Why does Implied Risk Aversion smile?

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Abstract: A few recent papers have derived estimates of the representative agent’s risk aversion by comparing the statistical density of asset returns and the state-price density. The implied risk aversion estimates obtained in these studies are puzzling, exhibiting (i) pronounced U-shaped patterns (a “smile”) and (ii) negative values. This paper analyzes three potential explanations for these phenomena: (i) heterogeneity in investor preferences, (ii) difficulties in estimating agents’ beliefs and (iii) heterogeneous beliefs among agents. Our results show that preferences alone cannot explain the patterns reported in the literature. Misestimation of investors’ beliefs caused by stochastic volatility and jumps in the return process cannot explain the smile either. The patterns of beliefs misestimation required to generate the empirical implied risk aversion estimates found in the literature suggest that heterogeneous beliefs are the most likely cause of the smile.

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Executive Summary

Equilibrium asset prices reflect investors’ preferences and beliefs. Historically, financial theory has made assumptions about investor preferences and beliefs in order to make predictions about asset prices. Recently, recognizing that preferences are not well-understood and hard to estimate compared to beliefs and asset prices, a new strand of research has emerged that estimates investors’ risk aversion by comparing their beliefs with asset prices. Estimates obtained in this fashion, which relate investors’ risk aversion to the level of aggregate wealth in the economy, are called implied risk aversion functions.

Implied risk aversion estimates exhibit two puzzling features: they are strongly U-shaped around the futures price (“smile”) and exhibit negative values. This paper analyzes three potential explanations for these patterns: (i) heterogeneity in investor preferences, (ii) misestimation of investors’ beliefs, and (iii) heterogeneous beliefs among investors.

The first possibility explored is whether heterogeneous preferences among investors aggregate in such a way as to lead to an oddly-behaved economy-wide risk aversion function. It turns out that preferences aggregate quite well. The economy-wide risk aversion function inherits some of the properties of individual agents’ risk aversion. Furthermore, risk-sharing among agents in the economy tends to even-out any oddly-behaved individual preferences, suggesting that heterogeneous preferences among investors cannot explain the implied risk aversion smile.

Implied risk aversion functions are very sensitive to misestimation of investors’ beliefs. Since beliefs are typically estimated using historical return distributions, a major potential source of misestimation is the presence of stochastic volatility and jumps in asset returns, which have been documented recently. However, it turns out that stochastic volatility and jumps are unable to explain the implied risk aversion smile. Using the relationship between implied risk aversion misestimation and beliefs misestimation, it is demonstrated that the patterns of beliefs misestimation that can be inferred from the implied risk aversion estimates are very peculiar and hard to reproduce in simple homogeneous beliefs situations.

Turning to heterogeneous beliefs as a potential explanation, it is shown that complex belief misestimation patterns and corresponding “smile effects” in implied risk aversion can be obtained easily when the assumption of homogeneous beliefs is relaxed. Fitting a simple model with two classes of investors with heterogeneous beliefs closely reproduces the empirical misestimation patterns, suggesting that heterogeneous beliefs are the most likely cause of the smile.
Why does Implied Risk Aversion smile?

1 Introduction

In a representative agent economy, equilibrium asset prices reflect the agent’s preferences and beliefs. As was shown by Rubinstein (1994), any two of the following imply the third: (i) the representative agent’s preferences, (ii) his subjective probability assessments, and (iii) the state-price density. Therefore, essentially any state-price density can be reconciled with the distribution of asset prices by using an appropriate set of preferences for the representative agent. Building on this insight, a few recent papers in the literature have derived estimates of the representative agent’s degree of risk aversion from the state-price density $Q$ and the subjective probability distribution $P$. In effect, the estimation of implied risk aversion functions reverses the classical direction of research, which was to move from assumptions on subjective probabilities and risk aversion to conclusions about the state-price density (Jackwerth and Rubinstein (2001)). The motivation for this new strand of research is that utility functions are not well-understood and hard to estimate compared to state-price densities and beliefs.

Letting $S$ denote the aggregate endowment in the economy, Aït-Sahalia and Lo (2000) derive a local estimator of the investor’s degree of relative risk aversion as $\rho(S) = S(P'(S)/P(S) - Q'(S)/Q(S))$, where $P$ denotes the statistical density of future asset prices and $Q$ the state-price density. In a related paper, Jackwerth (2000) shows that the representative agent’s implied absolute risk aversion is given by $\alpha(S) = P'(S)/P(S) - Q'(S)/Q(S)$. As a practical matter, $P$ is estimated from historical return realizations and $Q$ from traded option prices.

The empirical risk aversion estimates obtained by these authors are puzzling. Using S&P 500 index option prices and the historical density of index returns, Aït-Sahalia and Lo (2000) find that implied relative risk aversion is not constant across S&P 500 index values. Rather, it exhibits considerable variation, with values ranging from about 2 to 60, and is U-shaped around the futures price. Using minute-by-minute S&P 500 index option, index futures and index level quotes, Jackwerth (2000) finds that implied absolute risk aversion is U-shaped around the current forward price. He even finds that implied absolute risk aversion can become
significantly negative, with values as low as \(-15\).\(^1\)

This paper aims at explaining this “smile effect” in implied risk aversion. Although the smile effect in option implied volatility has received considerable attention, very few papers have sought to investigate the reasons for the *implied risk aversion* smile. Brown and Jackwerth (2001) is a notable exception. In a recent paper, these authors look for ways to reconcile Jackwerth’s (2000) empirical estimates of the pricing kernel and economic theory. The focus of their work lies in explaining why the pricing kernel is increasing in wealth (i.e. implied risk aversion is negative) for a range of index values centered on the current index level. Among the explanations they propose are crash-o-phobia – investors irrationally overestimating the likelihood of a market crash – and state-dependent utility.

Although the volatility smile and the implied risk aversion smile are closely related (Jackwerth (2000)), looking at implied risk aversion rather than the volatility smile can help us understand what is actually going on on financial markets. The reason is the link between the representative agent’s risk aversion, his subjective probability assessments and the state-price density alluded to above. An analysis of the causes of the implied risk aversion smile must therefore resort to one of these factors:

- The first possibility is that the representative agent may be a poor assumption. Heterogeneous preferences among investors may aggregate in such a way as to lead to an oddly-behaved aggregate risk aversion function for the representative agent.

- Second, agents’ subjective probability assessments may be misestimated, leading to a distortion in implied risk aversion estimates. Since agents’ beliefs are unobservable, a long tradition has emerged in financial economics that uses historical return realizations to estimate investors’ beliefs. As was noted by Brown and Jackwerth (2001), however, the problem with this approach is that the estimates thus obtained are *backward-looking*, while investor beliefs are by definition *forward-looking*. To the extent that the return process is nonstationary, belief estimates obtained from historical return frequency distributions will not match agents’ actual probability assessments. Another issue with estimates based on historical data is that they ignore the possibility of agents’ having heterogeneous beliefs.

\(^1\)Empirical estimates of risk aversion have also been obtained by Rosenberg and Engle (1999). Using the power function to approximate the empirical pricing kernel over the period 1991-1995, they show that risk aversion exhibits considerable variation through time, with values ranging from 1.21 to 57.99.
To the extent that such heterogeneity in beliefs exists, it will lead to further distortions in implied risk aversion estimates.

- The third possibility is that the state-price density may be misestimated and distort implied risk aversion estimates. Note, however, that the estimation of state-price densities does not suffer of the same pitfalls as the statistical density:

1. State-price densities are forward-looking estimates obtained directly from observed forward-looking variables, namely traded asset prices.

2. They are unique market prices, irrespective of whether investors have homogeneous or heterogeneous beliefs or preferences.

The analysis in this paper therefore focuses on investor preferences and beliefs as potential explanations for the “smile effect” in implied risk aversion. More specifically, the properties of implied risk aversion estimates are considered in three nested settings: (i) heterogeneous preferences among agents, (ii) misestimation of agents’ subjective beliefs by the researcher, and (iii) heterogeneous beliefs among agents. The question we seek to answer is whether each of these factors is sufficient to generate implied risk aversion functions that are consistent with the empirical smile patterns.

Our results show that a number of properties of individual agents’ risk aversion functions carry over to the representative agent’s risk aversion function, implying that heterogeneous preferences alone cannot explain the puzzling implied risk aversion patterns documented by Aït-Sahalia and Lo (2000) and Jackwerth (2000). Turning to the misestimation of agents’ beliefs as a potential explanation, we demonstrate that implied risk aversion estimates are very sensitive to errors in estimating the statistical density. In order to determine whether stochastic volatility and jumps in the return process can account for the smile effect, a simulation of Pan’s (2002) stochastic volatility and jumps model is performed. The implied risk aversion estimates obtained from this simulation still exhibit a smile, implying that stochastic volatility and jumps cannot explain the smile. We then derive the formal link between beliefs misestimation and implied risk aversion estimation errors and obtain the patterns of beliefs misestimation implied by the risk aversion smile. These patterns suggest that empirical estimates of agents’ beliefs based on historical data overestimate the probability of very high return realizations, and underestimate the probability of very low return realizations. We then show that these complex
misestimation patterns are hard to understand if agents have homogeneous beliefs. However, 
they can easily arise in a heterogeneous-beliefs economy, suggesting that heterogeneous beliefs 
are the most likely cause of the implied risk aversion smile.

The paper is organized as follows. Section 2 presents the model. Section 3 analyzes the 
properties of implied risk aversion functions and examines the role of preferences and beliefs in 
explaining the smile patterns obtained in the literature. Section 4 concludes.

2 The Model

In order to study the properties of implied risk aversion estimates in a setting with heterogeneous 
beliefs and preferences, consider a continuous-time economy with a large number of risk-averse 
agents, none of which have influence on equilibrium prices. Each agent \( i = 1 \ldots I \) lives for \( T \) 
periods and seeks to maximize his lifetime utility of consumption

\[
U^i(c) = E^{P_i} \left( \int_0^T u^i(c^i_s, s) ds \right)
\]

subject to the budget constraint

\[
E^Q \left( \int_0^T \exp \left( - \int_0^s r_u du \right) c^i_s ds \right) \leq W^i_t
\]

where \( P_i \) denotes agent \( i \)'s beliefs, \( Q \) the risk-neutral probability measure, \( c \) consumption, \( r \) the 
riskless interest rate and \( W \) wealth. Note that this formulation allows agents to differ both in 
their preferences and beliefs.

Using the Radon-Nikodym derivative \( \xi_i = P_i(S)/Q(S) \) and the Lagrange multiplier \( \lambda_i \), this 
maximization problem can be rewritten as

\[
\max_{c^i} E^{P_i} \left( \int_0^T \left( u^i(c^i_s, s) - \lambda_i \exp \left( - \int_0^s r_u du \right) c^i_s \frac{Q(S)}{P_i(S)} \right) ds \right)
\]

Maximizing time by time and state by state yields the first-order condition

\[
u^i_c(c^i_s, s) = \lambda_i \exp \left( - \int_0^s r_u du \right) \frac{Q(S)}{P_i(S)}
\]
Therefore, for each agent in the economy, the following relationship between his beliefs $P_i(S)$ and the state-price density $Q(S)$ holds:

$$\lambda_i \exp \left(- \int_t^s r_u du \right) Q(S) = u^i_c(c^i_s, s) P_i(S)$$

(5)

Differentiating this expression with respect to $S$ yields

$$\lambda_i \exp \left(- \int_t^s r_u du \right) Q'(S) = u^i_c(c^i_s, s) P'_i(S) + u^i_{cc}(c^i_s, s) \frac{\partial c^i_s}{\partial S} P_i(S)$$

(6)

or, using the first-order optimality condition (5),

$$\frac{P'_i(S)}{P_i(S)} - \frac{Q'(S)}{Q(S)} = -\frac{u^i_{cc}(c^i_s, s)}{u^i_c(c^i_s, s)} \frac{\partial c^i_s}{\partial S}$$

(7)

For each agent $i$ in the economy, equation (7) provides a link between his beliefs $P_i$, his risk aversion $-u^i_{cc}(c^i_s, s)/u^i_c(c^i_s, s)$ and market prices (the state-price density $Q$). For the reasons discussed in the introduction, assume that although market prices are observable and $Q$ can therefore be estimated accurately, the researcher does not observe each individual agent’s beliefs $P_i$. Rather, he only has a single estimate of the statistical density of asset prices. Letting $\hat{P}$ denote this estimate, the researcher’s implied ARA estimator $\alpha(S)$ is given by

$$\alpha(S) \equiv \left( \frac{\hat{P}'(S)}{\hat{P}(S)} - \frac{Q'(S)}{Q(S)} \right)$$

$$= -\frac{u^i_{cc}(c^i_s, s)}{u^i_c(c^i_s, s)} \frac{\partial c^i_s}{\partial S} + \left( \frac{\hat{P}'(S)}{\hat{P}(S)} - \frac{P'_i(S)}{P_i(S)} \right)$$

(8)

Thus, the researcher’s implied risk aversion estimate is equal to the agent’s actual risk aversion, scaled by the sensitivity of the agent’s consumption to shifts in aggregate consumption $\partial c^i_s/\partial S$, plus an estimation error arising from the fact that the researcher does not observe the agent’s beliefs perfectly.

The problem at this point is that the sensitivity of the agent’s consumption to shifts in aggregate consumption $\partial c^i_s/\partial S$ is unobservable. In order to derive the implied risk aversion in terms of the observable aggregate endowment $S$, one can resort to an equilibrium argument. First, solve (8) for $\partial c^i_s/\partial S$ to obtain

$$\frac{\partial c^i_s}{\partial S} = \left( \alpha(S) - \left( \frac{\hat{P}'(S)}{\hat{P}(S)} - \frac{P'_i(S)}{P_i(S)} \right) \right) \frac{1}{-u^i_{cc}(c^i_s, s)/u^i_c(c^i_s, s)}$$

(9)
Aggregating across all agents and requiring market clearing then yields

\[
\sum_{i=1}^{I} \frac{\partial c_i}{\partial S} = 1 = \alpha(S) \sum_{i=1}^{I} \frac{1}{-u_{cc}^i(c_s^i, s)/u_c^i(c_s^i, s)} - \sum_{i=1}^{I} \frac{\left( \frac{\hat{P}_i(S)}{P(S)} - \frac{\hat{P}'_i(S)}{P'(S)} \right)}{-u_{cc}^i(c_s^i, s)/u_c^i(c_s^i, s)}
\]

allowing to write the implied risk aversion estimator as

\[
\alpha(S) = \frac{1 + \sum_{i=1}^{I} \frac{\left( \frac{\hat{P}_i(S)}{P(S)} - \frac{\hat{P}'_i(S)}{P'(S)} \right)}{-u_{cc}^i(c_s^i, s)/u_c^i(c_s^i, s)}}{\sum_{i=1}^{I} \frac{1}{-u_{cc}^i(c_s^i, s)/u_c^i(c_s^i, s)}} = \frac{1 + \sum_{i=1}^{I} \frac{\left( \frac{\hat{P}_i(S)}{P(S)} - \frac{\hat{P}'_i(S)}{P'(S)} \right)}{\alpha_i(c_s^i)}}{\sum_{i=1}^{I} \frac{1}{\alpha_i(c_s^i)}}
\]

where \(\alpha_i(c_s^i) \equiv -u_{cc}^i(c_s^i, s)/u_c^i(c_s^i, s)\) denotes investor \(i\)'s degree of absolute risk aversion on his optimal consumption path. Implied aggregate absolute risk aversion \(\alpha(S)\) is thus the harmonic sum of individual investors’ degree of absolute risk aversion, plus an adjustment term that depends both on investors’ degree of risk aversion and on the divergence between individual agents’ actual beliefs and the researcher’s estimates. At this level of generality, almost any pattern of implied risk aversion is possible. The analysis below considers which patterns of implied risk aversion can emerge depending on (i) the degree of heterogeneity in individual agents’ preferences, (ii) the nature of the divergence between agents’ beliefs and the researcher’s estimates, and (iii) the degree of heterogeneity in beliefs among agents.

3 Properties of Implied Risk Aversion Functions

This section analyzes the equilibrium properties of implied risk aversion functions in three nested settings in order to determine what it takes in order to explain the risk aversion smile documented in the literature. Section 3.1 considers the special case in which only preferences differ among agents, i.e. where agents have homogeneous beliefs and these can be estimated accurately. The analysis demonstrates that in such a situation, individual agents’ preferences must have very peculiar properties in order to generate the implied risk aversion smile. Heterogeneous preferences among agents alone therefore seem insufficient to explain the smile. Section 3.2 then considers a somewhat more general situation in which agents have homogeneous beliefs, but these cannot be estimated accurately. It is shown that even a minor misestimation of agents’ beliefs has a significant impact on implied risk aversion functions. Stochastic volatility and jumps in returns, however, are shown to be unable to explain the implied risk aversion smile. The patterns of beliefs misestimation consistent with the implied risk aversion smile are
then derived. The peculiar patterns that arise from this analysis suggest that heterogeneous beliefs are the most likely cause of the smile, an issue addressed in section 3.3.

3.1 Homogeneous Beliefs and Perfect Estimation

This section considers the case in which all agents in the economy have homogeneous beliefs and these can be estimated accurately by the researcher. Under these circumstances, one can establish the following result:

**Proposition 1:** Suppose that beliefs are homogeneous and can be estimated accurately. Then, the implied absolute risk aversion function \( \alpha(S) \) is the harmonic sum of individual agents’ absolute risk aversion on the optimal consumption path, \( \alpha_i(c_s^i) \).

**Proof:** Under the assumption of homogeneous beliefs and accurate estimation, \( \hat{P}'(S)/\hat{P}(S) = P'_i(S)/P_i(S) \) for all \( i \), and the implied risk aversion estimator (11) becomes

\[
\alpha(S) = \frac{1}{\sum_{i=1}^{I} \frac{1}{1 - u_{i,1}(c_s^i)/u_i^*(c_s^i)}} = \frac{1}{\sum_{i=1}^{I} \frac{1}{\alpha_i(c_s^i)}} \tag{12}
\]

Equation (12) is the well-known result that risk tolerance is additive across agents, implying that the economy-wide risk tolerance equals the sum of individual agents’ risk tolerance (Wilson (1968)). This result has an important implication for empirical implied risk aversion estimates, which is stated as

**Corollary 1:** If all agents are risk-averse, have homogeneous beliefs and these beliefs are accurately estimated, then estimated implied risk aversion is strictly positive.

Furthermore, under homogeneous beliefs and accurate estimation, some additional properties of individual agents’ risk aversion carry over to implied risk aversion. The first follows immediately from Proposition 1, and is stated as

**Corollary 2:** If all agents have constant absolute risk aversion (CARA) utility, have homogeneous beliefs and these beliefs are accurately estimated, then estimated implied risk aversion also displays constant absolute risk aversion.

The other results are somewhat less immediate and are therefore stated as a separate proposition:
**Proposition 2:** Suppose all agents have increasing (decreasing) absolute risk aversion, have homogeneous beliefs and these beliefs are accurately estimated. Then, estimated implied risk aversion also displays increasing (decreasing) absolute risk aversion.

**Proof:** Differentiating (12) with respect to \( S \) yields

\[
\alpha'(S) = -\sum_{i=1}^{I} \frac{\alpha'(c^i_s) \partial c^i_s}{\sum_{i=1}^{I} \frac{1}{\alpha(c^i_s)}}\left(\sum_{i=1}^{I} \frac{1}{\alpha(c^i_s)}\right)^2 = \frac{\sum_{i=1}^{I} \frac{\alpha'(c^i_s) \partial c^i_s}{\alpha_i^2(c^i_s) S}}{\sum_{i=1}^{I} \frac{1}{\alpha(c^i_s)}}
\]

Since all agents are risk-averse and have homogeneous beliefs, risk-sharing among them implies that \( \partial c^i_s / \partial S > 0 \) for all \( i \). Therefore, (13) will be positive whenever \( \alpha'_i(c^i_s) > 0 \) for all \( i \) and negative whenever \( \alpha'_i(c^i_s) < 0 \) for all \( i \), establishing the result.

Proposition 2 implies that if all agents have CRRA utility, which exhibits decreasing absolute risk aversion, then the market-wide risk aversion function will exhibit decreasing absolute risk aversion as well. For the special case of CRRA utility, however, even stronger results can be established. As a first step, let us consider the properties of the implied relative risk aversion function, \( \rho(S) = S \alpha(S) \).

**Proposition 3:** Suppose that beliefs are homogeneous and are estimated accurately. Then, implied relative risk aversion \( \rho(S) \) is a harmonic weighted average of individual agents’ relative risk aversion, with the weights in this average given by each agent’s share of aggregate consumption at a given level of the aggregate endowment.

**Proof:** Relative risk aversion is given by \( \rho(S) = S \alpha(S) \). Thus, using (11), we have

\[
\rho(S) = S\frac{1}{\sum_{i=1}^{I} \frac{1}{-u_i^c(c^i_s,s)/u_i^c(c^i_s,s)}} = \frac{1}{\sum_{i=1}^{I} \frac{1}{-u_i^c(c^i_s,s)c^i_s/u_i^c(c^i_s,s)/S}} = \frac{1}{\sum_{i=1}^{I} \frac{1}{\rho_i(c^i_s)c^i_s/S}}
\]

where \( \rho_i(c^i_s) \equiv -u_i^c(c^i_s,s)c^i_s/u_i^c(c^i_s,s) \) denotes agent \( i \)'s relative risk aversion on the optimal consumption path.

This result, which was first derived by Benninga and Mayshar (2000) in a somewhat different setting, suggests that relative risk aversion will typically depend on the aggregate endowment, even if all agents have constant relative risk aversion. Benninga and Mayshar (2000) show that if agents have heterogeneous, CRRA preferences, then the economy-wide relative risk aversion will be decreasing in the aggregate endowment. This is so because as the aggregate endowment increases, relatively less risk averse agents’ share of aggregate consumption increases, driving
down the average in (14). Although we refer the reader to their paper for a formal proof, note that assuming CRRA utility for all agents in (14), the first derivative of implied relative risk aversion equals

\[
\rho'(S) = -\frac{\sum_{i=1}^{I} \frac{1}{\rho_i(c_i)} \frac{\partial}{\partial S} \left( c_i \right)}{\left( \sum_{i=1}^{I} \frac{1}{\rho_i(c_i)} \frac{c_i}{S} \right)^2} = \frac{1}{S} \sum_{i=1}^{I} \frac{1}{\rho_i(c_i)} \left( \frac{c_i}{S} - \frac{\partial c_i}{\partial S} \right)
\]

\[
= \alpha(S) \frac{\sum_{i=1}^{I} \frac{1}{\rho_i(c_i)} \left( \frac{c_i}{S} - \frac{\partial c_i}{\partial S} \right)}{\sum_{i=1}^{I} \frac{1}{\rho_i(c_i)} \frac{c_i}{S}} = \alpha(S) \left( 1 - \frac{\sum_{i=1}^{I} \frac{1}{\rho_i(c_i)} \frac{\partial c_i}{\partial S}}{\sum_{i=1}^{I} \frac{1}{\rho_i(c_i)} \frac{c_i}{S}} \right)
\]

which will indeed be negative when \( \partial c_i / \partial S > c_i / S \) for low risk aversion agents and \( \partial c_i / \partial S < c_i / S \) for high risk aversion agents. An alternate way to understand this result is to note that (15) can be rewritten as

\[
\rho'(S) = \alpha(S) (1 - \eta)
\]

where \( \eta \) denotes the average elasticity of agents’ consumption with respect to the aggregate endowment, with the weights in this average given by the inverse of agents’ relative risk aversion coefficients \( \rho_i \). \( \eta \) will exceed one because less risk averse agents’ share of aggregate consumption is increasing in \( S \), implying that market-wide relative risk aversion is decreasing in \( S \).

The results in this section demonstrate that some of the properties of individual agents’ utility functions carry over to the representative agent’s risk aversion. This has a number of important implications for our assessment of empirical estimates of implied risk aversion functions:

1. If we are willing to assume that all agents in the economy are risk averse, then the negative estimates obtained in some recent research (such as Jackwerth (2000)) cannot be caused by heterogeneity in investor preferences. Rather, they must result from divergences between estimates of the statistical density and agents’ actual beliefs or from heterogeneous beliefs among agents.

2. If we are also willing to accept that agents have nonincreasing absolute risk aversion (as postulated by Arrow (1970) and supported by everyday observation) or constant relative risk aversion, then the U-shaped estimates obtained by Jackwerth (2000) and Aït-Sahalia and Lo (2000) have a similar explanation.²

²Although Aït-Sahalia and Lo (2000) estimate a relative risk aversion function, it is easy to check from their Figure 4 that their representative agent also exhibits increasing absolute risk aversion by noting that the slope
3. If we are willing to accept that agents are risk-averse but not that they have nonincreasing absolute risk aversion, then U-shaped patterns as documented by Aït-Sahalia and Lo (2000) are theoretically possible. However, one would still need to explain why relative risk aversion reaches levels as high as 60 for index values about 15% below the current futures price and as high as 30 for index values 15% above. Clearly, factors such as habit persistence can be invoked to explain high risk aversion at low index levels. However, they cannot account for the increase in risk aversion at high index levels. A similar conclusion holds for state-dependent utility (as noted by Brown and Jackwerth (2001), in order to generate a U-shaped risk aversion with respect to the index, the relationship between the state variable and aggregate wealth must be non-monotonic).

Could heterogeneous preferences among agents help in this respect? Note that even if one did not assume nonincreasing absolute risk aversion, equation (14) would hold, and relative risk aversion would be a consumption-weighted average of individual agents’ relative risk aversion. Furthermore, in each state of nature, those agents with relatively low risk aversion would have a share of aggregate consumption exceeding their relative wealth and drive down the representative agent’s risk aversion. Therefore, the smile pattern in Aït-Sahalia and Lo (2000) could only arise if a very significant proportion of agents had extremely high risk aversion at high and low index values (and very few agents had low risk-aversion at those same index values). Although this is not inconceivable, it seems very unlikely.

The analysis in this section therefore suggests that heterogeneous preferences alone cannot account for the implied risk aversion smile – if anything, the effect of risk-sharing among agents with heterogeneous preferences on the properties of the economy-wide risk aversion function makes the smile even more puzzling.

### 3.2 Homogeneous Beliefs and Imperfect Estimation

This section considers the properties of implied risk aversion functions in an economy in which agents have homogeneous beliefs, but these beliefs cannot be estimated perfectly. In this setting, one has

![Image](https://via.placeholder.com/150)

of rays drawn through the origin of their diagram and the points on their implied risk aversion function is increasing for index values between 415 and 440, between 465 and 475 and above 500.
**Proposition 4:** Suppose that agents’ beliefs are homogeneous but imperfectly estimated. Then, the implied risk aversion function $\alpha(S)$ is given by

$$\alpha(S) = \frac{1}{\sum_{i=1}^{I} \frac{1}{\alpha_i(c_i)}} + \left( \frac{\hat{P}'(S)}{\hat{P}(S)} - \frac{P'(S)}{P(S)} \right) \equiv \frac{1}{\sum_{i=1}^{I} \frac{1}{\alpha_i(c_i)}} + \epsilon(S) \quad (17)$$

where $\epsilon(S)$ denotes an implied risk aversion estimation error and $P(S)$ agents’ common beliefs.

**Proof:** Under the assumption of homogeneity in beliefs, one has $P_i'(S)/P_i(S) = P'(S)/P(S)$ for all $i$, so the term $(\hat{P}'(S)/\hat{P}(S) - P'(S)/P(S))$ is common across agents and the implied risk aversion estimator (11) can be rewritten as

$$\alpha(S) = \frac{1}{\sum_{i=1}^{I} \frac{1}{\alpha_i(c_i)}} + \left( \frac{\hat{P}'(S)}{\hat{P}(S)} - \frac{P'(S)}{P(S)} \right) \sum_{i=1}^{I} \frac{1}{\alpha_i(c_i)}$$

Simplifying then gives (17).

Equation (17) demonstrates that at any aggregate wealth level $S$, the divergence between estimated implied risk aversion and the representative agent’s actual risk aversion will depend on the divergence between agents’ actual and estimated beliefs. This immediately raises a number of questions: How sensitive are implied risk aversion estimates to misestimation of the statistical density? Can factors that lead to difficulties in estimating the statistical density – such as stochastic volatility and jumps in the return process – account for the implied risk aversion smile? And if not, what kind of beliefs misestimation do the empirical implied risk aversion estimates suggest and how can it be explained?

### 3.2.1 The Sensitivity of Implied Risk Aversion to Beliefs Misestimation

Consider first the issue of the sensitivity of implied risk aversion estimates to misestimation of the statistical density. Do small beliefs misestimations only lead to small distortions in implied risk aversion estimates, or do they have a significant impact? The answer to this question is not immediately obvious, because the estimation error $\epsilon(S) = \hat{P}'(S)/\hat{P}(S) - P'(S)/P(S)$ depends not only on actual and estimated densities $P(S)$ and $\hat{P}(S)$, but also on their derivatives. Thus, large implied risk aversion estimation errors could arise even if $\hat{P}(S) - P(S)$ is small.

To gain some insight into the magnitude of this effect, consider a numerical example. Suppose that agents’ beliefs are lognormal with $E(\ln(S)) = \mu$ and $\text{Var}(\ln(S)) = \sigma^2$ and that the
The researcher seeks to estimate these two parameters. Thus, agents’ beliefs are given by

\[ P(S) = \frac{1}{S\sigma \sqrt{2\pi}} \exp \left( -\frac{(\ln(S) - \mu)^2}{2\sigma^2} \right) \]  

(19)

whereas the researcher estimates them as

\[ \hat{P}(S) = \frac{1}{S\hat{\sigma} \sqrt{2\pi}} \exp \left( -\frac{(\ln(S) - \hat{\mu})^2}{2\hat{\sigma}^2} \right) \]  

(20)

Therefore, one can write the implied risk aversion estimation error as

\[ \epsilon(S) = \frac{\hat{P}'(S)}{\hat{P}(S)} - \frac{P'(S)}{P(S)} = \frac{d \ln(\hat{P}(S))}{dS} - \frac{d \ln(P(S))}{dS} \]

\[ = \frac{1}{S} \left( \frac{\ln(S) - \mu}{\sigma^2} - \frac{\ln(S) - \hat{\mu}}{\hat{\sigma}^2} \right) \]

\[ = \frac{1}{S\sigma^2 \hat{\sigma}^2} \left( (\ln(S) - \mu)(\hat{\sigma}^2 - \sigma^2) + \sigma^2(\hat{\mu} - \mu) \right) \]  

(21)

Figure 1 depicts the actual and estimated statistical densities \( P(S) \) and \( \hat{P}(S) \), as well as the implied risk aversion estimation error \( \epsilon(S) \) that arises if the researcher slightly underestimates the variability of aggregate wealth \( \hat{\sigma} = 0.095 < \sigma = 0.1 \). The upper panel of Figure 1 shows that the small parameter estimation error of 5% used in this example only leads to minor differences between the actual and the estimated density. However, as can be seen in the lower panel, the magnitude of the implied risk aversion estimation error is sizable.

Of course, this argument goes the other way as well: statistical distributions inferred from the state-price density and some set of (assumed) investor preferences are not too sensitive to the particular specification of the market-wide risk-aversion function chosen. If we are willing to make certain assumptions about economy-wide investor preferences, we can therefore estimate the statistical density directly from the state-price density in order to determine what kind of beliefs were underlying the state-price density at a given point in time.\(^3\) Agents’ subjective probability assessments can be estimated as follows: supposing that agents’ preferences are known, we have \( \alpha(S) = \hat{P}'(S)/\hat{P}(S) - Q'(S)/Q(S) \). Rewriting this expression as

\[ \frac{d \ln(P(S))}{dS} = \frac{d \ln(Q(S))}{dS} + \alpha(S) \]  

(22)

we have, for an arbitrary reference point \( \underline{S} \),

\[ \ln \left( \frac{P(S)}{P(\underline{S})} \right) = \ln \left( \frac{Q(S)}{Q(\underline{S})} \right) + \int_{\underline{S}}^{S} \alpha(u)du \]  

(23)

\(^3\)Rubinstein (1994) shows how to perform this estimation in a discrete-state setting. His approach uses agents’ marginal utility directly, whereas the approach presented here uses the risk aversion coefficient.
or

\[ P(S) = \frac{P(S)}{Q(S)} Q(S) \exp \left( \int_S \alpha(u) du \right) = \gamma Q(S) \exp \left( \int_S \alpha(u) du \right) \]  

with \( \gamma \equiv P(S)/Q(S) \) a constant that ensures that \( P(S) \) integrates to 1. Thus, having knowledge of the state-price density \( Q \) and assuming a particular functional form for the market-wide risk aversion function \( \alpha(S) \), we can infer the statistical density \( P \).

What kind of beliefs do the empirical state-price densities suggest? To answer this question, the statistical density was computed from the S&P 500 empirical state-price density for three types of market-wide utility functions:

1. CARA, with a market-wide risk aversion coefficient of \( \bar{\alpha} \). In this case, (24) implies the estimator

\[ P(S) = \gamma Q(S) \exp \left( \int_S \bar{\alpha} du \right) = \gamma Q(S) \exp (\bar{\alpha}(S - \bar{S})) \]  

(25)

2. CRRA preferences with a market-wide relative risk aversion coefficient of \( \bar{\rho} \). Using (24) yields the estimator

\[ P(S) = \gamma Q(S) \exp \left( \int_S \alpha(u) du \right) = \gamma Q(S) \exp \left( \int_S \bar{\rho} \frac{du}{u} \right) = \gamma Q(S) \left( \frac{S}{\bar{S}} \right)^{\bar{\rho}} \]  

(26)

It is a well-known fact that if agents’ beliefs are lognormal, then the Black-Scholes model will obtain in this case.

3. DRRA preferences with \( \rho(S) = \bar{\rho}/S \), for which (24) implies the estimator

\[ P(S) = \gamma Q(S) \exp \left( \int_S \alpha(u) du \right) = \gamma Q(S) \exp \left( \int_S \bar{\rho} \frac{du}{u^2} \right) = \gamma Q(S) \exp \left( \bar{\rho} \left( \frac{1}{\bar{S}} - \frac{1}{S} \right) \right) \]  

(27)

This latter setting aims at capturing the case of agents with heterogeneous CRRA preferences described in section 3.1, which results in a decreasing market-wide relative risk aversion.

The estimation is performed as follows. In a first step, the state-price density is estimated using the semi-parametric approach and the data of Aït-Sahalia and Lo (2000). The data,
which is described in more detail in their paper, consists of 14431 S&P 500 index option prices for the period January 4, 1993 to December 31, 1993. The semi-parametric approach involves regressing option implied volatility nonparametrically on moneyness and time to expiration and then estimating the state-price density as the second derivative of the Black-Scholes option pricing formula with respect to the strike price, using the nonparametric volatility estimate as an input. As recommended in their paper, the kernel functions and bandwidth values are chosen so as to optimize the properties of the state-price density estimator (Gaussian kernel functions and bandwidths of 0.040 and 20.52 for moneyness and time to expiration, respectively). In a second step, the subjective probabilities are computed according to the above expressions, assuming a relative risk aversion at the current futures price of 450 of 4 for all three cases considered, implying \( \alpha = 4/450 \) for case 1, \( \overline{\rho} = 4 \) for case 2 and \( \overline{\rho} = 4 \cdot 450 \) for case 3.\(^4\)

The estimation results for the state-price density and the subjective probabilities corresponding to the above three cases are depicted in Figure 2 for time horizons of 1, 2, 4 and 6 months. Note that although there is a significant difference between the state-price density and the statistical densities reflecting the effect of risk aversion, the three statistical densities are almost indistinguishable for all time horizons considered. This confirms the results from Figure 1 and suggests that differences in the assumed investor preferences have very little impact on the beliefs that can be inferred from the state-price density.\(^5\) However, these beliefs do not tell us what beliefs estimates based on historical returns are potentially missing, the issue we wish to address now.

### 3.2.2 A Potential Explanation: Stochastic Volatility and Jumps in Returns

The above results demonstrate that implied risk aversion estimates are very sensitive to the underlying statistical density estimates and that minor density estimation errors can lead to large perturbations in implied risk aversion. Beliefs, however, are intrinsically hard to estimate.

As mentioned in the introduction, a long tradition has developed in financial economics to

\(^4\)The value of 4 is arbitrary but not out of line with existing empirical evidence. Based on an analysis of the demand for risky assets, Friend and Blume (1975) find that the average coefficient of relative risk aversion is probably well in excess of one and perhaps in excess of two. Using an analysis of deductibles in insurance contracts, Drèze (1981) finds somewhat higher values. When fitting a CRRA model to their data, Aït-Sahalia and Lo (2000) find a value of 12.7.

\(^5\)Rubinstein (1994) comes to the same conclusion.
estimate (unobservable) beliefs based on (observed) historical returns. Using *a time series* of past returns to estimate subjective probabilities at a given *point* in time can be problematic because historical returns do not necessarily tell us much about agents’ current beliefs\(^6\). A major potential source of beliefs misestimation are stochastic volatility and jumps, which have recently been documented by Pan (2002).

In order to determine whether stochastic volatility and jumps are sufficient to explain the implied risk aversion smile, the following empirical analysis is performed: first, the state-price density \( Q \) is again estimated using the semi-parametric method and the data of Aït-Sahalia and Lo (2000) for time horizons of 1, 2, 4 and 6 months. In a second step, the statistical density \( P \) is estimated by simulating the stochastic volatility and jumps model of Pan (2002) for S&P 500 index returns, and running a kernel density estimation on these return realizations. The starting values for the simulation are chosen so as to be consistent with those used when generating the state-price densities, i.e. a riskless interest rate of 3.10% and a dividend yield of 2.78% as reported in Table 3 of Aït-Sahalia and Lo (2000), an initial volatility equal to the implied volatility of an at-the-money option with a maturity equal to the time horizon of the simulation, and the initial cash price implied by the spot-futures parity for the reference futures price of 450 used in the state-price density estimation. The kernel density estimation is based on a Gaussian kernel and bandwidths selected using the Silverman’s rule, i.e. \( h = \sigma_S (4/3n)^{1/5} \), where \( n = 10000 \) denotes the number of simulation runs and \( \sigma_S \) the unconditional standard deviation of the index value at the end of the simulation horizon. The bandwidths for time horizons of 1, 2, 4 and 6 months are 2.22, 3.09, 4.49 and 5.64, respectively, and the resulting densities depicted in Figure 3. In a third step, the degree of relative risk aversion implied by these density estimates is computed as \( \rho(S) = S \left( P'(S)/P(S) - Q'(S)/Q(S) \right) \). As can be seen in Figure 4, even when accounting for stochastic volatility and jumps, implied risk aversion exhibits considerable variation for all four time horizons considered. Interestingly, the results in the fourth panel of Figure 4 are very similar to Figure 4 of Aït-Sahalia and Lo (2000) (who report results for the 6 months horizon): implied risk aversion is moderate around the current futures price, but “smiles” to reach values of almost 30 for index values about 15% above and below the futures price. This suggests that stochastic volatility and jumps cannot explain the

\(^6\)Although Jackwerth (2000) tries to address this problem by varying the length of the historical sample he uses from 2 to 10 years and shows that his results do not change significantly, the basic issue of using actual return realizations to estimate agents’ beliefs remains.
implied risk aversion smile documented by these authors.

The results in Figure 4 have another important implication. Observe that the implied risk aversion patterns are different depending on the time horizon considered. Although there is no conclusive experimental evidence on the issue, it is not unreasonable to argue that preferences should not depend strongly on the time horizon considered, especially for the short horizons of 1 to 6 months considered here. Similarly, one would not expect investor preferences to change abruptly through time, making the considerable time-variation in implied risk aversion documented, for instance, by Rosenberg and Engle (1999) a puzzling phenomenon. On the other hand, rational investors do update their beliefs frequently, and changes in investor beliefs therefore appear to be a more natural cause of these shifts in implied risk aversion through time than shifts in investor preferences.

3.2.3 What Patterns of Belief Misestimation does Implied Risk Aversion Suggest?

Since stochastic volatility and jumps in asset returns are unable to explain the implied risk aversion smile, beliefs estimates based on historical returns must be missing something else. Can we tell what? Do the implied risk aversion estimates suggest any particular patterns of beliefs misestimation, and if so, how can these patterns be explained?

To perform this analysis, let us reverse the perspective. Suppose we knew agent’s preferences. Then, by analyzing the relationship between implied risk aversion estimation error and beliefs estimation error, we would be able to determine what our beliefs estimates miss. The reason is that with knowledge of agents’ actual risk aversion, $\epsilon(S)$ is known, and the beliefs estimation error can be derived directly from equation (17). Rewriting this expression as

$$\frac{d\ln(\hat{P}(S))}{dS} = \frac{d\ln(P(S))}{dS} + \epsilon(S)$$

(28)

yields, for an arbitrary reference point $\underline{S}$,

$$\ln \left( \frac{\hat{P}(S)}{P(\underline{S})} \right) = \ln \left( \frac{P(S)}{P(\underline{S})} \right) + \int_{\underline{S}}^{S} \epsilon(u) du$$

(29)

or

$$\frac{\hat{P}(S)}{P(S)} = \frac{\hat{P}(\underline{S})}{P(\underline{S})} \exp \left( \int_{\underline{S}}^{S} \epsilon(u) du \right) = \gamma \exp \left( \int_{\underline{S}}^{S} \epsilon(u) du \right)$$

(30)
with $\gamma \equiv \hat{P}(S)/P(S)$ again a constant that ensures that $P(S)$ integrates to 1. The density misestimation factor given in equation (30) allows us to determine to what extent historical returns under- or overestimate the probability of certain states of nature based on a comparison of implied risk aversion estimates and an assessment of agents’ actual preferences. As a numerical illustration, suppose that implied risk aversion is quadratic, reaches a minimum of $-15$ at a wealth level of 1 and a value of zero at wealth levels of 0.97 and 1.03, the basic picture that emerges from Figure 3, Panel D of Jackwerth (2000), and assume that the true coefficient of absolute risk aversion is 4.\footnote{Although it seems restrictive, this assumption is innocuous. Similar patterns would arise with any assumed actual risk aversion function as long as it exhibits less curvature than the implied risk aversion function and the functions cross twice. Given the high curvature of the implied risk aversion function reported by Jackwerth (2000), this would be the case for a very wide set of investor preferences.} Together, these functions, which are depicted in the upper panel of Figure 5, result in an implied risk aversion estimation error of $\epsilon(S) = -19 + (15/0.03^2)(S - 1)^2$. Using this functional form for $\epsilon(S)$ and a reference level of $\underline{S} = 1$ yields

$$\frac{\hat{P}(S)}{P(S)} = \gamma \exp \left( \int_{\underline{S}}^{S} \epsilon(u) du \right) = \gamma \exp \left( -19(S - 1) + \frac{5}{0.03^2} (S - 1)^3 \right)$$

(31)

A plot of this function for $\gamma = 1$ is reported in the lower panel of Figure 5. The density misestimation factor suggests that historical return data underestimates agents’ assessment of the probability of very low aggregate wealth states (Brown and Jackwerth’s (2001) “crash-o-phobia”), slightly overestimates the probability of wealth states slightly below 1, slightly underestimates the probability of wealth states slightly above 1 and significantly overestimates the probability of very high wealth states. Note that the misestimation factor exhibits a very pronounced pattern, and it is therefore unlikely that it is merely the consequence of random factors in estimation, such as measurement error. Agents’ beliefs seem to differ in a systematic fashion from the historical return distribution, suggesting that something peculiar must be going on on the market.

What does it take to reproduce these patterns? To answer this question, let us consider a simple setting in which both agents’ beliefs and historical returns are lognormal and analyze the beliefs misestimation factor that arises when the mean and/or the variance differ. As illustrated in Figure 6, misestimation of the mean leads to a monotonic misestimation factor. On the other hand, misestimation of the variance leads to a bell- or U-shaped misestimation factor,
as illustrated in Figure 7. Even if both mean and variance are allowed to differ, lognormal densities are unable to reproduce the misestimation pattern documented in Figure 5. The resulting misestimation factor will have either a single maximum or minimum (i.e. will be either bell-shaped or U-shaped), but not both. This suggests that something more serious than mere misestimation of expected returns or variance is happening.

3.3 Heterogeneous Beliefs

The analysis in the preceding section shows that historical return realizations are a poor predictor of agents’ beliefs, but that taking stochastic volatility and jumps into account cannot explain the implied risk aversion smile. Moreover, simple homogeneous beliefs specifications are unable to account for the complex beliefs misestimation patterns implied by a comparison of implied risk aversion and a wide range of reasonable investor preference assumptions. This section derives the properties of the beliefs misestimation factor under heterogeneous beliefs and demonstrates that heterogeneous beliefs can easily give rise to complex misestimation patterns such as those reported in Figure 5, making them the most likely cause of the implied risk aversion smile.

To derive the beliefs misestimation function under heterogeneous beliefs, rewrite (11) as

\[ \alpha(S) = \frac{1}{\sum_{i=1}^{I} \frac{1}{\alpha_i(c_i^s)}} + \frac{\hat{P}'(S)}{P(S)} - \frac{\sum_{i=1}^{I} \left( \frac{\nu_i'(S)}{\nu_i(S)} \right)}{\sum_{i=1}^{I} \frac{1}{\alpha_i(c_i^s)}} \]  

(32)

Then, the implied risk aversion estimation error, \( \epsilon(S) \), is given by

\[ \epsilon(S) = \frac{\hat{P}'(S)}{P(S)} - \frac{\sum_{i=1}^{I} \left( \frac{\nu_i'(S)}{\nu_i(S)} \right)}{\sum_{i=1}^{I} \frac{1}{\alpha_i(c_i^s)}} \equiv \frac{\hat{P}'(S)}{P(S)} - \sum_{i=1}^{I} \phi_i(S) \frac{P_i'(S)}{P_i(S)} \]  

(33)

where \( \phi_i(S) \equiv 1/ \left( \alpha_i(c_i^s) \sum_{i=1}^{I} (1/\alpha_i(c_i^s)) \right) \) denotes a local measure of the share of risk borne by agent \( i \) in the economy and has the obvious property that \( \sum_i \phi_i(S) = 1 \) for all \( S \). For general preferences, \( \phi_i \) will be a function of the aggregate endowment \( S \) through the effect of the latter on agent \( i \)’s consumption \( c_i^s \), \( \phi_i = \phi_i(S) \). Rewriting,

\[ \epsilon(S) = \frac{d \ln(\hat{P}(S))}{dS} - \sum_{i=1}^{I} \phi_i(S) \frac{d \ln(P_i(S))}{dS} \]  

(34)
or, for an arbitrary reference point $S$,

$$\ln \left( \frac{\hat{P}(S)}{P(S)} \right) = \sum_{i=1}^{I} \int_{S}^{S} \phi_i(u) d \ln (P_i(u)) + \int_{S}^{S} \epsilon(u) du \quad (35)$$

Therefore,

$$\frac{\hat{P}(S)}{\exp \left( \sum_{i=1}^{I} \int_{S}^{S} \phi_i(u) d \ln (P_i(u)) \right)} = \gamma \exp \left( \int_{S}^{S} \epsilon(u) du \right) \quad (36)$$

with $\gamma \equiv \hat{P}(S)$ again a constant ensuring that the $P_i$'s integrate to 1. Similarly, by comparing (36) with (30) and (24), the corresponding expression linking the state-price density, preferences and beliefs can be seen to be given by

$$\frac{Q(S)}{\exp \left( \sum_{i=1}^{I} \int_{S}^{S} \phi_i(u) d \ln (P_i(u)) \right)} = \gamma \exp \left( - \int_{S}^{S} \alpha(u) du \right) \quad (37)$$

with $\gamma \equiv Q(S)$ again a constant.

In the special case of CARA utility, $\phi_i$ does not depend on $S$, and one can write (36) as

$$\frac{\hat{P}(S)}{\prod_{i=1}^{I} P_i(S) \phi_i} = \gamma \exp \left( \int_{S}^{S} \epsilon(u) du \right) \quad (38)$$

with $\gamma \equiv \hat{P}(S)/\prod_{i=1}^{I} P_i(S) \phi_i$. Note that the “misestimation factor” $\hat{P}(S)/\prod_{i=1}^{I} P_i(S) \phi_i$ is the ratio of the estimated probability to the geometric weighted average of the individual agents’ probability assessments, with the weights in this average given by the share of risk $\phi_i$ they bear in the economy. Therefore, an implied risk aversion smile could even arise in a simple setting in which agents have CARA utility and heterogeneous beliefs, even if the researcher’s probability assessment is equal to the arithmetic weighted average of individual agents’ probability assessments.

Would heterogeneous beliefs of this form be sufficient to explain the misestimation pattern reported in Figure 5 without making extreme assumptions about the nature of beliefs and/or their estimates? To answer this question, model (38) was fitted to the density misestimation factor (31) for the following simple setting: There are two types of CARA agents indexed by $i \in \{1, 2\}$, with respective weights $\phi$ and $1 - \phi$ in the economy. Both types view future asset prices as lognormally distributed, but (potentially) differ in their estimate of mean $\mu_i$ and variance $\sigma_i^2$. The researcher’s beliefs estimate $\hat{P}$ is unbiased on average in the sense of being equal to the weighted average of the two groups’ beliefs, $\hat{P} = \phi P_1 + (1 - \phi) P_2$. 

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Estimating this model using nonlinear least squares yields the parameter values $\phi = 0.6042$, $\mu_1 = 0.0806$, $\mu_2 = -0.0007$, $\sigma_1 = 0.0393$ and $\sigma_2 = 0.0203$. These results thus suggest that beliefs are indeed heterogeneous. About 60% of agents are relatively optimistic and have an estimate of expected returns of about 8%. The other group, representing roughly 40% of agents, is relatively pessimistic and estimates expected returns to be about zero. Note that using the expression $\phi = 1/(a_1(1/a_1 + 1/a_2))$ and the constraint that $1/(1/a_1 + 1/a_2) = 4$ implied by our initial assumptions about the representative agent’s preferences, one can recover the risk aversion coefficients of both groups, $a_1 = 4/\phi = 6.6203$ and $a_2 = 4/(1 - \phi) = 10.1061$. Thus, in this particular example, those agents which are most pessimistic also turn out to be those which have the highest degree of risk aversion.

Figure 8 depicts the actual and fitted misestimation factors. Note that in spite of the stringent CARA preferences assumption it makes, the simple setting used here is able to reproduce the density misestimation factor from Figure 5 quite closely. Even more importantly, no extreme or unreasonable assumptions about the shape of individual investors’ beliefs need to be made in order to obtain this result – it suffices to allow individual investors to have parameter estimates that differ somewhat and to assume that the researcher ignores beliefs heterogeneity in his estimation, even though his beliefs estimate is the weighted average of the individual agents’ beliefs. Taken somewhat more broadly, this analysis has the important implication that under heterogeneous beliefs, the density misestimation factor – and therefore estimates of implied risk aversion obtained under the implicit assumption of homogeneous beliefs – can take almost any shape. Heterogeneous beliefs – in contrast to mere heterogeneity in preferences among investors and “random” beliefs misestimation – therefore appear as the most likely cause of the smile effect in implied risk aversion.

4 Conclusion

This paper considers the properties of implied risk aversion estimators in three nested settings in order to explain the “smile effect” in implied risk aversion. The analysis demonstrates that if agents’ beliefs are homogeneous and can be estimated accurately, the implied risk aversion function will inherit some of the properties of agents’ utility functions. More specifically, if all agents are risk averse, then implied risk aversion will be strictly positive. Moreover, if all
agents display constant (decreasing, increasing) absolute risk aversion, so will the implied risk
aversion function. Finally, if all agents display constant relative risk aversion, implied relative
risk aversion will be decreasing.

In light of these aggregation results, heterogeneity in investor preferences is not sufficient to
explain the implied risk aversion smile documented in the literature. Some misestimation of
investors’ beliefs is therefore likely to be responsible for the smile. It is shown that even
a minor misestimation of investors’ beliefs will lead to sizable perturbations of implied risk
aversion estimates. Conversely, moderate differences in the market-wide risk aversion have no
significant impact on the beliefs that can be inferred from the state-price density.

An empirical analysis demonstrates that beliefs misestimation resulting from stochastic volatil-
ity and jumps cannot account for the smile. Furthermore, the patterns of beliefs misestimation
that can be inferred from the implied risk aversion estimates reported in the literature are very
peculiar and hard to reproduce in simple homogeneous beliefs situations. If the assumption of
homogeneity in investors’ beliefs is relaxed, however, complex belief misestimation patterns and
corresponding “smile effects” in implied risk aversion can be obtained easily. Fitting a simple
model with two classes of investors with heterogeneous, lognormal beliefs closely reproduces
the empirical beliefs misestimation patterns, suggesting that heterogeneity in investor beliefs is
the most likely cause of the implied risk aversion smile.
References


Figure 1: Effect of Beliefs Misestimation on Implied Risk Aversion. Implied absolute risk aversion estimation error $\epsilon(S) = \tilde{P}'(S)/\tilde{P}(S) - P'(S)/P(S)$, where $P$ denotes investors’ actual beliefs and $\tilde{P}$ estimated beliefs in the case of lognormally distributed asset prices. A slight underestimation of the standard deviation of asset prices by the researcher leads to a sizable implied risk aversion estimation error (value of the parameters: $\mu = \tilde{\mu} = 0$, $\sigma = 0.1$, $\tilde{\sigma} = 0.095$).
Figure 2: Inferring Beliefs from State Prices and Risk Aversion
Semi-parametric estimation of the state-price density (SPD) and the corresponding statistical densities for three different risk aversion settings: constant absolute risk aversion (CARA), constant relative risk aversion (CRRA) and decreasing relative risk aversion (DRRA). Each case assumes a coefficient of relative risk aversion of 4 at the current futures price of 450. The three statistical densities obtained in this fashion are almost indistinguishable.
Figure 3: State-Price Density and Statistical Density with Stochastic Volatility and Jumps. Semi-parametric estimation of the state-price density and nonparametric estimation of the statistical density based on a simulation of the stochastic volatility and jumps model of Pan (2002). Results are reported for time horizons of 1, 2, 4 and 6 months (top left to bottom right).
Figure 4: Implied Risk Aversion with Stochastic Volatility and Jumps. Implied risk aversion patterns obtained from the semi-parametric estimate of the state-price density and the nonparametric estimate of the statistical density based on a simulation of the stochastic volatility and jumps model of Pan (2002) and depicted in Figure 3. Results are reported for time horizons of 1, 2, 4 and 6 months (top left to bottom right). The risk aversion estimates exhibit a smile pattern comparable to that documented by Aït-Sahalia and Lo (2000), suggesting that stochastic volatility and jumps cannot account for the smile effect reported by these authors.
Figure 5: Link between Beliefs Misestimation and Risk Aversion Misestimation. Errors in estimating investors’ beliefs can be derived from implied risk aversion and assumptions about actual risk aversion. The U-shaped risk implied risk aversion patterns reported in the literature suggest that historical return data underestimates agents’ assessment of very low aggregate wealth states and overestimate the probability of very high wealth states.
**Figure 6: Expected Return Estimates and Beliefs Misestimation Factor.** Beliefs misestimation factor when the researcher’s estimate of expected returns exceeds that of investors and asset prices are lognormally distributed. The misestimation factor is strictly increasing (value of the parameters: $\mu = 0$, $\tilde{\mu} = 0.1$, $\sigma = \tilde{\sigma} = 0.2$).
Figure 7: Variance Estimates and Beliefs Misestimation Factor. Beliefs misestimation factor when the researcher’s estimate of the variance in asset returns is lower than that of investors and asset prices are lognormally distributed. The misestimation factor is single-peaked (value of the parameters: $\mu = \hat{\mu} = 0$, $\sigma = 0.2$, $\hat{\sigma} = 0.18$).
Figure 8: Heterogeneous Beliefs and Beliefs Misestimation Factor. Fitting a simple model with two classes of CARA investors with heterogeneous, lognormal beliefs reproduces the beliefs misestimation factor reported in Figure 5 quite closely. This occurs even though the researcher’s probability assessments are constrained to equal the average beliefs of the two groups of investors. Heterogeneous beliefs therefore appear to be the most likely cause of the implied risk aversion smile.