Multi-Asset Risk Measures

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Multi-asset risk measures \(^1\)

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Abstract

We study risk measures for financial positions in a multi-asset setting, representing the minimum amount of capital to raise and invest in eligible portfolios of traded assets in order to meet a prescribed acceptability constraint. We investigate finiteness and continuity properties of these multi-asset risk measures, highlighting the interplay between the acceptance set and the class of eligible portfolios. We develop a new approach to dual representations of convex multi-asset risk measures which relies on a characterization of the structure of closed convex acceptance sets. To avoid degenerate cases we need to ensure the existence of extensions of the underlying pricing functional which belong to the effective domain of the support function of the chosen acceptance set. We provide a characterization of when such extensions exist. Finally, we discuss applications to conical market models and set-valued risk measures, optimal risk sharing, and superhedging with shortfall risk.

Keywords: multi-asset risk measures, acceptance sets, dual representations, conical markets, optimal risk sharing, superhedging with shortfall risk

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1 Introduction

The objective of this paper is to provide a variety of new results for the class of risk measures introduced by Scandolo in [22] and Frittelli and Scandolo in [14] in the context of bounded measurable positions, and further investigated by Artzner, Delbaen, and Koch-Medina in [4] in a finite state space setting.

To understand our financial motivation consider an economy with dates \( t = 0 \) and \( t = T \), and let \( \mathcal{X} \) be a topological vector space ordered by a convex cone \( \mathcal{X}_+ \) and representing the space of capital positions of a given financial institution at time \( t = T \). The institution is deemed to be adequately capitalized if its position belongs to a pre-specified acceptance set \( \mathcal{A} \subset \mathcal{X} \), i.e. a proper set such that \( X \in \mathcal{A} \) and \( Y \geq X \) imply \( Y \in \mathcal{A} \). The institution has to deal with the following capital adequacy problem: Assuming a position is not acceptable, can it be made acceptable through appropriate management actions and, if so, at which cost? To turn this question into a mathematically tractable problem we need to specify a nonempty set \( A \) representing the “eligible” or “admissible” management actions, a “cost” function \( c : A \to \mathbb{R} \) assigning to each eligible management action its cost, and an “impact” function \( I : \mathcal{X} \times A \to \mathcal{X} \)
assigning to each position $X$ the new position $I(X, a)$ obtained after implementing $a$. The “minimal” cost of making $X$ acceptable can be then expressed as

$$
\rho(X) := \inf \{ c(a) \mid a \in A : I(X, a) \in \mathcal{A} \}.
$$

The general capital requirements defined in Scandolo [22] and studied in Frittelli and Scandolo [14] and Artzner, Delbaen, and Koch-Medina in [4] have this form. The corresponding class of eligible actions consists in a subset $\mathcal{S}$ of $\mathcal{X}$, interpreted as the set of payoffs of eligible strategies, equipped with a pricing functional $\pi : \mathcal{S} \to \mathbb{R}$. For $X \in \mathcal{X}$ and $Z \in \mathcal{S}$ the impact function has the form $I(X, Z) := X + Z$, leading to

$$
\rho_{\mathcal{S}, \mathcal{X}, \pi}(X) := \inf \{ \pi(Z) \mid Z \in \mathcal{S} : X + Z \in \mathcal{A} \}.
$$

In this paper we will refer to the map $\rho_{\mathcal{S}, \mathcal{X}, \pi} : \mathcal{X} \to \overline{\mathbb{R}}$ as a multi-asset risk measure. Clearly, this map can be seen as a generalized version of the standard risk measures introduced in Artzner, Delbaen, Eber, and Heath [3] for which the only allowed management action is to raise capital and invest it in a single pre-specified asset, typically taken to be cash.

Under the assumption that $\mathcal{S}$ is a linear space and $\pi : \mathcal{S} \to \mathbb{R}$ is a linear functional, we provide several new results on the finiteness and continuity of multi-asset risk measures. In the convex case we adopt a new approach to dual representations based on a careful analysis of the acceptance set. This new approach leads to a better insight into the structure of dual representations for risk measures. Finally, we apply our results to obtain dual representations in several important situations, including a dual representation for scalarized set-valued risk measures and a dual representation for the cost of superhedging relative to shortfall risk.

**Finiteness and continuity.** A natural first step when studying multi-asset risk measures of the form $\rho_{\mathcal{S}, \mathcal{X}, \pi}$ is to determine conditions under which they are finitely valued and continuous. This is undertaken in Section 3, where we start by discussing general acceptance sets in Proposition 3.1 and then narrow down our analysis to convex and coherent acceptance sets in Proposition 3.2 and Proposition 3.8, respectively. In this respect, a key result is the simple but far-reaching Lemma 2.6, the so-called Reduction Lemma. This result shows how to express a multi-asset risk measure as a risk measure with respect to a single asset by properly enlarging the acceptance set. The “augmented” acceptance set consists of all positions which are acceptable up to a zero-cost eligible strategy. The Reduction Lemma allows to transfer a variety of finiteness and continuity results from the single-asset to the multi-asset setting.

**Dual representations.** In Theorem 4.16 we provide a dual representation for a convex, lower semicontinuous multi-asset risk measure. In this case, if we denote by $\mathcal{X}_+^\prime$ the set of all positive, continuous linear functionals $\psi : \mathcal{X} \to \mathbb{R}$, we show that

$$
\rho_{\mathcal{S}, \mathcal{X}, \pi}(X) = \sup_{\psi \in \mathcal{E}_\pi(\mathcal{S})} \{ \sigma_{\mathcal{S}}(\psi) - \psi(X) \},
$$

where $\mathcal{E}_\pi(\mathcal{S})$ is the set of all extensions of the pricing functional in $\mathcal{X}_+^\prime$, i.e.

$$
\mathcal{E}_\pi(\mathcal{S}) := \{ \psi \in \mathcal{X}_+^\prime : \psi(Z) = \pi(Z), \forall Z \in \mathcal{S} \},
$$

3
and

$$
\sigma(\psi) := \inf_{A \in \mathcal{A}} \psi(A)
$$

is the support function of $\mathcal{A}$. In Proposition 4.20 we prove that the supremum is attained in (3) whenever $\rho_{\mathcal{A}, \mathcal{J}, \pi}$ is finite and continuous at $X$. Our approach to dual representations, which seems to be new, is based on the external characterization of closed convex sets and provides an alternative to using Fenchel-Moreau duality. We start by providing a dual representation of the underlying acceptance set in Theorem 4.4 and in Theorem 4.10. From these two results we obtain the representation formula (3) in a straightforward manner. A salient feature of this formula is that the individual roles of the acceptance set, the eligible space, and the pricing functional are made transparent in a natural way. In particular, the objective function in (3) is determined exclusively by the underlying acceptance set, highlighting its importance and simplifying the practical implementation when alternative choices for the eligible space need to be considered. This is in contrast to the usual Fenchel-Moreau representation, where the penalty function depends a priori on the triple $(\mathcal{A}, \mathcal{J}, \pi)$. Note also that a key ingredient in (3) is the set of extensions $\psi$ of the pricing functional for which the support function $\sigma_{\mathcal{A}}(\psi)$ is finitely valued. Accordingly, in Theorem 4.11 we provide equivalent conditions for the existence of such extensions. Incidentally, this result generalizes classical results on the extension of positive functionals.

Set-valued risk measures. In Section 5.1 we investigate the link between multi-asset risk measures and set-valued risk measures as defined by Hamel, Heyde and Rudloff in [16]. Set-valued risk measures are important tools when studying capital adequacy or margin requirement problems in conical market models characterized by proportional transaction costs. In Proposition 5.1 we show that the scalarization of a set-valued risk measure by means of a consistent pricing system can be expressed as a multi-asset risk measure. Hence, we can apply our general results and provide a dual representation for convex scalarized set-valued risk measures in Proposition 5.4. Finally, we focus on special acceptance sets satisfying the market compatibility condition introduced in [16] and we specify the dual representation to this case in Corollary 5.5. These results show the strong link between the framework developed in [16] and our multi-asset setting.

Optimal risk sharing. In Section 5.2 we show that the infimal convolution of single-asset risk measures can always be expressed as a multi-asset risk measure. Since infimal convolutions are important tools when studying optimal risk sharing across several business lines, our results find a natural application in this context. In particular, we provide a dual representation for the infimal convolution of convex risk measures in Proposition 5.8.

Superhedging relative to shortfall risk. In Section 5.3 we focus on superhedging relative to shortfall risk as studied by Arai in [2]. Using our approach to dual representations, we are able to extend to arbitrary (convex) classes of admissible trading strategies the key Lemma 5.1 in that paper which was established for so-called $W$-admissible strategies. As a result, combining the representation (3) with the continuity results in Section 3.2, we are able to provide in Proposition 5.9 a sharper dual representation for the cost of superhedging relative to shortfall risk.
Related literature

Risk measures of the form $\rho_{\mathcal{A},\mathcal{F}}$ seem to have been first considered by Scandolo [22] and then investigated by Frittelli and Scandolo [14] in the context of spaces of bounded positions. However, the main focus in [14] is on risk measures in a multi-period setting and the key results, when applied to a one-period setting, reduce to the standard single-asset, cash-additive case, as underlined in the comment after Definition 4.3 in that paper. The dual representation in [14] is based on standard Fenchel-Moreau duality and is established for finitely-valued risk measures. In particular, there is no result ensuring the existence of extensions of the pricing functional belonging to the effective domain of the acceptance set, which is a key condition to have non-degenerate dual representations.

Our approach to multi-asset risk measures is close in spirit to Artzner, Delbaen, and Koch-Medina [4], where the treatment, however, was specific to the case of coherent acceptance sets and finite state spaces. Finally, Kountzakis [19] also works with abstract spaces and proves a representation theorem for coherent multi-asset risk measures. The dual representation is stated under the restrictive assumption that $\mathcal{X}$ is a reflexive Banach space and that some eligible payoff in $\mathcal{F}$ is an interior point of the positive cone of $\mathcal{X}$. In particular, this result does not apply to spaces whose positive cone has empty interior, like $L^p$ spaces, $1 \leq p < \infty$, on nonatomic probability spaces.

2 Risk measures in a multi-asset setting

In this introductory section we investigate the basic properties of multi-asset risk measures defined on an ordered topological vector space.

2.1 Financial positions and acceptance sets

The space $\mathcal{X}$ of financial positions is assumed to be an ordered (Hausdorff) topological vector space over $\mathbb{R}$ with positive cone $\mathcal{X}_+$. We use the standard notation $Y \geq X$ whenever $Y - X \in \mathcal{X}_+$. Unless explicitly stated, we do not assume that $\mathcal{X}_+$ has nonempty interior. By $\mathcal{X}'$ we denote the topological dual space of $\mathcal{X}$, which we assume ordered by the positive cone $\mathcal{X}'_+$ of all functionals $\psi \in \mathcal{X}'$ such that $\psi(X) \geq 0$ whenever $X \in \mathcal{X}_+$. If $\mathcal{A}$ is a subset of $\mathcal{X}$, we denote by $\text{int}(\mathcal{A})$, $\overline{\mathcal{A}}$ and $\partial \mathcal{A}$ the interior, the closure and the boundary of $\mathcal{A}$, respectively. Moreover, by $\text{core}(\mathcal{A})$ we denote the core of $\mathcal{A}$, i.e. the set of all positions $X \in \mathcal{A}$ such that for each $Y \in \mathcal{X}$ there exists $\varepsilon > 0$ with $X + \lambda Y \in \mathcal{A}$ whenever $|\lambda| < \varepsilon$. The convex hull $\text{co}(\mathcal{A})$ of $\mathcal{A}$ is the smallest convex set containing $\mathcal{A}$. Note that $\mathcal{A}$ is said to be a cone if $\lambda \mathcal{A} \subset \mathcal{A}$ for every $\lambda \geq 0$.

We start by recalling the notion of an acceptance set.

**Definition 2.1.** A set $\mathcal{A} \subset \mathcal{X}$ is called an acceptance set whenever the following two conditions are satisfied:

(A1) $\mathcal{A}$ is a nonempty, proper subset of $\mathcal{X}$ (non-triviality);

(A2) if $X \in \mathcal{A}$ and $Y \geq X$ then $Y \in \mathcal{A}$ (monotonicity).
Later on we will focus on convex acceptance sets and coherent acceptance sets, i.e. acceptance sets that are convex cones. We refer to [13] for a financial interpretation and examples.

We recall a simple but fundamental and useful property of acceptance sets, see Lemma 2.14 in [11]: any positive halfspace containing an acceptance set must be generated by a positive functional. Here, the (positive) halfspace generated by a functional $\psi : \mathcal{X} \to \mathbb{R}$ at a level $\alpha \in \mathbb{R}$ is the set

$$\mathcal{H}^+(\psi, \alpha) := \{ X \in \mathcal{X} ; \psi(X) \geq \alpha \} .$$

**Lemma 2.2.** Let $\mathcal{A} \subset \mathcal{X}$ be an arbitrary acceptance set. If $\mathcal{A} \subset \mathcal{H}^+(\psi, \alpha)$ for some (not necessarily continuous) linear functional $\psi : \mathcal{X} \to \mathbb{R}$ and $\alpha \in \mathbb{R}$, then $\psi$ is positive.

### 2.2 Multi-asset risk measures

Consider a financial market described by a vector subspace $\mathcal{M} \subset \mathcal{X}$. The space $\mathcal{M}$ is called the *marketed space* and its elements represent the marketed payoffs. We think of marketed payoffs as the payoffs that can be replicated by executing “admissible” trading strategies involving a finite set of basic traded assets. We assume the existence of a linear pricing functional $\pi : \mathcal{M} \to \mathbb{R}$ such that, for every market payoff $Z \in \mathcal{M}$, the quantity $\pi(Z)$ represents the initial value of the replicating strategy $\delta$. Moreover, we assume that the market is free of arbitrage opportunities by requiring that $\pi$ is strictly positive, i.e. $\pi(Z) > 0$ for any nonzero positive $Z \in \mathcal{M}$.

If the financial position of a financial institution is not acceptable with respect to a given acceptance set $\mathcal{A} \subset \mathcal{X}$, it is natural to ask which management actions can turn it into an acceptable position and at which cost. In this paper we allow financial institutions to modify the acceptability profile of their capital position by raising capital and investing it in admissible portfolios with payoff in a fixed subspace $\mathcal{S}$ of the marketed space $\mathcal{M}$. The restriction to strategies with payoffs in $\mathcal{S} \subset \mathcal{M}$ is meant to capture situations where a financial institution may only be able to operate in a segment of the full market.

Throughout the paper we will use the following notation. The extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$ will be denoted by $\overline{\mathbb{R}}$. For a subspace $\mathcal{I} \subset \mathcal{M}$ and $m \in \mathbb{R}$, we will write

$$\mathcal{I}_m := \{ Z \in \mathcal{I} ; \pi(Z) = m \} .$$

For the triple $(\mathcal{A}, \mathcal{I}, \pi)$ where $\mathcal{A}$ is an arbitrary subset of $\mathcal{X}$ and $\mathcal{I}$ is a subspace of $\mathcal{M}$ we define the map $\rho_{\mathcal{A}, \mathcal{I}, \pi} : \mathcal{X} \to \overline{\mathbb{R}}$ by setting

$$\rho_{\mathcal{A}, \mathcal{I}, \pi}(X) := \inf\{ \pi(Z) ; Z \in \mathcal{I} : X + Z \in \mathcal{A} \} .$$

We say $\rho_{\mathcal{A}, \mathcal{I}, \pi}$ is a *multi-asset* risk measure whenever $\dim(\mathcal{I}) \geq 2$ and a *single-asset* risk measure if $\dim(\mathcal{I}) = 1$. In the latter case $\mathcal{I}$ is generated by a nonzero $U \in \mathcal{M}$ and, when $\pi(U) \neq 0$, we set

$$\rho_{\mathcal{A}, U, \pi}(X) := \rho_{\mathcal{A}, \mathcal{I}, \pi}(X) = \inf\left\{ m \in \mathbb{R} ; X + \frac{m}{\pi(U)} U \in \mathcal{A} \right\} .$$

---

1For our purposes it suffices to know that there is a linear pricing functional assigning market values to marketed payoffs. For more details on the underlying market models, also for the case of infinite dimensional marketed spaces, we refer to Clark [8] and Kreps [20].
Note that risk measures of the form (9) have been studied in [11].

The interpretation of the map \( \rho_{A,S,\pi} \) is clear. If \( A \subset X \) is an acceptance set and \( S \) a subspace of \( M \), then for a given \( X \in X \) the quantity \( \rho_{A,S,\pi}(X) \) represents, when finite, the “minimum” amount of capital that needs to be raised and invested at time \( t = 0 \) in portfolios with payoff in \( S \) in order to make the position \( X \) acceptable.

For \( \rho_{A,S,\pi} \) to be a sound risk measurement tool, we need to impose additional conditions on the couple \((A,S)\).

**Definition 2.3.** The couple \((A,S)\) is called a risk measurement regime if \( A \) is an acceptance set and \( S \) a subspace of \( M \) satisfying \( S \cap X_+ \neq \{0\} \).

We say that \((A,S)\) is a convex, respectively coherent, risk measurement regime when \( A \) is a convex, respectively coherent, acceptance set.

**Remark 2.4.** Let \((A,S)\) be a risk measurement regime. Requiring that \( S \) contains some nonzero positive payoff \( U \) is natural from a financial point of view. Moreover, since \( \pi(U) > 0 \), we can decrease the amount of required capital by adding capital invested in \( U \) given that \( \rho_{A,S,\pi}(X + \lambda U) = \rho_{A,S,\pi}(X) - \lambda \pi(U) \) holds for any \( \lambda \in \mathbb{R} \).

Before stating the basic properties of risk measures of the form \( \rho_{A,S,\pi} \), we recall some standard terminology. The effective domain of a map \( \rho : X \to \mathbb{R} \) is

\[
\text{dom}(\rho) := \{ X \in X ; \rho(X) < \infty \}.
\] (10)

The function \( \rho \) is said to be convex, subadditive, or positively homogeneous, whenever the epigraph \( \text{epi}(\rho) := \{(X,\alpha) \in X \times \mathbb{R} ; \rho(X) \leq \alpha\} \) is convex, closed under addition, or it is a cone, respectively. The map \( \rho \) is said to be decreasing if \( \rho(X) \geq \rho(Y) \) whenever \( X \leq Y \). Finally, if \( S \) is a subspace of \( M \), we say that \( \rho \) is \( S \)-additive whenever

\[
\rho(X + Z) = \rho(X) - \pi(Z) \quad \text{for all } X \in X, \ Z \in S.
\] (11)

The following lemma records a few straightforward properties of maps of the form \( \rho_{A,S,\pi} \) and is stated without proof.

**Lemma 2.5.** Let \( A \subset X \) be an arbitrary nonempty set and \( S \) a subspace of \( M \). Then \( \rho_{A,S,\pi} \) satisfies the following properties:

(i) \( \text{dom}(\rho_{A,S,\pi}) = A + S \);

(ii) \( \rho_{A,S,\pi} \) is \( S \)-additive;

(iii) if \( A \) is convex, closed under addition, or a cone, then \( \rho_{A,S,\pi} \) is convex, subadditive, or positively homogeneous, respectively;

(iv) if \( A \) is a monotone set, then \( \rho_{A,S,\pi} \) is decreasing;

(v) if \((A,S)\) is a risk measurement regime, the set \( \{ \pi(Z) ; \exists Z \in S : X + Z \in A \} \) is a (possibly empty) interval which is unbounded to the right for any \( X \in X \).
2.3 From several assets to a single asset

In this brief section we extend and complement the Corollary on page 112 in [4] and shows how to convert multi-asset risk measures into single-asset risk measures by properly augmenting the underlying acceptance set. This Reduction Lemma allows us to translate results for single-asset risk measures to our multi-asset setting.

**Lemma 2.6** (Reduction Lemma). Let $\mathcal{A}$ be an arbitrary nonempty set in $\mathcal{X}$ and $\mathcal{I}$ a subspace of $\mathcal{M}$. If $U \in \mathcal{I}$ is a nonzero positive payoff, then

$$\rho_{\mathcal{A},\mathcal{I},\pi}(X) = \rho_{\mathcal{A}+\mathcal{I}_0,U,\pi}(X)$$

for every $X \in \mathcal{X}$.

**Proof.** Note that $\mathcal{I}_m = \frac{m}{\pi(U)}U + \mathcal{I}_0$ for any $m \in \mathbb{R}$. It follows that for every $X \in \mathcal{X}$

$$\rho_{\mathcal{A},\mathcal{I},\pi}(X) = \inf\{m \in \mathbb{R}; (X + \mathcal{I}_m) \cap \mathcal{A} \neq \emptyset\}$$

$$= \inf \left\{ m \in \mathbb{R}; X + \frac{m}{\pi(U)}U \in \mathcal{A} + \mathcal{I}_0 \right\}$$

$$= \rho_{\mathcal{A}+\mathcal{I}_0,U,\pi}(X),$$

concluding the proof.

**Remark 2.7.** The augmented set $\mathcal{A} + \mathcal{I}_0$ is always nonempty and monotone but may fail to be an acceptance set since $\mathcal{A} + \mathcal{I}_0$ may not be a proper subset of $\mathcal{X}$.

## 3 No acceptability arbitrage, finiteness, and continuity

In this section we provide a variety of finiteness and continuity results for multi-asset risk measures.

We start by showing that $\rho_{\mathcal{A},\mathcal{I},\pi}(0) > -\infty$ if and only if the pricing functional is bounded from below on $\mathcal{A} \cap \mathcal{I}$. If this property is not satisfied, then there exist acceptable positions at arbitrarily negative cost allowing for what could be called “acceptability arbitrage”. As we will see, the absence of “acceptability arbitrage” is typically sufficient to ensure finiteness and continuity of multi-asset risk measures.

**Lemma 3.1.** Let $(\mathcal{A}, \mathcal{I})$ be a risk measurement regime. The following statements are equivalent:

(a) $\mathcal{A} \cap \{Z \in \mathcal{I}; \pi(Z) \leq m\} = \emptyset$ for some $m \in \mathbb{R}$;

(b) $\mathcal{A} \cap \mathcal{I}_m = \emptyset$ for some $m \in \mathbb{R}$;

(c) $\rho_{\mathcal{A},\mathcal{I},\pi}(0) > -\infty$.

**Proof.** Clearly, (b) and (c) are equivalent. Assume (a) holds. Then $\mathcal{A} \cap \{Z \in \mathcal{I}; \pi(Z) \leq m\} = \emptyset$ for some $m \in \mathbb{R}$. In particular, this implies $\mathcal{A} \cap \mathcal{I}_m = \emptyset$, showing (b). Conversely, assume (b) holds but $Z_k \in \mathcal{A} \cap \mathcal{I}_k$ for $k < m$. Take a positive $U \in \mathcal{I}$ with $\pi(U) = 1$. Then $Z_k + (m - k)U \in \mathcal{A} \cap \mathcal{I}_m$, contradicting (b). Hence, (b) implies (a) concluding the proof. 

\[ \square \]
Remark 3.2. In a pricing context, Jaschke and Küchler in [18] introduced the assumption \( \mathcal{A} \cap \mathcal{X}_0 \setminus \{0\} = \emptyset \) of absence of good deals (of the first kind). Clearly, this condition implies the “no acceptability arbitrage” condition (b) in Lemma 3.1.

### 3.1 General acceptance sets

First, we provide a finiteness and continuity result for a general acceptance set under the assumption that the eligible space contains an order unit. Recall that an element \( U \in X^+ \) is called an order unit if \( U \in \text{core}(X^+) \). Note that, if \( X \) admits an order unit, every acceptance set \( \mathcal{A} \subset X \) has nonempty core, since \( \mathcal{A} + X^0 \subset \mathcal{A} \).

**Remark 3.3.**

(i) If the positive cone \( X^+ \) has nonempty interior, the notion of order unit coincides with that of interior points of \( X^+ \).

(ii) Let \((\Omega, \mathcal{F})\) be a finite measurable space, and consider the space \( X \) of all measurable functions \( X : \Omega \rightarrow \mathbb{R} \). The interior of the positive cone in \( X \) is nonempty and consists of all \( X \in X \) such that \( X(\omega) > 0 \) for all \( \omega \in \Omega \).

(iii) Consider the space \( L^\infty \) over a general probability space \((\Omega, \mathcal{F}, P)\). The corresponding positive cone has nonempty interior, consisting of all elements \( X \) which are bounded away from zero, i.e. such that \( X \geq \varepsilon \) almost surely for some \( \varepsilon > 0 \).

(iv) Let \((\Omega, \mathcal{F}, P)\) be a nonatomic probability space. The positive cone of \( L^p \), for \( 1 \leq p < \infty \), has empty core. This is also the case for nontrivial Orlicz hearts \( H^\Phi \) with respect to an Orlicz function \( \Phi \).

**Proposition 3.4.** Let \((\mathcal{A}, \mathcal{I})\) be a risk measurement regime. Assume that \( \mathcal{I} \) contains an order unit \( U \). Then it holds:

(i) \( \rho_{\mathcal{A}, \mathcal{I}, \pi}(X) < \infty \) for every \( X \in \mathcal{I} \).

(ii) The following statements are equivalent:

   (a) \( \rho_{\mathcal{A}, \mathcal{I}, \pi}(X) > -\infty \) for every \( X \in \mathcal{I} \);

   (b) \( \rho_{\mathcal{A}, \mathcal{I}, \pi}(0) > -\infty \);

   (c) \( \text{core}(\mathcal{A}) \cap \mathcal{I}_m = \emptyset \) for some \( m \in \mathbb{R} \);

   (d) \( \mathcal{A} + \mathcal{I}_0 \neq \mathcal{I} \).

In particular, \( \rho_{\mathcal{A}, \mathcal{I}, \pi} \) is finitely valued if and only if \( \rho_{\mathcal{A}, \mathcal{I}, \pi}(0) > -\infty \). In this case, if \( U \) is an interior point of \( \mathcal{I}_+ \) then \( \rho_{\mathcal{A}, \mathcal{I}, \pi} \) is continuous.

**Proof.**

(i) Take \( X \in \mathcal{I} \) and \( Y \in \mathcal{A} \). Since \( U \) is an order unit, we have that \( Y - X \leq \lambda U \) for some \( \lambda > 0 \). As a result we obtain \( \rho_{\mathcal{A}, \mathcal{I}, \pi}(X) \leq \rho_{\mathcal{A}, \mathcal{I}}(Y) + \lambda \pi(U) < \infty \).

(ii) It is clear that (a) implies (b), which in turn implies (c). Assume now that (c) holds but \( \mathcal{A} + \mathcal{I}_0 = \mathcal{I} \). As a result, given \( Z_m \in \mathcal{I}_m \) we can find \( Y \in \mathcal{A} \) and \( Z_0 \in \mathcal{I}_0 \) with \( Z_m - U = Y + Z_0 \). But then
\[ Z_m - Z_0 = Y + U \in \text{core}(\mathcal{A}) \cap \mathcal{I}_m, \text{ contradicting (c). Hence, (c) implies (d). Finally, if (d) holds then } \mathcal{A} + \mathcal{I}_0 \text{ is an acceptance set so that (a) follows applying the Reduction Lemma and Proposition 3.1 in [11].} \]

Finally, assume that \( \rho(\mathcal{A}, \mathcal{I}_0, \pi)(0) > -\infty \) and \( U \) is an interior point of \( \mathcal{I}_+ \). Since \( \mathcal{A} + \mathcal{I}_0 \) is a proper subset of \( \mathcal{I} \), the continuity of \( \rho(\mathcal{A}, \mathcal{I}, \pi) \) follows again from Proposition 3.1 in [11]. \qed

### 3.2 Convex acceptance sets

In general \( L^p \) spaces or Orlicz hearts we cannot apply Proposition 3.4 to obtain finiteness and continuity results since, by Remark 3.3, the core of the positive cone in these spaces is empty. However, by the remark below, the positive cone in these spaces does possess strictly positive elements, i.e. positive elements \( U \) such that \( \psi(U) > 0 \) for every nonzero positive functional \( \psi \in \mathcal{I}' \). It is thus important that, whenever the acceptance set is convex, instead of requiring that \( \mathcal{I} \) contains an order unit it is sufficient to assume that it contains a strictly positive payoff.

**Remark 3.5.**

(i) If the positive cone \( \mathcal{I}_+ \) has nonempty interior, the notion of strictly positive elements coincides with that of order units and interior points of \( \mathcal{I}_+ \).

(ii) Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space. If we equip \( L^\infty \) with the \( \sigma(L^\infty, L^1) \) topology, the set of strictly positive elements is larger than the core of \( L^\infty_+ \) and consists of all positive \( U \) such that \( U > 0 \) almost surely.

(iii) Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space. The strictly positive elements in the space \( L^p, 1 \leq p < \infty \), are precisely all positive \( U \) such that \( U > 0 \) almost surely. The same is true for any nontrivial Orlicz heart \( H^\Phi \) with respect to an Orlicz function \( \Phi \).

**Proposition 3.6.** Let \( (\mathcal{A}, \mathcal{I}) \) be a convex risk measurement regime such that \( \mathcal{A} \) has nonempty interior. Assume \( \mathcal{I} \) contains a strictly positive element \( U \). Then it holds:

(i) \( \rho(\mathcal{A}, \mathcal{I}, \pi)(X) < \infty \) for every \( X \in \mathcal{I} \).

(ii) The following are equivalent:

\begin{enumerate}
  
  (a) \( \rho(\mathcal{A}, \mathcal{I}, \pi)(X) > -\infty \) for every \( X \in \mathcal{I} \);
  
  (b) \( \rho(\mathcal{A}, \mathcal{I}, \pi)(0) > -\infty \);
  
  (c) \( \text{int}(\mathcal{I}) \cap \mathcal{I}_m = \emptyset \) for some \( m \in \mathbb{R} \);
  
  (d) \( \mathcal{A} + \mathcal{I}_0 \neq \mathcal{I} \).
\end{enumerate}

In particular, \( \rho(\mathcal{A}, \mathcal{I}, \pi) \) is finitely valued if and only if \( \rho(\mathcal{A}, \mathcal{I}, \pi)(0) > -\infty \). In this case, \( \rho(\mathcal{A}, \mathcal{I}, \pi) \) is also continuous.

**Proof.** In the proof we will repeatedly use Lemma 2.2 without further mention.

(i) If \( \rho(\mathcal{A}, \mathcal{I}, \pi)(X) = \infty \) for some \( X \in \mathcal{I} \), then \( (X + \mathcal{I}) \cap \mathcal{A} = \emptyset \). By separation we find a nonzero positive \( \psi \in \mathcal{I}' \) such that \( \psi(X + Z) \leq \psi(A) \) for all \( Z \in \mathcal{I} \) and \( A \in \mathcal{A} \). In particular, \( \psi(X) + \lambda \psi(U) \leq \psi(A) \)
holds for every $\lambda \in \mathbb{R}$ and $A \in \mathcal{A}$. This implies $\psi(U) = 0$ which is impossible since $U$ is strictly positive. In conclusion, $\rho_{\mathcal{A},\mathcal{I},\pi}(X) < \infty$ for all $X \in \mathcal{I}$.

(ii) Clearly, (a) implies (b) which implies (c). Assume $\int(\mathcal{A}) \cap \mathcal{I}_m = \emptyset$ for some $m \in \mathbb{R}$. By separation there exists a nonzero positive $\psi \in \mathcal{J}'$ such that $\psi(Z_m) \leq \psi(A)$ for all $Z_m \in \mathcal{I}_m$ and $A \in \mathcal{A}$. If (d) does not hold, for every $X \in \mathcal{I}$ we find $A \in \mathcal{A}$ and $Z_0 \in \mathcal{I}_0$ with $X = A + Z_0$. Since $\psi(m\frac{U}{\pi(U)} - Z_0) \leq \psi(A)$, we have $\frac{m}{\pi(U)} \psi(U) \leq \psi(X)$. But this cannot hold for every $X \in \mathcal{I}$, hence we must have $\mathcal{A} + \mathcal{I}_0 \neq \mathcal{I}$. Thus, (c) implies (d).

In conclusion, assume (d) holds and let $\rho_{\mathcal{A},\mathcal{I},\pi}(X) = -\infty$ for some $X \in \mathcal{I}$. By the Reduction Lemma we have $X + mU \in \mathcal{A} + \mathcal{I}_0$ for all $m \in \mathbb{R}$. Take $Y \notin \mathcal{A} + \mathcal{I}_0$. Since $\mathcal{A} + \mathcal{I}_0$ is a convex acceptance set with nonempty interior, there exists a nonzero positive functional $\psi \in \mathcal{I}'$ separating $\mathcal{A} + \mathcal{I}_0$ from $Y$ so that $\psi(Y) \leq \psi(A + Z_0)$ for all $A \in \mathcal{A}$ and $Z_0 \in \mathcal{I}_0$. But then $\psi(Y) \leq \psi(X + mU)$ for every $m \in \mathbb{R}$, implying $\psi(U) = 0$ which is impossible since $U$ is strictly positive. We conclude that $\rho_{\mathcal{A},\mathcal{I},\pi}(X) > -\infty$ for every $X \in \mathcal{I}$, showing that (a) holds.

Finally, if $\rho_{\mathcal{A},\mathcal{I},\pi}$ is finitely valued then it is also continuous as a consequence of Theorem 3.12 in [11].

### 3.3 Coherent acceptance sets

For coherent acceptance sets we can obtain finiteness and continuity results under even weaker conditions on the eligible subspace $\mathcal{I}$ than those of Proposition 3.6. In this case, the key assumption is that the eligible space $\mathcal{I}$ contains a positive payoff belonging to the core of $\mathcal{A}$.

**Remark 3.7.** The condition that $\mathcal{I}$ contains a positive payoff belonging to $\text{core}(\mathcal{A})$ is indeed weaker that the requirement that $\mathcal{I}$ contains a strictly positive element. For example, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and fix $1 \leq p \leq \infty$. Consider the coherent acceptance set based on Tail Value-at-Risk at level $\alpha \in (0,1)$

$$\mathcal{A}^\alpha := \left\{ X \in L^p ; \frac{1}{\alpha} \int_0^\infty \text{VaR}_\beta(X) \, d\beta \leq 0 \right\},$$  \hspace{1cm} (16)

where

$$\text{VaR}_\alpha(X) := \inf \{ m \in \mathbb{R} ; \mathbb{P}[X + m < 0] \leq \alpha \}$$  \hspace{1cm} (17)

is the Value-at-Risk of $X$ at level $\alpha$. As shown by Lemma 4.5 in [11], a positive $U \in \mathcal{U}$ belongs to $\text{core}(\mathcal{A}^\alpha)$ if and only if $\mathbb{P}[U = 0] < \alpha$. In particular, $U$ need not be an order unit or a strictly positive element.

**Proposition 3.8.** Let $(\mathcal{A}, \mathcal{I})$ be a coherent risk measurement regime. Assume there exists a nonzero positive $U \in \mathcal{I}$ such that $U \in \text{core}(\mathcal{A})$. Then it holds:

(i) $\rho_{\mathcal{A},\mathcal{I},\pi}(X) < \infty$ for every $X \in \mathcal{I}$.

(ii) The following statements are equivalent:

(a) $\rho_{\mathcal{A},\mathcal{I},\pi}(X) > -\infty$ for every $X \in \mathcal{I}$;

(b) $\rho_{\mathcal{A},\mathcal{I},\pi}(0) > -\infty$;

11
In particular, \( \rho_{\mathcal{A},\mathcal{X},\pi} \) is finitely valued if and only if \( \rho_{\mathcal{A},\mathcal{X},\pi}(0) > -\infty \). In this case, if \( U \in \text{int}(\mathcal{A}) \) then \( \rho_{\mathcal{A},\mathcal{X},\pi} \) is continuous.

Proof. (i) Since \( U \in \text{core}(\mathcal{A}) \), for every \( X \in \mathcal{X} \) we find \( \lambda > 0 \) such that \( U + \lambda X \in \mathcal{A} \) or, equivalently, \( X + \frac{1}{\lambda} U \in \mathcal{A} \). Hence, \( \rho_{\mathcal{A},\mathcal{X},\pi}(X) < \infty \).

(ii) It is clear that (a) implies (b). Assume that (c) does not hold, so that there exists \( Z_0 \in \text{core}(\mathcal{A}) \cap \mathcal{X}_0 \). Take \( m \in \mathbb{R} \). For any \( Z_m \in \mathcal{X}_m \) we can find \( \lambda > 0 \) such that \( Z_0 + \lambda Z_m \in \mathcal{A} \), hence \( Z_m + \frac{1}{\lambda} Z_0 \in \mathcal{A} \cap \mathcal{X}_m \). It follows that \( \rho_{\mathcal{A},\mathcal{X},\pi}(0) = -\infty \). As a result, (b) implies (c).

Now assume (c) holds. By the algebraic separation in Theorem 5.61 in [1], we find a nonzero positive linear functional \( \psi : \mathcal{X} \to \mathbb{R} \) such that \( \psi(Z_0) \leq \psi(A) \) or, equivalently, \( 0 \leq \psi(A + Z_0) \) for all \( Z_0 \in \mathcal{X}_0 \) and \( A \in \mathcal{A} \). As a result, we must have \( \mathcal{A} + \mathcal{X}_0 \neq \mathcal{X} \) since otherwise \( 0 \leq \psi(X) \) would hold for every \( X \in \mathcal{X} \). It follows that (d) holds.

Finally, assume that (d) holds. Since \( \mathcal{A} + \mathcal{X}_0 \) is a coherent acceptance set, we can apply Theorem 3.19 in [11] combined with the Reduction Lemma to obtain (a). The same results imply that if \( U \) is an interior point of \( \mathcal{A} \), then \( \rho_{\mathcal{A},\mathcal{X},\pi} \) is continuous on \( \mathcal{X} \).

4 Dual representations

In this section we provide dual representation results for convex acceptance sets and convex multi-asset risk measures. We assume throughout that the space of financial positions \( \mathcal{X} \) is a locally convex ordered topological vector space.

4.1 Dual representation of closed convex acceptance sets

We start by recalling the notion of the support function of a given set \( \mathcal{A} \subset \mathcal{X} \).

**Definition 4.1.** The **support function** of a set \( \mathcal{A} \subset \mathcal{X} \) is the map \( \sigma_{\mathcal{A}} : \mathcal{X}' \to \mathbb{R} \cup \{-\infty\} \) defined by

\[
\sigma_{\mathcal{A}}(\psi) := \inf_{X \in \mathcal{A}} \psi(X) .
\]

The domain of finiteness of the support function is denoted by \( B(\mathcal{A}) \) and is called the **barrier cone** of \( \mathcal{A} \), i.e.

\[
B(\mathcal{A}) := \{ \psi \in \mathcal{X}' \mid \sigma_{\mathcal{A}}(\psi) > -\infty \} .
\]

We now list several simple properties of support functions that will be used in the sequel without further reference. Recall that \( \text{co}(\mathcal{A}) \) denotes the convex hull of the set \( \mathcal{A} \). Moreover, we will call **floor function**\(^2\) any map \( \gamma : \mathcal{X}'_+ \to \mathbb{R} \cup \{-\infty\} \) which is not identically equal to \( -\infty \) and such that there exists \( X \in \mathcal{X} \) satisfying \( \psi(X) \geq \gamma(\psi) \) for each \( \psi \in \mathcal{X}'_+ \).

\(^2\)In the literature on risk measures it is more common to work with **penalty functions**, which are the negatives of floor functions. However, in the context of acceptance sets, the representation (21) takes a more natural form when floor functions are used.
Lemma 4.2. For any nonempty subset $\mathcal{A}$ of $\mathcal{X}$ we have:

$$\overline{co}(\mathcal{A}) = \bigcap_{\psi \in \mathcal{X}'} H^+(\psi, \sigma_{\mathcal{A}}(\psi)) = \bigcap_{\psi \in B(\mathcal{A})} H^+(\psi, \sigma_{\mathcal{A}}(\psi)).$$

(20)

Moreover, the following statements hold for any nonempty subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{X}$:

(i) $\sigma_{\mathcal{A}}$ is positively homogeneous and concave, hence also superadditive;

(ii) $\sigma_{\mathcal{A} + \mathcal{B}} = \sigma_{\mathcal{A}} + \sigma_{\mathcal{B}}$;

(iii) if $\overline{co}(\mathcal{A}) \neq \mathcal{X}$ then $\sigma_{\mathcal{A}}$, when restricted to $\mathcal{X}'_+$, is a floor function;

(iv) if $\mathcal{A}$ is a cone, then $B(\mathcal{A}) = \{\psi \in \mathcal{X}'; \sigma_{\mathcal{A}}(\psi) = 0\}$.

Remark 4.3. All the above statements on support functions can be found in Section 5, Chapter 1 of [5]. Note, however, that support functions are typically defined using the supremum rather than the infimum. We have deviated from this convention because using the infimum seems more natural in our setting.

The following result provides a dual representation for closed, convex acceptance sets in a locally convex ordered topological vector space.

Theorem 4.4 (Dual representation of closed convex acceptance sets). A nonempty subset $\mathcal{A}$ of $\mathcal{X}$ is a closed, convex acceptance set if and only if there exists a floor function $\gamma : \mathcal{X}'_+ \rightarrow \mathbb{R} \cup \{-\infty\}$ such that

$$\mathcal{A} = \bigcap_{\psi \in \mathcal{X}'_+ \cap B(\mathcal{A})} H^+(\psi, \gamma(\psi)).$$

(21)

In (21) we can always choose the floor function to be $\sigma_{\mathcal{A}}$. Moreover, the floor function $\sigma_{\mathcal{A}}$ is maximal, i.e. if $\gamma$ is any other floor function satisfying (21), then $\sigma_{\mathcal{A}}(\psi) \geq \gamma(\psi)$ for all $\psi \in \mathcal{X}'_+$.

Proof. The “if” part is clearly true. In particular, the set $\mathcal{A}$ is a nonempty proper subset of $\mathcal{X}$ because $\gamma$ is a floor function. To prove the “only if” part note that, by Lemma 2.2, if $\psi \in \mathcal{X}'$ is not positive then $\sigma_{\mathcal{A}}(\psi) = -\infty$. Hence the result follows from Lemma 4.2 since, when taking intersections in (20), we can disregard all functionals that are not positive and obtain (21) with $\gamma := \sigma_{\mathcal{A}}$.

The maximality of $\sigma_{\mathcal{A}}$ is straightforward. Indeed, given a floor function $\gamma$ satisfying (21), we have $\psi(X) \geq \gamma(\psi)$ for each $\psi \in \mathcal{X}'_+$ and $X \in \mathcal{A}$. The assertion follows by taking the infimum over $X \in \mathcal{A}$.

By Lemma 4.2, whenever $\mathcal{A}$ is a conic acceptance set we have $\sigma_{\mathcal{A}}(\psi) = 0$ on $B(\mathcal{A})$. Hence, the dual representation (21) has a particularly simple structure for coherent acceptance sets.

Corollary 4.5. A nonempty subset $\mathcal{A}$ of $\mathcal{X}$ is a closed coherent acceptance set if and only if

$$\mathcal{A} = \bigcap_{\psi \in \mathcal{X}'_+ \cap B(\mathcal{A})} \{X \in \mathcal{X}; \psi(X) \geq 0\}.$$  

(22)
In (21) we can sometimes reduce the range of functionals over which intersections are taken by a suitable “normalization”.

**Proposition 4.6.** Let $\mathcal{A} \subset \mathcal{X}$ be a closed, convex acceptance set. Take $U \in \mathcal{X}_+$ and assume $\varphi(U) > 0$ for some $\varphi \in \mathcal{X}'_+ \cap B(\mathcal{A})$. Then we have

$$\mathcal{A} = \bigcap_{\psi \in \mathcal{X}'_+ \cap B(\mathcal{A})} \mathcal{H}^+ (\psi, \sigma_\mathcal{A} (\psi)),$$

(23)

where $\mathcal{X}'_+ : = \{ \psi \in \mathcal{X}'_+ ; \psi(U) = 1 \}$. 

**Proof.** First, note that we can normalize $\varphi$ to get $\varphi(U) = 1$, hence the set $\mathcal{X}'_+ \cap B(\mathcal{A})$ is nonempty due to the positive homogeneity of $\sigma_\mathcal{A}$. Now take $X \in \mathcal{X}$ such that $\phi(X) \geq \sigma_\mathcal{A} (\phi)$ for all $\phi \in \mathcal{X}'_+ \cap B(\mathcal{A})$, and fix an arbitrary $\psi \in \mathcal{X}'_+ \cap B(\mathcal{A})$. By Theorem 4.4 we only need to show that $\psi(X) \geq \sigma_\mathcal{A} (\psi)$. Assume $\psi(U) = 0$, otherwise the claim is trivial. For all positive integers $n$ set $\varphi_n := \varphi + n\psi$. As a consequence of the superlinearity of $\sigma_\mathcal{A}$, the functional $\varphi_n$ also belongs to $\mathcal{X}'_+ \cap B(\mathcal{A})$. Hence,

$$\frac{1}{n} \varphi_n(X) = \varphi_n(X) \geq \frac{1}{n} \sigma_\mathcal{A} (\varphi_n) \geq \frac{1}{n} \sigma_\mathcal{A} (\varphi) + \sigma_\mathcal{A} (\psi).$$

(24)

Letting $n \to \infty$ we obtain $\psi(X) \geq \sigma_\mathcal{A} (\psi)$, concluding the proof. 

**Remark 4.7.** The above proposition provides just one possible “normalized” version of Theorem 4.4. Another common type of normalization can be performed when $\mathcal{X}$ is an ordered normed space. In that case one can restrict the attention in (21) to functionals with norm 1.

For the dual representation of convex multi-asset risk measures we need a sharper dual representation of “augmented” acceptance sets of the form $\mathcal{A} + \mathcal{J}_0$. We start with a description of their support function, omitting the easy proof. Recall that the annihilator of a subspace $\mathcal{J} \subset \mathcal{X}$ is defined by $\mathcal{J}^\perp := \{ \psi \in \mathcal{X}' ; \psi(X) = 0, \forall X \in \mathcal{J} \}$.

**Lemma 4.8.** Let $(\mathcal{A}, \mathcal{J})$ be a risk measurement regime. Then for any $\psi \in \mathcal{X}'$

$$\sigma_\mathcal{A} + \mathcal{J}_0 (\psi) = \begin{cases} \sigma_\mathcal{A} (\psi) & \text{if } \psi \in \mathcal{J}^\perp_0, \\ -\infty & \text{otherwise}. \end{cases}$$

(25)

Moreover, $B(\mathcal{A} + \mathcal{J}_0) = B(\mathcal{A}) \cap \mathcal{J}^\perp_0$.

Given a subspace $\mathcal{J}$ of $\mathcal{M}$, we introduce the set $\mathcal{E}_\pi(\mathcal{J})$ consisting of all positive, continuous linear extensions of the pricing functional $\pi : \mathcal{J} \to \mathbb{R}$ to the whole space $\mathcal{X}$, i.e.

$$\mathcal{E}_\pi(\mathcal{J}) := \{ \psi \in \mathcal{X}'_+ ; \psi(Z) = \pi(Z), \forall Z \in \mathcal{J} \}.$$ 

(26)

We note that

$$\mathcal{E}_\pi(\mathcal{J}) = \{ \psi \in \mathcal{J}^\perp_0 ; \psi(U) = 1 \} = \mathcal{X}'_+ \cap \mathcal{J}^\perp_0,$$

(27)

for any positive $U \in \mathcal{J}$ with $\pi(U) = 1$ and with $\mathcal{X}'_+ \cap B(\mathcal{A})$ defined as in Proposition 4.6. Note also that we are taking extensions of $\pi$ restricted to $\mathcal{J}$. As a result the elements in $\mathcal{E}_\pi(\mathcal{J})$ need not be extensions of $\pi : \mathcal{M} \to \mathbb{R}$. 

14
Remark 4.9. Assume the market satisfies the no free lunch\(^3\) condition \(\mathcal{X}_+ \cap \mathcal{X}_0 - \mathcal{X}_+ \setminus \{0\} = \emptyset\). Then, using a separation argument, it is not difficult to show that \(\mathcal{E}_\pi(\mathcal{I})\) is nonempty.

**Theorem 4.10.** Let \((\mathcal{A}, \mathcal{I})\) be a convex risk measurement regime. The following statements hold.

(i) If \(\mathcal{E}_\pi(\mathcal{I}) \cap B(\mathcal{A})\) is nonempty, then
\[
\mathcal{A} + \mathcal{I}_0 = \bigcap_{\psi \in \mathcal{E}_\pi(\mathcal{I}) \cap B(\mathcal{A})} \mathcal{H}^+(\psi, \sigma_{\mathcal{A}}(\psi)).
\] (28)

(ii) If \(\mathcal{E}_\pi(\mathcal{I}) \cap B(\mathcal{A})\) is empty, then
\[
\mathcal{A} + \mathcal{I}_0 = \bigcap_{\psi \in \mathcal{J}_0^\perp \cap B(\mathcal{A})} \mathcal{H}^+(\psi, \sigma_{\mathcal{A}}(\psi)).
\] (29)

Proof. Since \((\mathcal{A}, \mathcal{I})\) is a risk measurement regime we find a positive payoff \(U \in \mathcal{I}\) such that \(\pi(U) = 1\).

(i) By (27) and Lemma 4.8 we have
\[
\mathcal{J}^t_{+, U} \cap B(\mathcal{A} + \mathcal{I}_0) = \mathcal{E}_\pi(\mathcal{I}) \cap B(\mathcal{A}).
\] (30)

Hence, (28) follows by applying Proposition 4.6.

(ii) Since \(\mathcal{I} = \mathcal{I}_0 + \mathbb{R}U\), it is easy to see that
\[
\mathcal{J}^t \cap B(\mathcal{A} + \mathcal{I}_0) = \mathcal{J}^t \cap \mathcal{I}_0^\perp \cap B(\mathcal{A}) = \mathcal{J}^t \cap \mathcal{I}_0 \cap B(\mathcal{A}).
\] (31)

Together with (21), this implies (29) and concludes the proof.

\[\square\]

### 4.2 Extending the pricing functional

Consider a convex risk measurement regime \((\mathcal{A}, \mathcal{I})\). Because of Theorem 4.10 it is important to investigate the existence of positive, continuous linear extensions of the pricing functional \(\pi\) which belong to the barrier cone \(B(\mathcal{A})\). In the following theorem we provide equivalent conditions for such extensions to exist. Since the acceptance set \(\mathcal{A}\) is only required to be convex, this result can be seen as a generalization of classical extension results for positive functionals given by Namioka in Theorems 2.2 and 4.4 in [21], by Bauer in Theorem 2 in [7], and by Hustad in Theorem 2 in [17]. Moreover, as shown in the next section, the conditions below establish the natural threshold for a non-degenerate theory of convex, lower semicontinuous multi-asset risk measures.

**Theorem 4.11 (Extensions of the pricing functional).** Let \((\mathcal{A}, \mathcal{I})\) be a convex risk measurement regime satisfying \(\mathcal{A} \cap \mathcal{I} \neq \emptyset\). The following statements are equivalent:

(a) \(\mathcal{E}_\pi(\mathcal{I}) \cap B(\mathcal{A})\) is nonempty;

\[\text{For background on this condition we refer to [20] and the discussion in [8].}\]
(b) \( \pi \) is bounded from below on \( \mathcal{A} + \mathcal{I}_0 \cap \mathcal{I} \);

(c) \( \pi \) is bounded from below on \( (\mathcal{A} + \mathcal{U}) \cap \mathcal{I} \) for some neighborhood of zero \( \mathcal{U} \).

**Proof.** We first prove that (a) and (b) are equivalent. Assume (a) holds and take \( \psi \in \mathcal{E}_\pi(\mathcal{I}) \cap B(\mathcal{A}) \). Then, by Theorem 4.10, we have \( \pi(Z) = \psi(Z) \geq \sigma_\mathcal{A}(\psi) \) for all \( Z \in \mathcal{A} + \mathcal{I}_0 \cap \mathcal{I} \), hence (b) holds.

Assume now that (b) holds but \( \mathcal{E}_\pi(\mathcal{I}) \cap B(\mathcal{A}) \) is empty. Then \( \mathcal{I} \subset \mathcal{A} + \mathcal{I}_0 \) by Theorem 4.10. Indeed, taking \( W \in \mathcal{A} \cap \mathcal{I} \subset \mathcal{A} + \mathcal{I}_0 \) we have \( \sigma_\mathcal{A}(\psi) \leq \psi(W) = 0 \) for all \( \psi \in \mathcal{I}_+ \cap \mathcal{I}^\perp \). Consequently, \( \sigma_\mathcal{A}(\psi) \leq 0 = \psi(Z) \) for all \( Z \in \mathcal{I} \) and \( \psi \in \mathcal{I}_+ \cap \mathcal{I}^\perp \), yielding \( \mathcal{I} \subset \mathcal{A} + \mathcal{I}_0 \). But this implies \( \mathcal{A} + \mathcal{I}_0 \cap \mathcal{I} = \mathcal{I} \), contradicting (b). It follows that (a) and (b) are equivalent.

To see that (a) implies (c), take \( \psi \in \mathcal{E}_\pi(\mathcal{I}) \cap B(\mathcal{A}) \). If \( \mathcal{U} := \{ X \in \mathcal{I} ; \psi(X) > -1 \} \), then \( \pi(Z) = \psi(Z) > \sigma_\mathcal{A}(\psi) - 1 \) for all \( Z \in (\mathcal{A} + \mathcal{U}) \cap \mathcal{I} \).

We conclude by proving that (c) implies (a). Take a positive \( U \in \mathcal{I} \) with \( \pi(U) = 1 \). Then, \( \mathcal{I} = \mathbb{R} + \mathcal{I}_0 \). Without loss of generality we can take \( \mathcal{U} \) to be open and convex so that \( \mathcal{A} + \mathcal{U} \) is an open and convex acceptance set. Note that \( (\mathcal{A} + \mathcal{U}) \cap \{ Z \in \mathcal{I} ; \pi(Z) \leq m \} \) is empty for some \( m < 0 \) and that \( \{ Z \in \mathcal{I} ; \pi(Z) \leq m \} = \{ U + Z_0 ; \lambda \leq m \text{ and } Z_0 \in \mathcal{I}_0 \} \). Hence, we find by separation and Lemma 2.2 a nonzero positive \( \psi \in \mathcal{I}' \) such that

\[
\lambda \psi(U) + \psi(Z_0) \leq \psi(A + X)
\]  

(32)

for all \( A \in \mathcal{A} \), \( X \in \mathcal{U} \), \( \lambda \leq m \) and \( Z_0 \in \mathcal{I}_0 \). Since \( \mathcal{I}_0 \) is a subspace, (32) implies that \( \psi \in \mathcal{I}_0^\perp \). Furthermore \( \psi(U) > 0 \). Indeed, since \( \psi(U) \geq 0 \), we would otherwise have \( \psi(U) = 0 \) and, hence, \( \psi \in \mathcal{I}^\perp \). But then, taking \( A \in \mathcal{A} \cap \mathcal{I} \) we obtain from (32) that \( 0 \leq \psi(X) \) for \( X \in \mathcal{U} \) which is impossible since \( \mathcal{U} \) is a neighborhood of zero and \( \psi \) is nonzero. Rescaling \( \psi \) to satisfy \( \psi(U) = \pi(U) \) we see, by (27), that \( \psi \in \mathcal{E}_\pi(\mathcal{I}) \). Finally, (32) also implies that \( \inf_{A \in \mathcal{A}} \psi(A) > -\infty \) so that \( \psi \in B(\mathcal{A}) \).

**Remark 4.12.** Let \( (\mathcal{A}, \mathcal{I}) \) be a convex risk measurement regime.

(i) It is easy to see that, if \( \mathcal{A} + \mathcal{I}_0 \) is closed, the conditions in the previous theorem are equivalent to the “no acceptability arbitrage” condition \( \mathcal{A} \cap \{ Z \in \mathcal{I} ; \pi(Z) \leq m \} \) for some \( m \in \mathbb{R} \) in Lemma 3.1. In particular, they are equivalent to \( \rho_{\mathcal{A}, \mathcal{I}, \pi}(0) > -\infty \).

(ii) Requiring that \( \mathcal{A} \cap \mathcal{I} \neq \emptyset \) is equivalent to requiring \( \rho_{\mathcal{A}, \mathcal{I}, \pi}(0) < \infty \). This is reasonable since the zero position should either be acceptable in the first place or capable of being made acceptable by some eligible strategy. Moreover, if \( \mathcal{A} \cap \mathcal{I} = \emptyset \) then \( \rho_{\mathcal{A}, \mathcal{I}, \pi}(Z) = \infty \) for all \( Z \in \mathcal{I} \).

### 4.3 Dual representation of convex multi-asset risk measures

In this section we derive dual representation theorems for convex, lower semicontinuous multi-asset risk measures. We refer to Section 3 for conditions ensuring continuity under various assumptions on the regime \( (\mathcal{A}, \mathcal{I}) \), and to the appendix for a general discussion about semicontinuity of (multi-asset) risk measures.

As already mentioned, the condition \( \mathcal{E}_\pi(\mathcal{I}) \cap B(\mathcal{A}) \neq \emptyset \) provides a threshold for a sound theory of convex, lower semicontinuous multi-asset risk measures. Indeed, if any of those conditions is not fulfilled, then a convex risk measure cannot take finite values at any point of lower semicontinuity.
**Proposition 4.13.** Let \((\mathcal{A}, \mathcal{I})\) be a convex risk measurement regime. If \(E_\pi(\mathcal{I}) \cap B(\mathcal{A})\) is empty, then \(\rho_{\mathcal{A},\mathcal{I},\pi}\) cannot take finite values at any point \(X \in \mathcal{I}\) of lower semicontinuity.

**Proof.** Let \(X\) be a point of lower semicontinuity for \(\rho_{\mathcal{A},\mathcal{I},\pi}\). By Proposition 6.1 we can assume that \(\mathcal{A} + \mathcal{I}_0\) is closed. Take a positive \(U \in \mathcal{I}\) with \(\pi(U) = 1\). As a consequence of Theorem 4.10, we have for any \(m \in \mathbb{R}\) that \(X + mU \in \mathcal{A} + \mathcal{I}_0\) if and only if \(X \in \mathcal{A} + \mathcal{I}_0\). Hence, the Reduction Lemma implies that \(\rho_{\mathcal{A},\mathcal{I},\pi}(X) = -\infty\) if \(X \in \mathcal{A} + \mathcal{I}_0\) and \(\rho_{\mathcal{A},\mathcal{I},\pi}(X) = \infty\) if \(X \notin \mathcal{A} + \mathcal{I}_0\).

As a corollary of the above result we obtain a characterization of when a convex, (globally) lower semicontinuous multi-asset risk measure never takes the value \(-\infty\). In particular, this is the case whenever there is no “acceptability arbitrage” as defined in Section 3.

**Corollary 4.14.** Let \((\mathcal{A}, \mathcal{I})\) be a convex risk measurement regime with \(\mathcal{A} \cap \mathcal{I} \neq \emptyset\). Assume \(\rho_{\mathcal{A},\mathcal{I},\pi}\) is lower semicontinuous. Then the following statements are equivalent:

1. \(\rho_{\mathcal{A},\mathcal{I},\pi}(X) \in \mathbb{R}\) for some \(X \in \mathcal{I}\);
2. \(E_\pi(\mathcal{I}) \cap B(\mathcal{A})\) is nonempty;
3. \(\rho_{\mathcal{A},\mathcal{I},\pi}(X) > -\infty\) for every \(X \in \mathcal{I}\);
4. \(\rho_{\mathcal{A},\mathcal{I},\pi}(0) > -\infty\).

**Proof.** By Proposition 6.1 we can assume that \(\mathcal{A} + \mathcal{I}_0\) is closed. Clearly (a) implies (b) by Proposition 4.13. Assume (b) holds and take \(X \in \mathcal{I}\) and \(\psi \in E_\pi(\mathcal{I}) \cap B(\mathcal{A})\). Take a positive \(U \in \mathcal{I}\) with \(\pi(U) = 1\). Since \(\psi(X) + m = \psi(X + mU) \geq \sigma_\mathcal{A}(\psi)\) cannot hold for every \(m \in \mathbb{R}\), Theorem 4.10 implies that \(X + mU \notin \mathcal{A} + \mathcal{I}_0\) for some \(m \in \mathbb{R}\). Hence, \(\rho_{\mathcal{A},\mathcal{I},\pi}(X) > -\infty\) by the Reduction Lemma, proving (c). Clearly, (c) implies (d). Finally, (d) implies (a) because \(\rho_{\mathcal{A},\mathcal{I},\pi}(0) < \infty\) by Remark 4.12.

**Remark 4.15.** This corollary can be seen as a sharper version of the well-known Proposition 2.4 in [10] in the context of risk measures. That result implies that, on a locally convex topological vector space, lower semicontinuous convex functions which are not identical to \(\infty\) take some finite value if and only if they do not assume the value \(-\infty\). For a risk measure with regime \((\mathcal{A}, \mathcal{I})\) satisfying \(\mathcal{A} \cap \mathcal{I} \neq \emptyset\), this is the case if and only if it does not assume the value \(-\infty\) at 0.

We are now ready to prove our main dual representation theorem for convex multi-asset risk measures at points of lower semicontinuity. The proof draws on the dual representation of convex acceptance sets obtained in Theorem 4.10 and the pointwise characterization of lower semicontinuity provided in Proposition 6.1. Thus, our approach differs from the usual application of Fenchel-Moreau techniques. For an example of a risk measure which is lower semicontinuous at some but not at every position, see Remark 6.2.

**Theorem 4.16** (Pointwise dual representation). Let \((\mathcal{A}, \mathcal{I})\) be a convex risk measurement regime such that \(E_\pi(\mathcal{I}) \cap B(\mathcal{A})\) is nonempty. If \(\rho_{\mathcal{A},\mathcal{I},\pi}\) is lower semicontinuous at \(X \in \mathcal{I}\), then

\[
\rho_{\mathcal{A},\mathcal{I},\pi}(X) = \sup_{\psi \in E_\pi(\mathcal{I})} \{\sigma_\mathcal{A}(\psi) - \psi(X)\}.
\]
Proof. Note that the augmented acceptance set \( \mathcal{A} + \mathcal{I}_0 \) is convex and that, by Proposition 6.1, we may assume that it is closed. Fix now a positive \( U \in \mathcal{I} \) with \( \pi(U) = 1 \). Then the Reduction Lemma and Theorem 4.10 imply
\[
\rho_{\mathcal{A},\mathcal{I},\pi}(X) = \inf \{ m \in \mathbb{R} ; X + mU \in \mathcal{A} + \mathcal{I}_0 \}
\]
(34)
\[
= \inf \{ m \in \mathbb{R} ; \psi(X) + m\psi(U) \geq \sigma_{\mathcal{A}}(\psi), \forall \psi \in \mathcal{E}_\pi(\mathcal{I}) \}. \tag{35}
\]
Since \( \psi(U) = \pi(U) = 1 \) for all \( \psi \in \mathcal{E}_\pi(\mathcal{I}) \), the representation (33) immediately follows. \( \square \)

Remark 4.17. Let \((\mathcal{A}, \mathcal{I})\) be a convex risk measurement regime such that \( \rho_{\mathcal{A},\mathcal{I},\pi} \) is lower semicontinuous. The above dual representation (33) highlights the different roles played by \( \mathcal{A}, \mathcal{I} \) and \( \pi \) in the determination of the risk measure \( \rho_{\mathcal{A},\mathcal{I},\pi} \). The acceptance set \( \mathcal{A} \) determines, through the support function \( \sigma_{\mathcal{A}} \), the objective function and the eligible subspace \( \mathcal{I} \) together with the pricing functional \( \pi \) determines the optimization domain \( \mathcal{E}_\pi(\mathcal{I}) \). In particular, modifying the eligible subspace while maintaining the acceptance set, only requires changing the optimization domain, but not the objective function. This property, which is also useful from an operational perspective, is naturally unveiled through our approach and would have to be guessed and proved if the dual representation had been obtained by applying Fenchel-Moreau duality. For this reason, we believe the new approach adopted in this paper provides a better insight into the structure of the dual representation for risk measures. Further evidence of the advantages of our approach is provided by the results in Section 5.3.

In case of coherent risk measurement regimes we immediately obtain a simplified representation as a consequence of Corollary 4.5.

Corollary 4.18. Assume \((\mathcal{A}, \mathcal{I})\) is a coherent risk measurement regime such that \( \mathcal{E}_\pi(\mathcal{I}) \cap B(\mathcal{A}) \) is nonempty. Then for every \( X \in \mathcal{X} \) at which \( \rho_{\mathcal{A},\mathcal{I},\pi} \) is lower semicontinuous we have
\[
\rho_{\mathcal{A},\mathcal{I},\pi}(X) = \sup_{\psi \in \mathcal{D}} \psi(-X) \tag{36}
\]
where
\[
\mathcal{D} := \{ \psi \in \mathcal{E}_\pi(\mathcal{I}) ; \psi(Y) \geq 0, \forall Y \in \mathcal{A} \}. \tag{37}
\]
A natural and important question to ask is when the supremum in the representation formula (33) is attained. In contrast to the standard results on attainability, which typically rely on compactness, the result below exploits simple geometrical properties of the underlying acceptance set. In particular, we use the properties of support points and the conditions for upper semicontinuity stated in Proposition 6.1. Recall that a point \( X \) on the boundary \( \partial \mathcal{A} \) of a subset \( \mathcal{A} \subset \mathcal{X} \) is called a support point of \( \mathcal{A} \) if there exists a nonzero \( \psi \in \mathcal{X}' \) such that \( \psi(X) = \sigma_{\mathcal{A}}(\psi) \). In this case, \( \psi \) is called supporting functional.

The role of support points for attainability in the dual representation (33) is clear. Indeed, assume \((\mathcal{A}, \mathcal{I})\) is a convex risk measurement regime and \( U \in \mathcal{I} \) is a positive element with \( \pi(U) = 1 \). Then, under the assumptions of Theorem 4.16, the supremum in (33) is attained at \( X \) with \( \rho_{\mathcal{A},\mathcal{I},\pi}(X) \in \mathbb{R} \) if and only if \( X + \rho_{\mathcal{A},\mathcal{I},\pi}(X)U \) is a support point of \( \mathcal{A} + \mathcal{I}_0 \) with supporting functional in \( \mathcal{E}_\pi(\mathcal{I}) \). The following example shows that the supremum may be attained for some positions but not for others.
Example 4.19. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be an infinite probability space and take \(\mathcal{A} := L^1\) and \(\mathcal{A} := L^1_+\). Let \(U\) be an arbitrary nonzero positive payoff and set \(\pi(U) := 1\). Hence, the coherent risk measure \(\rho_{\mathcal{A}, \mathcal{F}, \mathbb{P}}\) is globally lower semicontinuous. Since \(\rho_{\mathcal{A}, \mathcal{F}, \mathbb{P}}(0) = 0\), Proposition 4.13 implies that \(\mathcal{E}(\mathcal{A}) \cap B(\mathcal{A})\) is nonempty. Using the representation formula in Corollary 4.18 we conclude that the supremum in (33) is attained at \(X = 0\). Consider now \(X \in L^1_+\) such that \(X > 0\) almost surely and assume, in addition, that \(\mathbb{P}(X < \lambda U) > 0\) for each \(\lambda > 0\). Since \(\rho_{\mathcal{A}, \mathcal{F}, \mathbb{P}}(X) = 0\) we have \(X + \rho_{\mathcal{A}, \mathcal{F}, \mathbb{P}}(X)U = X\). However, as is easily seen, \(X\) is not a support point of \(L^1_+\). Hence, the supremum in the dual representation is not attained at \(X\).

The next result shows that if \((\mathcal{A}, \mathcal{F})\) is a convex risk measurement regime, then the supremum in formula (33) is attained at every point where \(\rho_{\mathcal{A}, \mathcal{F}, \mathbb{P}}\) is continuous.

**Proposition 4.20.** Assume \((\mathcal{A}, \mathcal{F})\) is a convex risk measurement regime. Then for any point of finiteness and continuity \(X \in \mathcal{A}\) we have

\[
\rho_{\mathcal{A}, \mathcal{F}, \mathbb{P}}(X) = \max_{\psi \in \mathcal{E}(\mathcal{A})} \{\sigma_{\mathcal{A}}(\psi) - \psi(X)\}.
\] (38)

**Proof.** Take a positive payoff \(U \in \mathcal{F}\) with \(\pi(U) = 1\), and let \(X\) be a point of finiteness and continuity for \(\rho_{\mathcal{A}, \mathcal{F}, \mathbb{P}}\). Together with Proposition 6.1 this implies \(\text{int}(\mathcal{A} + \mathcal{F}_0) \neq \emptyset\) and \(X + mU \in \text{int}(\mathcal{A} + \mathcal{F}_0)\) for every \(m > \rho_{\mathcal{A}, \mathcal{F}, \mathbb{P}}(X)\). Fix such an \(m\). Since \(X + \rho_{\mathcal{A}, \mathcal{F}, \mathbb{P}}(X)U\) belongs to the boundary of \(\mathcal{A} + \mathcal{F}_0\), it follows from Lemma 7.7 in [1] that it is also a support point of \(\mathcal{A} + \mathcal{F}_0\). Let \(\psi \in \mathcal{F}'\) be the corresponding supporting functional which, by Lemma 2.2, must be positive. Since \(\psi(X + \rho_{\mathcal{A}, \mathcal{F}, \mathbb{P}}(X)U) = \sigma_{\mathcal{A} + \mathcal{F}_0}(\psi)\), we must have \(\psi \in \mathcal{F}'_0\). Moreover \(\psi(U) > 0\), as otherwise \(\psi(X + mU) = \psi(X + \rho_{\mathcal{A}, \mathcal{F}, \mathbb{P}}(X)U) = \sigma_{\mathcal{A} + \mathcal{F}_0}(\psi)\) contradicting the fact that \(X + mU \in \text{int}(\mathcal{A} + \mathcal{F}_0)\). Hence we can assume that \(\psi(U) = 1\), concluding that \(\rho_{\mathcal{A}, \mathcal{F}, \mathbb{P}}(X) = \sigma_{\mathcal{A}}(\psi) - \psi(X)\).

5 Applications

In this final section we apply our previous results to the conical market model adopted by Hamel, Heyde, and Rudloff in [16] as the underlying setting for their set-valued risk measures, and to the market model considered by Arai in [2] in connection to superhedging problems. Moreover, we show that multi-asset risk measures appear naturally in the context of optimal risk sharing across different business lines.

5.1 Risk measures on conical market models

In this section we investigate the link between multi-asset risk measures and set-valued risk measures as introduced by Hamel, Heyde and Rudloff in [16]. We start by briefly recalling the setting of that paper. We consider a one-period economy with dates \(t = 0\) and \(t = T\) where uncertainty is captured by a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We will consider risk measures for random portfolios of \(d\) traded assets. Random portfolios are described by random vectors \(X = (X_1, \ldots, X_d)^t \in L^p_{\mathbb{R}^d}\) for some \(1 \leq p \leq \infty\), where \(X_i \in L^p\) represents the (random) number of units of asset \(i\) held at time \(T\). The space \(L^p_{\mathbb{R}^d}\) inherits the order structure of \(\mathbb{R}^d\) in the almost surely sense. The topology on \(L^p_{\mathbb{R}^d}\) is the usual norm topology if \(p < \infty\) and the \(\sigma(L^\infty_{\mathbb{R}^d}, L^1_{\mathbb{R}^d})\) topology if \(p = \infty\).
Let $A \subset L^p_d$ be an acceptance set. A random portfolio can be made acceptable by adding “eligible” deterministic portfolios at time 0. The space of eligible portfolios is represented by a subspace $M \subset \mathbb{R}^d$, which is identified with the subspace of $L^p_d$ given by $\{(u_11_\Omega, \ldots, u_d1_\Omega)^t; \ u \in M\}$. Following [16], we assume that $M$ contains some nonzero positive element so that $(A, M)$ is a risk measurement regime. The associated set-valued risk measure is defined as the set-valued map

$$R_A(X) := \{u \in M; \ X + u \in A\}.$$  

(39)

For set-valued risk measures to be useful as capital requirements it is necessary to specify a procedure through which, roughly speaking, an optimal element in $R_A(X)$ can be chosen. Such a procedure is called scalarization and is explained in Section 5 of [16] and, in greater detail, in [15]. To illustrate this procedure, we recall the notion of the solvency cone.

Let $(\sigma_{ij}^0)$ be a $d \times d$ bid-ask matrix as defined by Schachermayer [23], i.e. $\sigma_{ij}^0$ represents the number of units of asset $i$ that are required to purchase one unit of asset $j$ at time 0. The solvency cone $K_0$ at time 0 is the convex cone in $\mathbb{R}^d$ consisting of all solvent deterministic portfolios, i.e. all portfolios $u \in \mathbb{R}^d$ admitting a $d \times d$ matrix $(\alpha_{ij})$ with $\alpha_{ij} \geq 0$ and $\alpha_{ii} = 0$ for $i, j = 1, \ldots, d$ such that

$$u_i + \sum_{j=1}^d \alpha_{ij} \sigma_{ij}^0 - \sum_{j=1}^d \alpha_{ji} \geq 0 \quad \text{for all } i = 1, \ldots, d,$$

(40)

where the coefficient $\alpha_{ij}$ is to be interpreted as the number of units of asset $j$ that are used to modify the position $i$ in the portfolio. Hence, the solvency cone $K_0$ contains all portfolios which can be converted at time 0 into portfolios with non-negative components.

The elements in the polar of the cone $-K_0$

$$K_0^+ := \left\{ \xi \in \mathbb{R}^d; \ \sum_{i=1}^d \xi_i u_i \geq 0, \ \forall \ u \in K_0 \right\}$$

(41)

are called consistent pricing systems, and are easily seen to be strictly positive vectors when nonzero. The scalarization of a set-valued risk measure $R_A$ at $X \in L^p_d$ by means of $\xi \in K_0^+$ is then defined as

$$\varphi_{R_A, \xi}(X) := \inf \left\{ \sum_{i=1}^d \xi_i u_i; \ u \in R_A(X) \right\}.$$  

(42)

We start by showing the intimate link between scalarized set-valued risk measures and multi-asset risk measures. This allows for a fruitful exchange of results between the theory developed in [16] and our multi-asset framework. In particular, we show that scalarized set-valued risk measures appear naturally as multi-asset risk measures. Hence, the results in this paper can be used to prove finiteness and continuity properties and to provide dual representations for scalarized set-valued risk measures. Note that, in this respect, only basic results are provided in [16].

**Proposition 5.1.** For every $\xi \in K_0^+$ there exists a linear functional $\pi : M \to \mathbb{R}$ such that

$$\varphi_{R_A, \xi}(X) = \rho_{A, M, \pi}(X)$$

(43)

for every $X \in L^p_d$.  

20
Proof. Define \( \pi : M \to \mathbb{R} \) by \( \pi(u) := \sum_{i=1}^{d} \xi_i u_i \). Then it is immediate to see that
\[
\varphi_{RA,\xi}(X) = \inf\{\pi(u) : u \in M : X + u \in A\} = \rho_{A,M,\pi}(X)
\] (44)
holds for every \( X \in L^p_d \).

A less compelling converse of the previous result is also possible if we consider multi-asset risk measures on \( L^p \) with respect to finite-dimensional eligible spaces.

**Proposition 5.2.** Let \((\mathcal{A}, \mathcal{F})\) be a risk measurement regime in \( L^p \), \( 1 \leq p \leq \infty \), and assume \( \dim(\mathcal{F}) = d \). Then there exist an acceptance set \( A \subset L^p_{d+1} \), a linear space \( M \subset \mathbb{R}^{d+1} \), a convex cone \( K_0 \subset \mathbb{R}^{d+1} \) and \( \xi \in K_0^+ \) such that
\[
\rho_{\mathcal{A},\mathcal{F},\pi}(X) = \varphi_{RA,\xi}((0, \ldots, 0, X)^t)
\] (45)
for every \( X \in L^p \).

**Proof.** Let \( \mathcal{F} \) be the span of \( Z_1, \ldots, Z_d \in L^p \), and define the acceptance set
\[
A := \left\{ X \in L^p_{d+1} : \sum_{i=1}^{d} X_i Z_i + X_{d+1} \in \mathcal{A} \right\}.
\] (46)
Moreover, set \( M := \{ u \in \mathbb{R}^{d+1} : u_{d+1} = 0 \} \) and \( K_0 := \{ u \in \mathbb{R}^{d+1} : \sum_{i=1}^{d} \pi(Z_i) u_i + u_{d+1} \geq 0 \} \). Taking \( \xi := (\pi(Z_1), \ldots, \pi(Z_d), 1)^t \in K_0^+ \), it is easy to see that (45) holds for every \( X \in L^p \).

**Remark 5.3.** The transition from a scalarized set-valued risk measure to a multi-asset risk measure in (43) is fairly natural in that the original underlying framework can be retained: the underlying space, the acceptance set, and the eligible space do not change. By contrast, the transition from a multi-asset risk measure to a scalarized set-valued risk measure seems to be more “formal” in character since we need to artificially enlarge the dimension of the eligible space by a “cash” asset and define a new acceptance set. Moreover, the new cash asset essentially plays the role of a numéraire asset since all positions \( X \in L^p \) are expressed as random vectors \((0, \cdots, 0, X)^t\).

Next, we provide a dual representation for scalarized set-valued risk measures based on the multi-asset dual representation (33). As usual, we identify the dual of \( L^p_d \) with the space \( L^q_d \) where \( 1 \leq q \leq \infty \) satisfies \( 1/p + 1/q = 1 \). Moreover, we denote by \( \mathcal{P}_d^q \) the set of all \( d \)-dimensional vectors \( \tilde{Q} \) whose components \( Q_i \) are probability measures on \((\Omega, \mathcal{F})\) which are absolutely continuous with respect to \( \mathbb{P} \) and such that \( \frac{dQ_i}{d\mathbb{P}} \in L^q \).

**Proposition 5.4.** Assume \( A \subset L^p_d \) is convex. Let \( \xi \in K_0^+ \) and define \( \pi(u) := \sum_{i=1}^{d} \xi_i u_i \) on \( M \). If \( \mathcal{E}_\pi(M) \cap B(A) \neq \emptyset \) and \( \varphi_{RA,\xi} \) is lower semicontinuous at \( X \), then
\[
\varphi_{RA,\xi}(X) = \sup_{(Q, w) \in \mathcal{D}^q} \left\{ \sum_{i=1}^{d} w_i E_{Q_i}[-X_i] + \inf_{Y \in A} \sum_{i=1}^{d} w_i E_{Q_i}[Y_i] \right\}
\] (47)
where
\[
\mathcal{D}^q := \left\{ (Q, w) \in \mathcal{P}_d^q \times \mathbb{R}_d^+ : \sum_{i=1}^{d} w_i u_i = \sum_{i=1}^{d} \xi_i u_i, \forall u \in M \right\}.
\] (48)
Proof. By (33) and (43) we immediately obtain
\[ \varphi_{RA,\xi}(X) = \sup_{\psi \in \mathcal{E}_\pi(M)} \{ \sigma_A(\psi) - \psi(X) \}. \] (49)

Note that every functional \( \psi \) in \( \mathcal{E}_\pi(M) \) can be identified with a positive element \( W \in L_d^P \) such that \( \psi(X') = \sum_{i=1}^d E[W_iX'_{i}] \) for every \( X' \in L_d^P \) and \( \sum_{i=1}^d w_iE[W_i] = \sum_{i=1}^d \xi_iu_i \) for all \( u \in M \). In turn, every \( W \) of this form can be identified with \( (\xi, w) \) in \( \mathcal{D}^q \) by setting \( w_i := E[W_i] \) and \( \frac{d\xi}{dp_i} := \frac{1}{w_i} W_i \), or \( Q_i := \mathbb{P} \) if \( w_i = 0 \), for any \( i = 1, \ldots, d \). This concludes the proof. \[ \square \]

The random character of the market at time \( t = T \) is reflected by the fact that the bid-ask matrix \( (\sigma_{ij}^T) \) at time \( T \) is random. When defining the corresponding “random” solvency cone \( K_T \) at time \( T \) we may proceed as in (40) but requiring that the “transition” matrix \( (\alpha_{ij}) \) is also random. The convex cone \( K_T^+ \) is defined analogously. Using the notation of [16], we denote by \( L_d^P(K_T) \), respectively \( L_d^P(K_T^+) \), the convex cone of all \( X \in L_d^P \) such that \( X \in K_T \), respectively \( X \in K_T^+ \), almost surely.

Consider an acceptance set \( A \subset L_d^P \), and let \( X \in L_d^P \) be a random portfolio of assets. From a capital adequacy perspective, if we want to account for the possibility of trading at time \( t = T \) in order to adjust for acceptability it is natural to consider the set
\[ R_{A+L_d^P(K_T)}(X) = \{ u \in M ; \ X + u \in A + L_d^P(K_T) \}. \] (50)

Indeed, the condition \( X + u \in A + L_d^P(K_T) \) means that \( X + u \) will be exchangeable at time \( t = T \) into an acceptable portfolio, after paying the transaction costs defined by \( K_T \). For this reason, the authors in [16] have considered acceptance sets \( A \) which are \( K_T \)-compatible, i.e. such that \( A = A + L_d^P(K_T) \). We conclude this section providing a dual representation for scalarized set-valued risk measures satisfying this compatibility condition.

**Corollary 5.5.** Assume \( A \subset L_d^P \) is convex and \( K_T \)-compatible. For \( \xi \in K_0^+ \) define \( \pi(u) := \sum_{i=1}^d \xi_iu_i \) on \( M \). If \( \mathcal{E}_\pi(M) \cap B(A) \neq \emptyset \) and \( \varphi_{RA,\pi} \) is lower semicontinuous at \( X \), then
\[ \varphi_{RA,\pi}(X) = \sup_{(Q, w) \in \mathcal{D}^q(K_T)} \left\{ \sum_{i=1}^d w_iE_{Q_i}[ -X_i ] + \inf_{Y \in A} \sum_{i=1}^d w_iE_{Q_i}[Y_i] \right\} \] (51)

where
\[ \mathcal{D}^q(K_T) := \left\{ (Q, w) \in \mathcal{D}^q ; \left( w_1 \frac{dQ_1}{dp_1}, \ldots, w_d \frac{dQ_d}{dp_d} \right)^t \in L_d^P(K_T^+) \right\} \] (52)

**Proof.** Since \( A \) is \( K_T \)-compatible, we have \( \sigma_A = \sigma_A + \sigma_{L_d^P(K_T)} \). Hence, we can restrict the optimization domain in (49) to all functionals \( \psi \in \mathcal{E}_\pi(M) \cap B(L_d^P(K_T)) \). But \( L_d^P(K_T) \) is a cone and therefore \( \psi \in B(L_d^P(K_T)) \) if and only if \( \sigma_{L_d^P(K_T)}(\psi) = 0 \), or equivalently \( \psi(Z) \geq 0 \) for every \( Z \in L_d^P(K_T) \). Using the dual variables in (47), we see that this is equivalent to replacing \( \mathcal{D}^q \) with \( \mathcal{D}^q(K_T) \). \[ \square \]
5.2 Optimal risk sharing with multi-asset risk measures

In this section we show how multi-asset risk measures arise naturally in the context of optimal risk sharing amongst several business lines. We focus on two business lines for ease of notation, the extension to more than two business lines being straightforward.

Throughout this section $\mathcal{X}$ is a locally convex ordered topological vector space, $\mathcal{M}$ the marketed subspace and $\pi : \mathcal{M} \to \mathbb{R}$ the pricing functional. As above, we assume that the market is free of arbitrage by requiring $\pi$ to be strictly positive. Assume two business lines have different types of capital requirements represented respectively by $\rho_{\mathcal{A}, \pi}$ and $\rho_{\mathcal{B}, \pi}$ where $\mathcal{A}$ and $\mathcal{B}$ are acceptance sets in $\mathcal{X}$ and $U$ and $V$ are nonzero positive payoffs in $\mathcal{M}$. We assume a “risk” $X \in \mathcal{X}$ can be shared amongst the two business lines, i.e. for any $Y \in \mathcal{X}$ we can assign $Y$ to the first and $X - Y$ to the second business line. The total required capital is then $\rho_{\mathcal{A}, \pi}(Y) + \rho_{\mathcal{B}, \pi}(X - Y)$. Optimal risk sharing is about finding the optimal $Y$. Hence, one naturally arrives at the infimal convolution between $\rho_{\mathcal{A}, \pi}$ and $\rho_{\mathcal{B}, \pi}$ at $X$ given by

$$\rho_{\mathcal{A}, \pi} \square \rho_{\mathcal{B}, \pi}(X) := \inf \{ \rho_{\mathcal{A}, \pi}(Y) + \rho_{\mathcal{B}, \pi}(X - Y) ; Y \in \mathcal{X} \} .$$

(53)

This quantity represents the “minimum” total required capital across all possible allocations $(Y, X - Y)$ of the aggregated position $X$. For more details on the applications of infimal convolutions in the theory of risk measures we refer to Barrieu and El Karoui [6].

First, we show that the infimal convolution of single-asset risk measures can be expressed as a multi-asset risk measure. Recall that a map $\rho : \mathcal{X} \to \mathbb{R}$ is said to be proper if it cannot assume the value $-\infty$ and its effective domain is nonempty.

**Proposition 5.6.** Assume $\rho_{\mathcal{A}, \pi}$ and $\rho_{\mathcal{B}, \pi}$ are proper. If $\mathcal{I}$ is the span of $U$ and $V$, then for all $X \in \mathcal{X}$

$$\rho_{\mathcal{A}, \pi} \square \rho_{\mathcal{B}, \pi}(X) = \rho_{\mathcal{A} + \mathcal{B}, \pi, \mathcal{I}}(X).$$

(54)

**Proof.** Take $X \in \mathcal{X}$. To show that $\rho_{\mathcal{A}, \pi} \square \rho_{\mathcal{B}, \pi}(X) \leq \rho_{\mathcal{A} + \mathcal{B}, \pi, \mathcal{I}}(X)$, assume $X + Z \in \mathcal{A} + \mathcal{B}$ for some $Z \in \mathcal{I}$. Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $X + Z = A + B$. Since $Z = \alpha U + \beta V$ for some $\alpha, \beta \in \mathbb{R}$, we obtain

$$\rho_{\mathcal{A}, \pi} \square \rho_{\mathcal{B}, \pi}(X) \leq \rho_{\mathcal{A} + \mathcal{B}, \pi, \mathcal{I}}(A - \alpha U + \rho_{\mathcal{B}, \pi}(B - \beta V) \leq \pi(\alpha U) + \pi(\beta V) = \pi(Z) .$$

(55)

The inequality follows by taking the infimum over all $Z \in \mathcal{I}$ with $X + Z \in \mathcal{A} + \mathcal{B}$. To show the converse inequality, assume $Y + \alpha U \in \mathcal{A}$ and $X - Y + \beta V \in \mathcal{B}$ for some $Y \in \mathcal{X}$, $\alpha, \beta \in \mathbb{R}$. Then, clearly, $X + \alpha U + \beta V \in \mathcal{A} + \mathcal{B}$ and, therefore, $\rho_{\mathcal{A} + \mathcal{B}, \pi, \mathcal{I}}(X) \leq \pi(\alpha U) + \pi(\beta V)$. Taking the infimum over all $\alpha$ and $\beta$ such $Y + \alpha U \in \mathcal{A}$ and $X - Y + \beta V \in \mathcal{B}$ yields $\rho_{\mathcal{A} + \mathcal{B}, \pi, \mathcal{I}}(X) \leq \rho_{\mathcal{A}, \pi}(Y) + \rho_{\mathcal{B}, \pi}(X - Y)$. We obtain the desired inequality after taking the infimum over all $X \in \mathcal{X}$. $\square$

The following corollary follows immediately from the preceding result and shows that every multi-asset risk measure with respect to a finite dimensional eligible space is in fact an infimal convolution of single-asset risk measures.

23
Corollary 5.7. Assume $\rho_{\mathcal{A},U,\pi}$ and $\rho_{\mathcal{A},V,\pi}$ are proper, and let $\mathcal{F}$ be the span of $U$ and $V$. Then for every $X \in \mathcal{X}$

$$\rho_{\mathcal{A},\mathcal{F},\pi}(X) = \rho_{\mathcal{A},U,\pi} \square \rho_{\mathcal{A},V,\pi}(X).$$

(56)

If $\mathcal{A}$ is coherent, then for all $X \in \mathcal{X}$

$$\rho_{\mathcal{A},\mathcal{F},\pi}(X) = \rho_{\mathcal{A},U,\pi} \square \rho_{\mathcal{A},V,\pi}(X).$$

(57)

As a final result, we provide a dual representation for infimal convolutions of convex single-asset risk measures.

Proposition 5.8. Assume $\mathcal{A}$ and $\mathcal{B}$ are convex with $\text{int}(\mathcal{A})$ nonempty. Let $\rho_{\mathcal{A},U,\pi}$ and $\rho_{\mathcal{B},V,\pi}$ be proper. Moreover, assume the span $\mathcal{F}$ of $U$ and $V$ contains a strictly positive element. If $\rho_{\mathcal{A},U,\pi} \square \rho_{\mathcal{B},V,\pi}(0) > -\infty$, then for all $X \in \mathcal{X}$

$$\rho_{\mathcal{A},U,\pi} \square \rho_{\mathcal{B},V,\pi}(X) = \max_{\psi \in \mathcal{D}} \left\{ \sigma_{\mathcal{A}}(\psi) + \sigma_{\mathcal{B}}(\psi) - \psi(X) \right\}.$$  

(58)

In particular, if $\mathcal{A}$ and $\mathcal{B}$ are coherent then for all $X \in \mathcal{X}$

$$\rho_{\mathcal{A},U,\pi} \square \rho_{\mathcal{B},V,\pi}(X) = \max_{\psi \in \mathcal{D}} \psi(-X)$$

(59)

where

$$\mathcal{D} := \left\{ \psi \in E_\pi(\mathcal{F}) : \psi(Y) \geq 0, \forall Y \in \mathcal{A} \cup \mathcal{B} \right\}.$$  

(60)

Proof. Note that $\mathcal{A} + \mathcal{B}$ is a convex acceptance set with nonempty interior. Then Proposition 3.6 and Proposition 5.6 imply $\rho_{\mathcal{A},U,\pi} \square \rho_{\mathcal{B},V,\pi}$ is continuous. Hence, (58) follows from Proposition 4.20. The representation (59) in the coherent case is a consequence of Corollary 4.18 and the fact that $B(\mathcal{A} + \mathcal{B}) = B(\mathcal{A}) \cap B(\mathcal{B})$.

5.3 Shortfall risk measures and superhedging price

In this section we focus on the superhedging problem studied by Arai in [2]. Based on our approach to dual representations, we provide a sharper dual representation of the superhedging price defined in that paper. For the background on Orlicz hearts and Orlicz spaces we refer to [9].

Fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, for $0 \leq t \leq T$, and let $S = (S_t)$ be a $d$-dimensional semimartingale representing the price dynamics of $d$ given assets. We assume $(\mathcal{F}_t)$ satisfies the usual conditions and $\mathcal{F}_T = \mathcal{F}$. Denote by $\Theta$ a convex set of $d$-dimensional, predictable, $S$-integrable processes $(\vartheta_t)$. The elements of $\Theta$ will be called admissible strategies. In addition, consider a loss function $\ell : \mathbb{R} \to \mathbb{R}$, which is assumed to be nonconstant, increasing, convex, and such that $\ell(0) = 0$. The reference space is taken to be the Orlicz heart $H^\Phi$ associated to the Orlicz function $\Phi$ defined by $\Phi(x) := \ell(|x|)$ for $x \in \mathbb{R}$.

From now on, we assume that $\int_0^T \vartheta_t dS_t \in H^\Phi$ for every admissible strategy $\vartheta \in \Theta$.

The main goal in Arai [2] is to provide dual representations, under specific assumptions on the class $\Theta$, for the map $\rho_\ell : H^\Phi \to \overline{\mathbb{R}}$ defined by

$$\rho_\ell(X) := \inf \left\{ m \in \mathbb{R} : \exists \vartheta \in \Theta : \mathbb{E} \left[ \ell \left( -X - m - \int_0^T \vartheta_t dS_t \right) \right] \leq \alpha \right\}.$$  

(61)
where $\alpha > 0$ is a pre-specified loss level. Note that the map $\rho_\ell$ can be regarded as a single-asset risk measure. Indeed, if we introduce the convex acceptance set

$$\mathcal{A}_\ell := \{X \in H^\Phi; \, \mathbb{E}[\ell(-X)] \leq \alpha\}$$  \hspace{1cm} (62)

and the convex set

$$\mathcal{C} := \left\{ \int_0^T \vartheta_t dS_t; \, \vartheta \in \Theta \right\},$$  \hspace{1cm} (63)

it is easy to see that $\rho_\ell = \rho_{\mathcal{A}_\ell - \mathcal{C}, U, \pi}$ with $U := 1_\Omega$ and $\pi(U) := 1$.

The financial motivation is given by the fact that the quantities $-\rho_\ell(X)$ and $\rho_\ell(-X)$ can be interpreted as pricing bounds for the claim $X \in H^\Phi$ which are compatible with the absence of “good deals”. We refer to [2] for a detailed explanation.

The main dual representation provided under the standing assumption $\rho_\ell(0) > -\infty$ is Proposition 3.5 in [2]. This representation is then specified to various situations including the case where $\Theta$ is a linear space (Section 4 in [2]) and $\Theta$ is the convex cone of $W$-admissible strategies (Section 5 in [2]). The key ingredient for the dual representation in the $W$-admissible case is Lemma 5.1 in that paper.

Our objective is to show that this key lemma holds for any choice of the class of admissible strategies $\Theta$. As a result, we provide a general dual representation sharpening Proposition 3.5 in [2].

Given a map $f : \mathbb{R} \to \mathbb{R}$, we denote by $f^* : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ the Fenchel conjugate of $f$ defined by

$$f^*(y) := \sup_{x \in \mathbb{R}} \{xy - f(x)\}. \hspace{1cm} (64)$$

Recall that the dual space $(H^\Phi)^*$ can be identified with the Orlicz space $L^{\Phi^*}$. Moreover, we denote by $\mathcal{P}^\Phi$ the set of all probability measures $Q$ on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to $\mathbb{P}$ and such that $dQ/d\mathbb{P} \in L^{\Phi^*}$.

**Proposition 5.9.** Assume $\rho_\ell(0) > -\infty$. The risk measure $\rho_\ell$ is finitely valued and continuous on $H^\Phi$. Moreover, for every $X \in H^\Phi$ we have

$$\rho_\ell(X) = \max_{Q \in \mathcal{P}^\Phi} \left\{ \mathbb{E}_Q[-X] - \sup_{\vartheta \in \Theta} \mathbb{E}_Q \left[ \int_0^T \vartheta_t dS_t \right] - \inf_{\lambda > 0} \frac{1}{\lambda} \left\{ \alpha + \mathbb{E} \left[ \ell^*(\lambda dQ/d\mathbb{P}) \right] \right\} \right\}. \hspace{1cm} (65)$$

**Proof.** Since $\alpha > 0$, it follows from Lemma 4.8 in [11] that $\mathcal{A}_\ell$ has nonempty interior, hence the convex acceptance set $\mathcal{A}_\ell - \mathcal{C}$ also has nonempty interior. Moreover, note that $1_\Omega$ is strictly positive in $H^\Phi$. Since $\rho_\ell(0) > -\infty$, we can apply Proposition 3.6 to ensure that $\rho_\ell$ is finitely valued and continuous. Furthermore, by the dual representation in Proposition 4.20 we have for all $X \in H^\Phi$

$$\rho_\ell(X) = \max_{\psi \in (H^\Phi)^*, \psi(1_\Omega) = 1} \left\{ \sigma_{\mathcal{A}_\ell}(\psi) + \sigma_\mathcal{C}(\psi) - \psi(X) \right\}$$

$$\begin{align*}
= & \max_{Q \in \mathcal{P}^\Phi} \left\{ -\inf_{\lambda > 0} \frac{1}{\lambda} \left\{ \alpha + \mathbb{E} \left[ \ell^*(\lambda dQ/d\mathbb{P}) \right] \right\} + \inf_{\vartheta \in \Theta} \mathbb{E}_Q \left[ -\int_0^T \vartheta_t dS_t \right] - \mathbb{E}_Q [X] \right\}.
\end{align*}$$
Indeed, every functional $\psi \in (H^\Phi)'_+$ with $\psi(1_\Omega) = 1$ can be identified with some $Q \in \mathcal{P}^\Phi$ such that $\psi(X') = E_Q[X']$ for all $X' \in H^\Phi$. The equivalent formulation for the term $\sigma_{\mathcal{A}_t}$ follows from Theorem 10 in [12], whose proof extends easily to the space $H^\Phi$.

Remark 5.10. The above result shows that, even in the familiar context of cash-additive risk measures, our approach based on support functions provides a direct insight into their dual representation. Indeed, Proposition 3.5 in [2] provides a dual representation for $\rho_t$ which require the introduction of two auxiliary set ($\tilde{A}$ and $A_1$). As highlighted by our dual representation (65), which is derived directly using the structure of $A - C$, the introduction of these two sets turns out not to be necessary and results in a non-optimal representation.

6 Appendix: Semicontinuity properties

In this appendix we list for easy reference a variety of (semi)continuity result for multi-asset risk measures. Recall that a function $\rho : \mathcal{X} \to \mathbb{R}$ is said to be lower semicontinuous at a point $X \in \mathcal{X}$ if for every $\varepsilon > 0$ there exists a neighborhood $\mathcal{W}$ of $X$ such that $\rho(Y) \geq \rho(X) - \varepsilon$ for all $Y \in \mathcal{W}$. We say that $\rho$ is (globally) lower semicontinuous if it is lower semicontinuous at every $X \in \mathcal{X}$. Upper semicontinuity properties of $\rho$ are defined by requiring the corresponding lower semicontinuity properties of $-\rho$. Note finally that a function $\rho : \mathcal{X} \to \mathbb{R}$ is continuous at a point $X \in \mathcal{X}$ if and only it is both lower and upper semicontinuous at $X$.

The following characterization of semicontinuity follows from a corresponding result for single-asset risk measures in [11] by means of the Reduction Lemma and is stated without proof.

**Proposition 6.1.** Assume $(\mathcal{A}, \mathcal{S})$ is a risk measurement regime. The following statements hold:

(i) $\rho_{\mathcal{A}, \mathcal{S}, \pi}$ is lower semicontinuous at $X$ if and only if $X + Z \notin \mathcal{A} + \mathcal{S}_0$ for any $Z \in \mathcal{S}$ with $\pi(Z) < \rho_{\mathcal{A}, \mathcal{S}, \pi}(X)$;

(ii) $\rho_{\mathcal{A}, \mathcal{S}, \pi}$ is upper semicontinuous at $X$ if and only if $X + Z \in \text{int}(\mathcal{A} + \mathcal{S}_0)$ for any $Z \in \mathcal{M}(\mathcal{S})$ with $\pi(Z) > \rho_{\mathcal{A}, \mathcal{S}, \pi}(X)$.

The following example shows that risk measures can be lower semicontinuous at some points without being globally lower semicontinuous.

**Example 6.2.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a nonatomic probability space. The set

$$\mathcal{A} := \{X \in L^1 ; \ E(X) \geq 0, \ \exists \ \lambda \in \mathbb{R} : X \geq \lambda\}$$

(66)

is a convex acceptance set. Note that $\mathcal{A}$ has empty interior and $\overline{\mathcal{A}} = \{X \in L^1 ; \ E(X) \geq 0\}$. Consider the payoff $U := 1_\Omega$, and set $\pi(U) := 1$. The risk measure $\rho_{\mathcal{A}, U, \pi}$ is not (globally) lower semicontinuous. Otherwise, $\rho_{\mathcal{A}, U, \pi}(X) = \rho_{\overline{\mathcal{A}}, U, \pi}(X) = -E[X]$ for all $X \in L^1$ as a consequence of the above proposition, implying that $\rho_{\mathcal{A}, U, \pi}$ is continuous. But this is not possible because $\mathcal{A}$ has empty interior and thus $\rho_{\mathcal{A}, U, \pi}$ cannot be (globally) upper semicontinuous, again due to the preceding proposition. However, $\rho_{\mathcal{A}, U, \pi}$ is lower semicontinuous at 0 since $\rho_{\mathcal{A}, U, \pi}(0) = 0$ and $mU \notin \overline{\mathcal{A}}$ for all $m < 0$. 

26
Given a risk measurement regime \((\mathcal{A}, \mathcal{F})\), it follows from Proposition 6.1 that \(\rho_{\mathcal{A}, \mathcal{F}, \pi}\) is always lower semicontinuous whenever the “augmented” acceptance set \(\mathcal{A} + \mathcal{F}_0\) is closed. In case \(\mathcal{F}\) is one dimensional we have \(\mathcal{F}_0 = \{0\}\), hence for single-asset risk measures the closedness of \(\mathcal{A}\) itself implies lower semicontinuity. Unfortunately, as the following example shows, closedness of \(\mathcal{A}\) no longer suffices to ensure lower semicontinuity in the multi-asset case.

**Example 6.3.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be an infinite probability space and take \(A \subset \Omega\) with \(0 < \mathbb{P}(A) < 1\). Let \(\mathcal{X} := L^1\) and \(\mathcal{A} := L^1_{\mathbb{F}}\) and assume \(U \in L^1_{\mathbb{F}}\) is unbounded from above on \(A\) and equal to 1 on \(A^c\). Taking \(\mathcal{F}\) to be the vector space spanned by \(U\) and \(V := U + 1_{A}\), define \(\pi: \mathcal{F} \rightarrow \mathbb{R}\) by setting \(\pi(U) = \pi(V) := 1\). It is easy to see that \(\mathcal{A} + \mathcal{F}_0\) consists of all positions \(X\) that are bounded from below on \(A\) and that are non-negative on \(A^c\). Therefore \(\mathcal{A} + \mathcal{F}_0 = \{X \in L^1; X1_{A^c} \geq 0\}\). Now, take \(X := -U1_A\). Then \(\rho_{\mathcal{A}, \mathcal{F}, \pi}(X) = 1\) while \(\rho_{\mathcal{A} + \mathcal{F}_0, U, \pi}(X) = 0\), hence \(\rho_{\mathcal{A}, \mathcal{F}, \pi}\) is not lower semicontinuous at \(X\) as a consequence of the Reduction Lemma and Proposition 6.1.

In the next result we provide a sufficient condition for \(\mathcal{A} + \mathcal{F}_0\) to be closed in case \(\mathcal{A}\) is also closed, ensuring the lower semicontinuity of the associated risk measure.

**Proposition 6.4.** Let \(\mathcal{A} \subset \mathcal{X}\) be a closed acceptance set with \(0 \in \mathcal{A}\). Assume that \(\mathcal{A}\) is either a cone or convex and let \(\mathcal{F}\) be a finite dimensional subspace of \(\mathcal{M}\). If \(\mathcal{A} \cap \mathcal{F}_0 \setminus \{0\}\) is closed, then \(\mathcal{A} + \mathcal{F}_0\) is closed.

**Proof.** If \(\mathcal{F}_0 = \{0\}\) the assertion is trivial. Assume \(\mathcal{F}_0\) is generated by a nonzero payoff \(Z \in \mathcal{M}\). Let \((A_{\alpha} + \lambda_{\alpha}Z)\) be a net in \(\mathcal{A} + \mathcal{F}_0\) converging to some \(X \in \mathcal{X}\). If \((\lambda_{\alpha})\) is unbounded, then we can find a subnet \((\lambda_{\beta})\) diverging to either \(\infty\) or \(-\infty\). Moreover, without loss of generality, we may assume \(|\lambda_{\beta}| > 1\).

It follows that \((A_{\beta}/\lambda_{\beta})\) has limit \(-Z\), respectively \(Z\). Since \(A_{\beta}/\lambda_{\beta}\) belongs to \(\mathcal{A}\), respectively \(-\mathcal{A}\), by the closedness of \(\mathcal{A}\) we conclude that \(\mathcal{A} \cap \mathcal{F}_0\) contains a nonzero element, contradicting the assumption. Therefore \((\lambda_{\alpha})\) must be bounded. Passing to a converging subnet, it is easy to show that the limit \(X\) lies in \(\mathcal{A}\), implying \(\mathcal{A} + \mathcal{F}_0\) is closed. We can conclude by induction on the dimension of \(\mathcal{F}_0\).

**Remark 6.5.** Note that the condition \(\mathcal{A} \cap \mathcal{F}_0 \setminus \{0\} = \emptyset\) in the above result is equivalent to the absence of good deals (of the first kind) introduced by Jaschke and Küchler [18] in a pricing context.

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