetilde \eta^\varepsilon_t \). As the expectation of the first jump time of \( (N_1^t, N_2^t) \) is of order \( O(\varepsilon^{\theta}) \) (and the integrands in the last line of (4.16) are uniformly of order \( O(\varepsilon^{2(1-\theta)}) \)), this gives

\[
E \left[ \int_0^T \frac{1}{2} U''(x_0) \sigma_t^2 \left( (\eta^\varepsilon_t - \beta 1_A_1^t - \beta 1_A_2^t)^2 + (\widetilde \eta^\varepsilon_t - \beta 1_A_1^t - \beta 1_A_2^t)^2 \right) \, dt \right] = o(\varepsilon^{2(1-\theta)}) + O(\varepsilon^{2-\theta})
\]

\[
= o(\varepsilon^{2(1-\theta)}).
\]

**Step 2:** Let \( (\psi_0^0, \psi_1^0)_{t \in [0,T]} \) be the portfolio process of an arbitrary self-financing strategy in the limit order market with \( (\psi_0^0, \psi_0) = (x_0, 0) \). By (4.7) and Step 1 in the proof of [21, Proposition 1], we have

\[
\psi^0_t + \psi_1 1_{\{\psi_r \geq 0\}}(1 - \varepsilon_t) S_t + \psi_1 1_{\{\psi_r < 0\}}(1 + \varepsilon_t) S_t \leq x_0 + \int_0^t \psi_s d\tilde S_s.
\]

(4.17)
Now take the strategy (3.2). For the corresponding portfolio process \((\varphi^0_\iota, \varphi_\iota)_{\iota \in [0,T)}\) we have 
\[ \eta^\kappa_t = \varphi_t S_{t-} \] and – by construction of the strategy and \(S_t - \phi^\kappa_t\) holds with equality for \((\varphi^0_t, \varphi_t) = (\varphi^0_t, \varphi_t)\). Together with (4.16) and the approximate optimality of \(\eta^\kappa_t\) in the \(S_t\)-market, this yields the assertion.

\section{Price Impact of Exogenous Orders}

In our model, bid and ask prices remain unaffected by the execution of incoming orders. This is the most optimistic scenario for liquidity providers because – modulo inventory risk – it allows them to earn the full spread between alternating buy and sell trades. Disregarding price impact is reasonable for small noise traders whose orders do not carry any information. For strategic and possibly informed counterparties, however, it is questionable. For these, prices are expected to rise after purchases and fall after sales, respectively (compare, e.g., [10, 22]). Similarly, larger orders of other market participants also move market prices in the same directions by depleting the order book.

Our basic model can be extended to incorporate the price impact of incoming orders in reduced form\footnote{This is similar in spirit to the Almgren-Chriss model \cite{Ac} from the optimal execution literature, in that we also do not attempt to specify the dynamics of the whole order book, but instead directly model the price moves caused by executions.} Indeed, suppose that the mid price follows 
\[ dS_t = \rightarrow S_t = \kappa \varepsilon_t dN^1_t + \kappa \varepsilon_t dN^2_t, \]
for some price impact parameter \(\kappa \in [0, 1]\). With the information \(\mathcal{F}_{t-}\) our small investor is allowed to place limit buy orders at the bid price \((1 - \varepsilon_t)S_{t-}\) and limit sell orders at the ask price \((1 + \varepsilon_t)S_{t-}\) immediately before the jump of \(S_t\). However, bid and ask prices jump down after exogenous sell orders are executed at the jump times of \(N^1_t\), and jump up after exogenous buy orders arrive at the jump times of \(N^2_t\). (Note that this happens irrespective of the liquidity our small investor chooses to provide.) The parameter \(\kappa\) represents the fraction of the half-spread \(\varepsilon_t\) by which prices are moved\footnote{Note that, as in [9], price impact is permanent here. Tracking an exogenous benchmark in a limit order market with transient price impact as in [25] is studied by [15].} \(\kappa = 0\) corresponds to the model without price impact studied above. Conversely, \(\kappa \approx 1\) leads to a model in the spirit of Madhavan et al. [22], where liquidity providers do not earn the spread, but only a small exogenous compensation for their services\footnote{Indeed, after a successful execution of a limit order the mid price jumps close to the limit price of the order if \(\kappa \approx 1\). This means that the liquidity provider actually trades at similar prices as in a frictionless market with price process \(S_t\). If moreover \(\alpha^1_t = \alpha^2_t\), the mid price is still a martingale and expected profits vanish.}.

In this extension of our model, the optimal policy is similar to the one in the baseline version without price impact. Indeed, one still trades to some position limits \(\overline{\beta}_t, \underline{\beta}_t\) whenever limit orders are executed. However, since the adverse effect of price impact diminishes the incentive to provide liquidity, \(\overline{\beta}_t, \underline{\beta}_t\) are reduced accordingly. If executions move bid and ask prices by a fraction \(\kappa\) of the current half-spread \(\varepsilon_t\), then 
\[ \overline{\beta}_t = \frac{2\varepsilon_t(1 - \frac{\kappa}{2}) \alpha^1_t}{2 \rightarrow (x_0) \sigma^2_t}, \quad \underline{\beta}_t = -\frac{2\varepsilon_t(1 - \frac{\kappa}{2}) \alpha^1_t}{2 \rightarrow (x_0) \sigma^2_t}, \]
given that \((1 - \frac{\kappa}{2}) \alpha^1_t - \frac{\kappa}{2} \alpha^2_t\) and \((1 - \frac{\kappa}{2}) \alpha^1_t - \frac{\kappa}{2} \alpha^2_t\) are positive. In the symmetric case \(\alpha^1_t = \alpha^2_t\), i.e., if buy and sell orders arrive at the same rates, this holds if and only of \(\kappa < 1\). In this case, 
\[ \overline{\beta}_t = \frac{2\varepsilon_t(1 - \kappa) \alpha_t}{2 \rightarrow (x_0) \sigma^2_t}, \quad \underline{\beta}_t = -\frac{2\varepsilon_t(1 - \kappa) \alpha_t}{2 \rightarrow (x_0) \sigma^2_t}, \]
so that price impact equal to a fraction \( \kappa \) of the current half-spread \( \varepsilon_t \) simply reduces liquidity provision by a factor of \( 1 - \kappa \). In particular, if \( \kappa \approx 1 \), then the boundaries can be of order \( o(\varepsilon^{1-\theta}) \). As a result, arrival rates of a higher order than \( \varepsilon^{-\theta}, \theta \in (0, 1) \) can be used without implying nontrivial profits as the spread collapses to zero. In any case, the formula (1.2) for the corresponding leading-order certainty equivalent remains the same after replacing the trading boundaries accordingly.

In addition to reducing the target positions for limit order trades, price impact also alters the rebalancing strategy between these. Recall that price impact increases bid-ask prices after the liquidation by market orders and stops trading altogether if the risky position exits before purchases. Hence, immediately rebalancing strategy between these. Recall that price impact increases bid-ask prices after the liquidation by market orders and stops trading altogether if the risky position exits before purchases. Hence, immediately starting to trade by market orders to keep the inventory in \([\beta_t, \beta_t]\) and assume that the quotient \( \sigma_t \) construct a frictionless shadow price process \( \tilde{S}_t \) for which the optimal strategy trades at the same prices as in the limit order market. Define

\[
\tilde{S}_t = (1 - \varepsilon_t)S_{t-} = \frac{1 - \varepsilon_t}{1 - \kappa \varepsilon_t} S_t \quad \text{if} \quad \Delta N_t^1 > 0, \quad \text{and} \quad \tilde{S}_t = (1 + \varepsilon_t)S_{t-} = \frac{1 + \varepsilon_t}{1 + \kappa \varepsilon_t} S_t \quad \text{if} \quad \Delta N_t^2 > 0,
\]

and assume that the quotient \( \tilde{S}_t / S_t \) is piecewise constant between the jump time of \( N_t^1, N_t^2 \). These properties are satisfied by the solution of

\[
\frac{d\tilde{S}_t}{S_{t-}} = \sigma_t dW_t + \frac{\kappa - 1}{(1 - \varepsilon_t)(1 - \kappa \varepsilon_t)} d\varepsilon_t + \frac{(\kappa - 1)\kappa}{(1 - \varepsilon_t)(1 - \kappa \varepsilon_t)^2} d\langle \varepsilon, \varepsilon \rangle_t + \frac{(\kappa - 1)\sigma_t}{1 - \varepsilon_t(1 - \kappa \varepsilon_t)} d\langle W, \varepsilon \rangle_t
\]

\[
+ 1 \frac{dA_t^1}{S_{t-}} \left( \frac{1 - \varepsilon_t}{1 + \kappa \varepsilon_t} - 1 \right) dN_t^1 + \kappa \varepsilon_t dN_t^2
\]

\[
+ 1 \frac{dA_t^2}{S_{t-}} \left( \frac{1 - \varepsilon_t}{1 + \kappa \varepsilon_t} - 1 \right) dN_t^1 + \kappa \varepsilon_t dN_t^2
\]

with \( \tilde{S}_0 := (1 + \varepsilon_0)(1 + \kappa \varepsilon_0)^{-1} S_0 \). Here, the terms in the second and fourth line of (5.2) ensure that \( \tilde{S}_t \) coincides with \( (1 - \varepsilon_t)(1 - \kappa \varepsilon_t)^{-1} S_t \) resp. \( (1 + \varepsilon_t)(1 + \kappa \varepsilon_t)^{-1} S_t \) between jumps even for stochastic spreads \( \varepsilon_t = \varepsilon \varepsilon_t \). As without price impact, these term do not contribute at the leading order for \( \varepsilon \to 0 \).

This frictionless price process matches the execution prices of limit orders in the original limit order market, as limit orders are executed at their limit prices which are fixed before orders are executed. However, the corresponding jumps due to price impact – which occur simultaneously with executions in the limit order market – are only accounted for at the next trade in the frictionless

\[16\] Alternatively, as in [2], one could consider models where the liquidity provider can post orders deeper in the order book to mitigate the adverse price impact.

\[17\] The same modification could also have been used in the baseline model without price impact. There, however, the exact optimal strategy keeps the inventory between \( \beta_t, \beta_t \) by market orders in simple settings [21], so that we stick to a strategy of that type there.
shadow market. Hence, market orders to manage the investor’s inventory – which naturally consist of sales after limit order purchases and vice versa – can be carried out at strictly more favorable price with $\tilde{S}_t$. Hence, trading $\tilde{S}_t$ is generally strictly more favorable than the original limit order market, and equally favorable only for limit order trades.

As in Section 4.2 one verifies that a risky position that oscillates between $\beta_t$, $\tilde{\beta}_t$ at the jump times of $N^1_t$, $N^2_t$ is optimal at the leading order for $\tilde{S}_t$. Similarly as in Section 4.3 one then checks that the same utility can be obtained in the original limit order market by using the policy proposed above. Indeed, the corresponding limit order trades are executed at the same prices as for $\tilde{S}_t$. For the potential liquidating trade by market orders, there is a single additional loss of order $O(\varepsilon^{2-\eta}) = o(\varepsilon^{2(1-\theta)})$, which is negligible at the leading order $O(\varepsilon^{2(1-\theta)})$. The utility lost due to terminating trading early is of order $O(\varepsilon^{2(1-\theta)})$, because it is bounded by its counterpart for $\tilde{S}_t$, and it follows similarly as in the proof of Lemma 4.1 that the probability for a premature termination tends to zero as $\varepsilon \to 0$. As a result, the total utility loss due to early termination is therefore also not visible at the leading order $O(\varepsilon^{2(1-\theta)})$. In summary, the policy proposed above matches the optimal utility in the superior frictionless market $\tilde{S}_t$ at the leading order, and is therefore optimal at the leading order in the original limit order market as well.

References


