Back to Basics: a new approach to the discrete dividend problem*

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1 Introduction

Stocks frequently pay dividends, which has implications for the value of options on these stocks. For options on a large portfolio of stocks, one can approximate discrete dividend payouts with a dividend yield and use the generalized Black-Scholes-Merton (BSM) model. For options on one stock, this is not a viable approximation, and the discreteness of the dividend has to be explicitly modelled.[1] We discuss how to properly make the necessary

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[1]An alternative to discrete cash dividend is discrete dividend yield. Implementation of discrete dividend yield is well known and straight forward using recombining lattice models (see for instance Haug, 1997; Hull, 2000). Typically, however, at least the first dividend is known in advance with some confidence. Discrete cash dividend models consequently seem to have been the models of choice among practitioners, despite having to deal with a more complex modelling problem. For very long term options the predictability of future dividends is less pronounced, and dividends should be somewhat correlated with the stock price level. Moreover, cash dividends tend to be reduced following a significant stock price decrease. If the stock price rallies, on the other hand, it indicates that the company is doing better than expected, which again can

adjustments.

It might come as a surprise to many readers that we write an entire paper about a supposedly mundane issue—which is thoroughly treated in any decent derivatives text book (including, but not limited to Cox and Rubinstein, 1985; Chriss, 1997; Haug, 1997; Hull, 2000; McDonald, 2003; Stoll and Whaley, 1993; Wilmott, 2000). It turns out, however, that some of the adjustments suggested in the extant literature admit arbitrage—which is fine if all your competitors use these models, but you know how to do the arbitrage-free adjustment.

Existing methods

Escrowed dividend model: The simplest escrowed dividend approach makes a simple adjustment to the BSM formula. The adjustment consists of replacing the stock price S_0 by the stock price minus the present value of the dividend $S_0 - \mathrm{e}^{-rt_D}D$, where D is the size of the cash dividend to be paid at time t_D. Because the stock price is lowered, the approach will typically lead to too little absolute price volatility ($\sigma_t S_t$) in the period before the dividend is paid. Moreover, it is just an approximation used to fit the ex-dividend price process into the geometric Brownian motion (GBM) assumption of the BSM formula. The approach will in general undervalue call options, and the mispricing is larger the later in the option's lifetime the dividend is paid. The approximation suggested by Black (1975) for American options suffers from the same problem, as does the Roll-Geske-Whaley (RGW) model (Roll, 1977; Geske, 1979, 1981; Whaley, 1981). The RGW model uses this approximation of the stock price process, and applies a compound option approach to take into account the possibility of early exercise. Not only does this yield a poor approximation in certain circumstances, but it can open up arbitrage opportunities!

result in higher cash dividends. For very long term options then, it is likely that the discrete dividend yield model can be a competing model to the discrete cash dividend model.

Several papers discuss the weakness of the escrowed dividend approach. In the case of European options, suggested fixes are often based on adjustments of the volatility, in combination with the escrowed dividend adjustment. We shortly discuss three such approaches, all of which assume that the stock price can be described by a GBM:

1. An adjustment popular among practitioners is to replace the volatility σ with $\sigma_2 = \frac{\sigma S}{S - D\mathrm{e}^{-rt_D}}$, (see for instance Chriss, 1997). This approach increases the volatility relative to the basic escrowed divided process. However, the adjustment yields too high volatility if the dividend is paid out early in the option's lifetime. The approach typically overprices call options in this situation.

2. A more sophisticated volatility adjustment also takes into account the timing of the dividend (Haug and Haug, 1998; Beneder and Vorst, 2001). It replaces σ with σ_2 as before, but not for the entire lifetime of the option. The idea behind the approximation is to leave volatility unchanged in the time before the dividend payment, and to apply the volatility σ_2 after the dividend payment. Since the BSM model requires one volatility as input, some sort of weight must be assigned to each of σ and σ_2. This is accomplished by looking at the period after the dividend payment as a forward volatility period (Haug and Haug, 1996). The single input volatility is then computed as

$$\hat{\sigma} = \sqrt{\frac{\sigma^2 t_D + \sigma_2^2 (T - t_D)}{T}},$$

   where T is the time of expiration for the option. The adjustment in the presence of multiple dividends is described in Appendix A. This is still simply an adjustment to *parameters* of the GBM price process, that ensures the adjusted price process remains a GBM—at odds with the true ex-dividend price process. The adjustment therefore remains an approximation. One can easily show numerically that this method performs particularly poorly in the presence of multiple dividends.

3. Bos et al. (2003) suggest a more sophisticated volatility adjustment, described in Appendix B. Still this is just another "quick fix" to try to get around the problems with the escrowed dividend price process. Numerical calculations show that this approach offers quite accurate values provided the dividend is small to moderate. For very high dividends the method performs poorly. The poor performance seemingly also occurs for long term options with multiple dividends.

4. A slightly different way to implement the escrowed dividend process is to adjust the stock price and strike (Bos and Vandermark, 2002). Even if this approach seems to work better than the approximations mentioned above, it suffers from approximation errors for large dividends just like approximation 3.

Lattice models: An alternative to the escrowed dividend approximation is to use non-recombining lattice methods (see for instance Hull, 2000). If implemented as a binomial tree one builds a new tree from each node on each dividend payment date. A problem with all non-recombining lattices is that they are time consuming to evaluate. This problem is amplified with multiple dividends. Schroder (1988) describes how to implement discrete dividends in a recombining tree. The approach is based on the escrowed dividend process idea, however, and the method will therefore significantly misprice options. Wilmott et al. (1993, p. 402) indicate what seems to be a sounder approach to ensure a recombining tree for the spot price process with a discrete dividend.

Problems and weaknesses of current approaches

In fact, the admission of arbitrage is merely the most egregious example among problems or weaknesses in current approaches. Of course, we cannot claim to have seen every paper written on this subject, but of those we have seen, we note one or more of the following weaknesses:

(i). *Logical flaws.* Many approaches use the idea of an escrowed dividend process, as discussed above. The idea is to break up the stock price process into two pieces, a risky part and an escrowed dividend part. The admirable goal is to "guarantee" that the declared dividend will be received. The logical flaw is that the resulting stock price process changes with the option expiration. Whatever the stock price process is, it cannot depend upon which option you happen to be considering. The fact that this is a logical fallacy is, to our thinking, under-appreciated.

(ii). *Ill-defined stock price processes.* Some treatments just don't "get it" that a constant dividend D can't be paid at arbitrarily low stock prices. This tends to be a problem with discussions of "non-recombining trees". The problem is (a) the discussion "never" mention that something must be said about this issue.[2] A related issue, which we have seen in some recent models is that (b) negative stock prices are explicitly allowed. The reason that authors can get away with (a) is that, in many computations, the problematic region is a very low probability event. But, this is no excuse for failing to completely define your model. The reader should be able to say similar bad things about (b).

(iii). *Only geometric Brownian motion* is discussed. This is understandable in the early literature, but not now-days. The weaknesses of GBM are well-know. Some fixes include stochastic volatility, jumps, and other ideas. A dividend treatment needs to accommodate these models.

(iv). *Arbitrage issues.* This has been mentioned above and an example is given below. All

[2]Except, to our knowledge (which, unlike the universe, is limited) Wilmott et al. (1993, p. 399), who mention this problem and suggest to let the company go bankrupt if the dividend is larger than the asset price. This approach avoids negative stock prices. Moreover, McDonald (2003, p. 352) points out this problem, and suggests using the approach of Schroder (1988). As we have already pointed out, this method has other flaws.

we will add here is that the source of this problem is (i) or (ii) above. Finally, to paraphrase the physicist Sidney Coleman, who was speaking in another context, just because a theory is dead wrong, doesn't mean it can't be highly accurate. In other words, the reader will observe below many instances of highly accurate numerical agreement between our current approach and some existing approximations that suffer from (ii) for example. Nevertheless, we would argue that a theory that predicts negative stock prices must be 'dead wrong' in Coleman's sense.[3]

Arbitrage example: In the case of European options, the above techniques are *ad hoc*, but get the job done (in most cases) when the corrections are properly carried out. To give you an idea of when it really goes wrong, consider the model of choice for American call options on stocks whose cum dividend price is a GBM. The Roll-Geske-Whaley (RGW) model has for decades been considered a brilliant closed form solution to price American calls on dividend paying stocks. Consider the case of an initial stock price of 100, strike 130, risk free rate 6%, volatility 30%, one year to maturity, and an expected dividend payment of 7 in 0.9999 years. Using this input the RGW model posits a value of 4.3007. Consider now another option, expiring just before the dividend payment, say in 0.9998 years. Since this in effect is an American call on a non-dividend paying stock it is not optimal to exercise it before maturity. In the absence of arbitrage the value must therefore equal the BSM price of 4.9183. This is, however, an arbitrage opportunity! The arbitrage occurs because the RGW model is mis-specified, in that the dynamics of the stock price process depends on the timing of the dividend. Similar examples have been discussed by Beneder and Vorst (2001) and Frishling (2002). This is not just an esoteric example, as several well known software systems use the RGW model and other similar mis-specified models. In more complex situations than described in this example the arbitrage will not necessarily be quite so obvious, and one

[3]Coleman (1985), speaking of Fermi's theory of the 'Weak force': *Phrased another way, the Fermi theory is obviously dead wrong because it predicts infinite higher-order corrections, but it is experimentally near perfect, because there are few experiments for which lowest-order Fermi theory is inadequate.*

would need an accurate model to confidently take advantage of it. It is precisely such a model we present in the present paper.

2 General Solution

A single dividend

You wish to value a Euro-style or American-style equity option on a stock that pays a discrete (point-in-time) dividend at time $t = t_D$. The simpler problem is to first specify a price process whereby any dividends are reinvested immediately back into the security—this is the so-called cum-dividend process $S_t = S(t)$. In general, S_t is not the market price of the security, but instead is the market price of a hypothetical mutual fund that only invests in the security. To distinguish the concepts, we will write the market price of the security at time t as Y_t, which we will sometimes call the ex-dividend process. Of course, if there are no dividends, then $Y_t = S_t$ for all t. Even if the company pays a dividend, we can always arrange things so that $Y_0 = S_0$, which guarantees (by the law of one price) that $Y_t = S_t$ for all $t < t_D$.

In our treatment, we allow S_t to follow a very general continuous-time stochastic process. For example, your process might be one of the following (to keep things simple, we suppose a world with a constant interest rate r):

Example (cum-dividend) processes

(P1). GBM: $dS_t = rS_t\,dt + \sigma S_t\,dB_t$, where σ is a constant volatility and B is a standard Brownian motion.

(P2). Jump-diffusion: $dS_t = (r - \lambda k) S_t \, dt + \sigma S_t \, dB_t + S_t \, dJ_t$, where dJ_t is a Poisson driven jump process with mean jump arrival rate λ, and mean jump size k.

(P3). Jump-diffusion with stochastic volatility: $dS_t = (r - \lambda k) S_t \, dt + \sigma_t S_t \, dB_t + S_t \, dJ_t$, where σ_t follows its own separate, possibly correlated, diffusion or jump-diffusion.

Consider an option at time t, expiring at time T, and assume for a moment that there are no dividends so that $Y_t = S_t$ for all $t \leq T$. In that case, clearly, models (P1) and (P2) are one-factor models: the option value $V(S_t, t)$ depends only upon the current state of one random variable. Model (P3) is a two-factor model, $V(S_t, \sigma_t, t)$. Obviously, "n-factor" models are possible in principle, for arbitrary n, and our treatment will apply to those too. Note that we leave the dependence on many parameters implicit.

What the examples have in common is that the stock price, with zero or more additional factors, jointly form a Markov process,[4] in which the discounted stock price is a martingale. Also, for simplicity, we will consider only time-homogenous processes; this means that transition densities for the state variables depend only on the length of the time period in question, and not on the beginning and end dates of the time period.

Choosing a dividend policy

Now we want to create an option formula for the case where the company declares a single discrete dividend of size D, where the "ex-dividend date" is at time t_D during the option holding period. We consider an unprotected Euro-style option, so that the option holder will not receive the dividend. Since option prices depend upon the market price of the security, for one factor models, we must now write $V(Y_t, t)$.

Note that we are being very careful with our choice of words; we have said that the company

[4] The Markov assumption is for simplicity. Perhaps it can be relaxed. We leave this issue open.

"declares" a dividend D. What we mean by that is that it is the company's stated intention to pay the amount D *if that is possible*. When will it be impossible? We assume that the company cannot pay out more equity than exists. For simplicity, we imagine a world where there are no distortions from taxes or other frictions, so that a dollar of dividends is valued the same as a dollar of equity. In addition, we always assume that there are no arbitrage opportunities. In such a world, if the company pays a dividend D, the stock price at the ex-dividend date must drop by the same amount: $Y(t_D) = Y(t_D^-) - D = S(t_D^-) - D$. Our notation is that t_D^- is the time instantaneously before the ex-dividend date t_D (in a world of continuous trading). Since stock prices represent the price of a limited liability security, we must have $Y(t_D) \geq 0$, so we have a contradiction between these last two concepts if $S(t_D^-) < D$.

This is the fundamental contradiction that every discrete dividend model must resolve. We resolve it here by the following minimal modification to the dividend policy. We assume that the company will indeed pay out its declared amount D if $S^- > D$, abbreviating $S^- = S(t_D^-)$. However, in the case where $S^- < D$, we assume that the company pays some *lesser* amount $\Delta(S^-)$ whereby $0 \leq \Delta(S^-) \leq S^-$. That is our general model, a "minimally modified dividend policy." In later sections, we show numerical results for two natural policy choices, namely $\Delta(S^-) = S^-$ (liquidator), and $\Delta(S^-) = 0$ (survivor). The first case allows liquidation because the ex-dividend stock price (at least in all of the sample models P1–P3 above) would be absorbed at zero. We will assume that a zero stock price is always an absorbing state. The second case (and, indeed any model where $\Delta(S) < S$) allows survival because the stock price process can then attain strictly positive values after the dividend payment.

These choices, liquidation versus survival, sound dramatically different. In cases of financial distress, where indeed the stock price is very low, they would be. But such cases are relatively rare. As a practical matter, we want to stress that for most applications, the choice of

$\Delta(S)$ for $S < D$ has a negligible financial effect; the main point is that *some* choice must be made to fully specify the model. There is little financial effect in most applications because the probability that an initial stock price S_0 becomes as small as a declared dividend D is typically negligible; if this is not obvious, then a short computation with the log-normal distribution should convince you. In any event, we will demonstrate various cases in numerical examples.

To re-state what we have said in terms of an SDE for the security price process, our general model is that the actual dividend paid becomes the random variable $\mathcal{D}(S)$, where

$$\mathcal{D}(S) = \begin{cases} D, & \text{if } S > D \\ \Delta(S) \leq S, & \text{if } S \leq D \end{cases}. \tag{1}$$

In (1) D is the declared (or projected) dividend—a constant, independent of S. The functional form for $\mathcal{D}(S)$ is any function that preserves limited liability. Then, the market price of the security evolves, using GBM as the prototype, as the SDE:

$$dY_t = \left[rY_t - \delta(t - t_D)\mathcal{D}(Y_{t_D^-}) \right] dt + \sigma Y_t \, dB_t, \tag{P1a}$$

where $\delta(t - t_D)$ is Dirac's delta function centered at t_D. The same SDE drift modification occurs for (P2), (P3), or any other security price process you wish to model.

It's worth stressing that the Brownian motion B_t that appears in (P1) and (P1a) have *identical realizations*. You might want to picture a realization of B_t for $0 \leq t \leq T$. Your mental picture will ensure that $Y_t = S_t$ for all $t < t_D$ and $Y_{t_D} = S_{t_D} - \mathcal{D}(S_{t_D})$. Note that Y_t is completely determined by knowledge of S_t alone for all $t \leq t_D$ (the fact that $Y_{t_D} = f(S_{t_D})$, where f is a deterministic function, will be crucial later). What about $t > t_D$? For those (post ex-dividend date) times, little can be said about Y_t given only knowledge of S_t (all you can say is that $Y_t < S_t$ if $D > 0$).

For our results to be useful, you need to be able to solve your model in the absence of dividends. By that, we mean that you know how to find the option values and the transition density for the stock price (and any other state variables) to evolve. Note that you need not have these functions in so-called "closed-form", but merely that you have some method of obtaining them. This method may be an analytic formula, a lattice method, a Monte Carlo procedure, a series solution, or whatever.

It's awkward to keep placeholders for arbitrary state variables, so we will simply write $\phi(S_0, S_t, t)$ (the cum-dividend transition density), with the understanding that additional state variable arguments should be inserted if your model needs them. To be explicit, the transition density is the probability density for an initial state (stock price plus other state variables) S_0 to evolve to the final state S_t in a time t. This evolution occurs under the risk-adjusted, cum-dividend process (or measure) such as the ones given under "Example (cum-dividend) processes" above. For GBM, $\phi(S_0, S_t, t)$ is the familiar log-normal density. Option formulas are similarly displayed only for the one-factor model, with additional state variables to be inserted by the reader if necessary.

With this discussion, let's collect all of our stated assumptions in one place:

(A1) Markets are perfect (frictionless, arbitrage-free), and trading is in continuous-time.

(A2) After risk adjustment, every cum-dividend stock price S_t, jointly with $n \geq 0$ additional factors (which are suppressed), evolve under a time-homogenous, (Markov) stochastic process. S_t is non-negative; if $S_t = 0$ is reached, it's a trap state (absorbing). All these statements also apply to the market price (ex-dividend process) Y_t.

(A3) If a company declares (or you project) a discrete dividend D, this is promoted to a random dividend policy $\mathcal{D}(S)$ in the minimal manner of (1). This causes the market price of the stock to drop instantaneously on an ex-dividend date, as prototyped by

(P1a).

Our Main Result

We write $V_E(S_t, t; D, t_D)$ for the time-t fair value of a European-style option that expires at time T, in the presence of a discrete dividend D paid at time t_D. The last two arguments are the main parameters in the fully specified dividend policy $\{t_D, \mathcal{D}(S)\}$ where $t < t_D < T$. If there is no dividend between time t and the option expiration T, we simply drop the last two arguments and write $V_E(S_t, t)$. So, to be clear about notation, when you see an option value $V(\cdot)$ that has *only two arguments*, this will be a formula that you know in the absence of dividends, like the BSM formula. Again, the strike price X, option expiration T, and other parameters and state variables have been suppressed for simplicity. Then, here is our main result:

Proposition 1. *Under assumptions **(A1)–(A3)**, the adoption by the company of a single discrete dividend policy $\{t_D, \mathcal{D}(S)\}$, causes the fair value of a Euro-style option to change from $V_E(S_0, 0)$ to $V_E(S_0, 0; D, t_D)$, where*

$$V_E(S_0, 0; D, t_D) = \mathrm{e}^{-rt_D} \int_0^\infty V_E(S - \mathcal{D}(S), t_D) \phi(S_0, S, t_D) \, \mathrm{d}S. \tag{2}$$

Proof. A very elaborate argument is:

(i) Let S be the cum-dividend process, and Y the ex-dividend process as before. Assume dividend policy $\mathcal{D}(S)$, paid on date t_D. Then $Y_t = S_t$ for all $t < t_D$, $Y_{t_D} = S_{t_D} - \mathcal{D}(S_{t_D})$, and typically $Y_t \neq S_t$ for $t > t_D$. Assume the distribution function F_S for S and that the option pays off $g(Y_T)$ at time T.

(ii) Relative to time $t < T$ the payoff from the option is *a random variable*/uncertain cash flow. The absence of arbitrage then implies that the price at time t for this cash flow is

$$\mathrm{e}^{-r(T-t)}\mathrm{E}_t\left\{g(Y_T)\right\}.$$

Let's call this value $V(Y_t, t)$, which again is *a random variable* relative to any time $t' < t$. For any $t \geq t_D$ this random variable is the price of an option on a non-dividend paying stock, and therefore assumed known for any value of Y_t.

(iii) We're interested in the value of the option on Y at time 0. Since $V(Y_t, t)$ is simply a random variable relative to today (time 0), we can use the exact same argument as in (ii) above. Its value at time 0 must therefore, in the absence of arbitrage, be

$$\mathrm{e}^{-rt}\mathrm{E}_0\left\{V(Y_t, t)\right\}. \tag{3}$$

So far, the only result we've used is the (almost) equivalence of no arbitrage and the martingale property of prices (Harrison and Kreps, 1979; Harrison and Pliska, 1981; Delbaen and Schachermayer, 1995, among others), and we haven't said anything about what specific date $t \geq t_D$ is (the argument would hold for $t < t_D$ too, but we want $V(Y_t, t)$ to be "known").

(iv) Assuming sufficient regularity, we can write (3) in integral form wrt. a distribution function

$$\mathrm{E}_0\left\{V(Y_t, t)\right\} = \int_0^\infty V(y, t)\, dF_Y(y; Y_0, 0, t).$$

The trouble with this integral is that we typically do not know the distribution function F_Y unless $t < t_D$—but in that case V is unknown (orthogonal knowledge ;-).

(v) The main insight is now that by considering $t = t_D$ we *know* that $Y_{t_D} = S_{t_D} - \mathcal{D}(S_{t_D})$. This means that we do not need to know F_Y, since by the "Law of the Unconscious Statistician"

$$\begin{aligned}\mathrm{E}_0\left\{V(Y_{t_D}, t_D)\right\} &= \int_0^\infty V(y, t_D)\, dF_Y(y; Y_0, 0, t_D) \\ &= \int_0^\infty V(s - \mathcal{D}(s), t_D)\, dF_S(s; S_0, 0, t_D).\end{aligned}$$

In other words, at time $t = t_D$ we know both how to compute V and how to compute its expectation $\mathrm{E}_0\left\{V(Y_{t_D}, t_D)\right\}$, and this is the only date for which this holds. With a time-homogeneous transition density $dF_S(s; S_0, 0, t_D) = \phi(S_0, s, t_D)\, ds$, and we arrive at (2). $\square$

An Example: Take GBM, where the dividend policy is $\Delta(S) = S$ (liquidator) for $S \leq D$. Then (2) for a call option becomes

$$C_E(S_0, 0; D, t_D) = \mathrm{e}^{-rt_D} \int_D^\infty C_E(S - D, t_D)\phi(S_0, S, t_D)\, \mathrm{d}S. \tag{4}$$

Note that the call price in the integrand of (2) is zero for $S - \mathcal{D}(S) = 0$ ($S \leq D$). In (4), $\phi(S_0, S, t)$ is simply the (no-dividend) log-normal density and $C_E(S - D, t_D)$ is simply the no-dividend BSM formula with time-to-go $T - t_D$. For example, suppose $S_0 = X = 100$, $T = 1$ (year), $r = 0.06$, $\sigma = 0.30$, and $D = 7$. Then, consider two cases; (i) $t_D = 0.01$, and (ii) $t_D = 0.99$. We find from (4) the high precision results: (i) $C_E(100, 0; 7, 0.01) = 10.59143873835989$ and (ii) $C_E(100, 0; 7, 0.99) = 11.57961536099359$.

American-style options

It is well-known, and easily proved that, for an American-style call option with a discrete dividend, early exercise is only optimal instantaneously prior to the ex-dividend date (Merton, 1973). This result of course applies to the present model. Hence, to value an American-style call option with a single discrete dividend, you merely replace (2) with

$$C_A(S_0, 0; D, t_D) = \mathrm{e}^{-rt_D} \int_0^\infty \max\{(S - X)^+, C_E(S - \mathcal{D}(S), t_D)\} \phi(S_0, S, t_D)\, \mathrm{d}S, \qquad (5)$$

Early exercise is never optimal unless there is a finite solution S^* to $S^* - X = C_E(S^* - D, t_D)$, where we are assuming that $X > D$ (a virtual certainty in practice). In this case, the reader may want to break up the integral into two pieces, but we shall just leave it at (5).

For American-style put options, as is also well-known, it can be optimal to exercise at any time prior to expiration, even in the absence of dividends. So, in this case, you are generally forced to a numerical solution, evolving the stock price according to your model. This is the well-known backward iteration. What may differ from what you are used to is that you must allow for an instantaneous drop of $\mathcal{D}(S)$ on the ex-date.

One down, $n - 1$ to go

With the sequence of dividends $\{(D_i, t_i)\}_{i=1}^n$, $t_1 < t_2 < \ldots < t_n$, the argument leading to formula (2) still holds. Simply repeat it iteratively, starting at time t_{n-1} by applying (2) to the last dividend (D_n, t_n). While straight forward, this procedure involves evaluating an n-fold integral. We therefore show a simpler way to compute it in Section 4.

3 Dividend Models

It is now time to compare specific dividend models, $\Delta(S)$. We consider the two extreme cases for $\Delta(S)$, use these to develop an inequality for an arbitrary dividend model, and then illustrate the impact on option prices.

Liquidator: We consider first the situation where the dividend is reduced to S_{t_D} when $S_{t_D} < D$, i.e., $\Delta(S) = \Delta^l(S) = S$ for $S < D$. This is tantamount to the firm being liquidated if the cum dividend stock price falls below the declared dividend. Although this might seem an extreme assumption, keep in mind that for most reasonable parameter values, this will be close to a zero-probability event. It is a simple approximation that succeeds in ensuring a non-negative ex-dividend stock price. In this case (2) reduces to

$$\begin{aligned} V_E^l(S_0, 0; D, t_D) &= \mathrm{e}^{-rt_D} \int_D^\infty V_E(S - D, t_D)\phi(S_0, S, t_D)\,\mathrm{d}S \\ &\quad + \mathrm{e}^{-rt_D} \int_0^D V_E(S - \Delta^l(S), t_D)\phi(S_0, S, t_D)\,\mathrm{d}S, \qquad (6) \\ &= \mathrm{e}^{-rt_D} \left\{ \int_D^\infty V_E(S - D, t_D)\phi(S_0, S, t_D)\,\mathrm{d}S + V_E(0, t_D) \int_0^D \phi(S_0, S, t_D)\,\mathrm{d}S \right\}. \end{aligned}$$

In this decomposition the "tail value" $\mathrm{e}^{-rt_D} V_E(0, t_D) \int_0^D \phi(S_0, S, t_D)\,\mathrm{d}S$ will vanish for a call option, but not for a put option.

Survivor: Consider next the situation where the dividend is canceled when $S_{t_D} < D$, i.e., $\Delta(S) = \Delta^s(S) = 0$ for $S \leq D$. This approximation also succeeds in ensuring a non-negative ex-dividend stock price, and also allows the firm to live on with probability one after the dividend payout. The option price is now similar to the one for the liquidator dividend, with a slight modification to the tail value of the option contract:

$$\begin{aligned} V_E^s(S_0, 0; D, t_D) = \mathrm{e}^{-rt_D} \Bigg\{ &\int_D^\infty V_E(S - D, t_D)\phi(S_0, S, t_D)\,\mathrm{d}S \\ &+ \int_0^D V_E(S, t_D)\phi(S_0, S, t_D)\,\mathrm{d}S \Bigg\}. \qquad (7) \end{aligned}$$

From (6) and (7) we can now establish a result that should enable you to sleep better at night if you are concerned with your choice of dividend policy $\Delta(\cdot)$.

Corollary 1. *Let $C_E(S_0, 0; D, t_D)$ be given by (2) for a generic dividend policy $\Delta(S)$ such that $0 \leq \Delta(S) \leq S$. For European call options*

$$C_E^l(S_0, 0; D, t_D) \leq C_E(S_0, 0; D, t_D) \leq C_E^s(S_0, 0; D, t_D).$$

If $S_{t_D} < D$ with positive probability then additionally $C_E^l(S_0, 0; D, t_D) < C_E^s(S_0, 0; D, t_D)$.

Proof. The $\int_D^\infty$-integral is identical for all three options. Since $0 \leq \Delta(S) \leq S$ it follows that

$$\begin{aligned}
&\int_0^D C_E(S - \Delta^l(S), t_D)\phi(S_0, S, t_D)\,\mathrm{d}S \\
&\qquad\leq \int_0^D C_E(S - \Delta(S), t_D)\phi(S_0, S, t_D)\,\mathrm{d}S \\
&\qquad\qquad\leq \int_0^D C_E(S - \Delta^s(S), t_D)\phi(S_0, S, t_D)\,\mathrm{d}S.
\end{aligned}$$

The strict inequality follows from a similar argument. □

We could easily have established the same inequality for American call options, by simply using more cumbersome notation that takes early exercise into account. The interesting part of the result is the weak inequality. It tells us that if there's a negligible difference in prices between the liquidator and survivor dividend policies, then it doesn't matter what assumption you make about $\Delta(S)$ as long as it satisfies limited liability.

We end this section with an illustration of the relevance of the dividend model $\Delta(S)$. We do so by illustrating the pricing implications of the two extreme dividend policies, liquidator and survivor, in a specific case.

A financial fairy "tail": A long-lived financial service firm, let's call them Ye Olde Reliable Insurance, paid a hefty dividend once a year, which they liked to declare well in advance. Once declared, they had never missed a payment, not once in their 103-year history.

As their usual practice, they went ex-dividend every June 30 and declared their next dividend in November. One November, with their stock approaching the \$100 mark, the Ye Olde board decided that 6% seemed fair and easily do-able, so they declared a \$6 dividend for the next June 30.

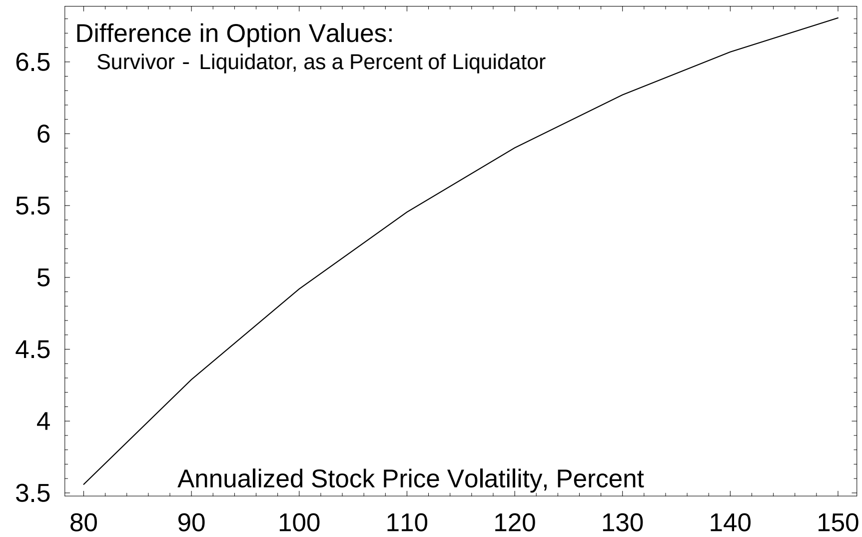
Unfortunately, during the very next month (December), the outbreak of a mysterious new virus coupled with an 8.2 temblor centered in Newport Beach devastated both their property/casualty and health insurance subs and their stock plummeted to the \$10 range.

The CBOE dutifully opened a new option series striking at \$10 with a leaps version expiring one year later. To keep things simple, we will imagine this series expires exactly in one year with an ex-dividend date at exactly the 1/2 year mark. Of course, there was much speculation and uncertainty about the declared \$6 dividend. The company's only comment was a terse press release saying "the board has spoken."

So, the potential option buyer was faced with a contract with $S = 10$, $X = 10$, $T = 1$ year and $t_D = 0.5$. If they paid in full, then $D = 6$. Interest rates were at 6%.

Our "liquidator" option model postulates that the company would pay in full unless the stock price S_{t_D} at mid-year was below $D = 6$, in which case they would pay out all the remaining equity, namely S_{t_D}. The skeptics said "no way" and proposed a new "survivor" option model in which the board would completely drop the dividend if $S_{t_D} \leq D$. The call price in this new model was larger by a "tail value". The big debate in the Wilmott forums became what volatility should one use to compute these values and, in the end, of course, no one knew. But everyone could agree that the stock price sure was volatile. So one way to proceed was to compute the option price in both models for volatilities ranging from 80% to 150% and the results are shown in the figure. Interestingly, the results only differed by

Figure 1: Relative tail value for a European call option



3.5 to 7% even in this extreme scenario with a doubtful yield, if paid, in the 60% range.

4 Applications

To illustrate the application of the pricing formula we now specialize the option contracts as well as the stock price process.

European call and put options: It is straight forward to derive the following put-call parity:

Proposition 2. *For a general cum-dividend price process S_t and dividend policy $\mathcal{D}(S)$ as in* (1),

$$C_E(S_0, 0; D, t_D) + \mathrm{e}^{-rT} X + \mathrm{e}^{-rt_D} \bar{D} = P_E(S_0, 0; D, t_D) + S_0, \tag{8}$$

where

$$\bar{D} = D - \int_0^D \phi(S_0, S, t_D)(D - \Delta(S))\,\mathrm{d}S$$

is the expected received dividend.

Proof. The idea of the argument is standard (Merton, 1973): Since $C_E(Y,T;D,t_D) + X = (Y-X)^+ + X = P_E(Y,T,D,t_D) + Y$, these two portfolios must have the same value today. Consider first the LHS. Its value is $\mathrm{e}^{-rT}\mathrm{E}_0\left\{(Y-X)^+ + X\right\} = C_E(S_0,0;D,t_D) + \mathrm{e}^{-rT}X$. Consider next the RHS. Its value is similarly given by $\mathrm{e}^{-rT}\mathrm{E}_0\left\{(X-Y)^+ + Y\right\} = P_E(S_0,0;D,t_D) + \mathrm{e}^{-rT}\mathrm{E}_0\left\{(Y+\bar{D}_T) - \bar{D}_T\right\} = P_E(S_0,0;D,t_D) + S_0 + \mathrm{e}^{-rT}\mathrm{E}_0\left\{\bar{D}_T\right\}$, where $\bar{D}_T$ is the future value (at time T) of the dividend received (a random variable). Since

$$\mathrm{E}_0\left\{\bar{D}_T\right\} = \mathrm{e}^{r(T-t_D)}\int_0^D \Delta(S)\phi(S_0,S,t_D)\,\mathrm{d}S + \mathrm{e}^{r(T-t_D)}D\int_D^\infty \phi(S_0,S,t_D)\,\mathrm{d}S$$

the result follows. $\square$

For the case of GBM stock price and liquidator dividend, $\Delta(S) = S$ for $S < D$, the value of a European call option can be written explicitly as

$$C_E(S_0,0;D,t_D) = \mathrm{e}^{-rt_D}\int_d^\infty C_E\left(S_0\mathrm{e}^{\left[r-\sigma^2/2\right]t_D+\sigma\sqrt{t_D}x} - D, t_D\right)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x, \qquad (9)$$
$$d = \frac{\ln\left(D/S_0\right) - \left(r-\sigma^2/2\right)t_D}{\sigma\sqrt{t_D}}.$$

A similar expression can be written down for the put option, but this is really not necessary in light of (8). Tables 1 and 2 report option prices for European call options for small and large dividends. The tables use the symbols:

BSM is the plain vanilla Black-Scholes-Merton model.

M73 is the BSM model with $S - \mathrm{e}^{-rt_D}D$ substituted for S—the escrowed dividend adjustment (Merton, 1973).

Vol1 is identical to M73, but with an adjusted volatility. The volatility of the asset is replaced with $\sigma_2 = \sigma \frac{S}{S - e^{-rt_D} D}$ (see for instance Chriss, 1997).

Vol2 is a slightly more sophisticated volatility adjustment than Vol1 (see Appendix A for a short description of this technique).

Vol3 is the volatility adjustment suggested by Bos et al. (2003) (see Appendix B for a short description of this adjustment).

BV adjusts the strike and stock price, to take into account the effects of the discrete dividend payment (Bos and Vandermark, 2002).

Num is a non-recombining binomial tree with 500 time steps, and no adjustment to prevent the event that $S - D < 0$ (see for instance Hull, 2000, for the idea behind this method).

HHL(4) is the *exact* solution in (4).

Table 1: European calls with dividend of 7
($S = 100$, $T = 1$, $r = 6\%$, $\sigma = 30\%$)

	BSM	Mer73	Vol1	Vol2	Vol3	BV	Num	HHL(4)
t				$X = 100$				
0.0001	14.7171	10.5805	11.4128	10.5806	10.5806	10.5806	10.5829	10.5806
0.5000	14.7171	10.6932	11.5001	11.1039	11.0781	11.0979	11.1079	11.1062
0.9999	14.7171	10.8031	11.5855	11.5854	11.5383	11.5887	11.5704	11.5887
				$X = 130$				
0.0001	4.9196	3.0976	3.7403	3.0977	3.0977	3.0977	3.0987	3.0977
0.5000	4.9196	3.1437	3.7701	3.4583	3.4203	3.4159	3.4368	3.4383
0.9999	4.9196	3.1889	3.7993	3.7993	3.6949	3.7263	3.7140	3.7263
				$X = 70$				
0.0001	34.9844	28.5332	28.9113	28.5332	28.5332	28.5332	28.5343	28.5332
0.5000	34.9844	28.7200	29.0832	28.9009	28.9047	28.9350	28.9218	28.9215
0.9999	34.9844	28.9016	29.2504	29.2504	29.2920	29.3257	29.3140	29.3257

Table 1 illustrates that the M73 adjustment is inaccurate, especially in the case when the dividend is paid close to the option's expiration. Moreover the Vol1 adjustment, often used by practitioners, gives significantly inaccurate values when the dividend is close to the beginning of the option's lifetime. Both Vol2 and BV do much better at accurately pricing the options. Vol3 yields values very close to the BV model. The non-recombining tree (Num) and our exact solution (HHL(4)) give very similar values in all cases. However, the non-recombining tree is not ensured to converge to the true solution (HHL(4)) in all situations, unless the non-recombining tree is set up to prevent negative stock prices in the nodes where $S - D < 0$. This problem will typically be relevant only with a very high divided, as we discussed in Section 3. For low to moderate cash dividends one can assume that even the "naive" non-recombining tree and our exact solution agree to economically significant accuracy.

Table 2: European calls with dividend of 50
($S = 100$, $T = 1$, $r = 6\%$, $\sigma = 30\%$)

	BSM	Mer73	Vol1	Vol2	Vol3	BV	Num	HHL(4)
t				$X = 100$				
0.0001	14.7171	0.1282	2.9961	0.1283	0.1282	0.1283	0.1273	0.1283
0.5000	14.7171	0.1696	3.0678	1.4323	0.5755	0.8444	1.0687	1.0704
0.9999	14.7171	0.2192	3.1472	3.1469	1.1566	2.1907	2.1825	2.1908
				$X = 130$				
0.0001	4.9196	0.0094	1.3547	0.0094	0.0094	0.0094	0.0092	0.0094
0.5000	4.9196	0.0133	1.3556	0.4313	0.0947	0.1516	0.2264	0.2279
0.9999	4.9196	0.0184	1.3609	1.3607	0.2510	0.6120	0.6072	0.6120
				$X = 70$				
0.0001	34.9844	1.6510	7.0798	1.6517	1.6513	1.6514	1.6515	1.6517
0.5000	34.9844	1.9982	7.3874	4.9953	3.3697	4.2808	4.7304	4.7299
0.9999	34.9844	2.3780	7.7100	7.7096	4.9966	7.2247	7.2122	7.2248

Table 2 shows that the BV and the non-recombining tree have significant differences when there's a significant dividend in the middle of the option's lifetime. The latter is closer to the true value. The Vol3 model strongly underprices the option when the dividend is this high.

American call and put options: Most traded stock options are American. We now do a numerical comparison of stock options with a single cash dividend payment. Tables 3–5 use the following models that differ from the European options considered above:

B75 is the approximation to the value of an American call on a dividend paying stock suggested by Black (1975). This is basically the escrowed dividend method, where the stock price in the BSM formula is replaced with the stock price minus the present value of the dividend. To take into account the possibility of early exercise one also compute an option value just before the dividend payment, without subtracting the dividend. The value of the option is considered to be the maximum of these values.

RGW is the model of Roll (1977); Geske (1979); Whaley (1981). It is considered a closed form solution for American call options on dividend paying stocks. As we already know, the model is seriously flawed.

HHL(5) it the exact solution in (5), again using the liquidator policy.

Table 3: American calls with dividend of 7
($D = 7, S = 100$, $T = 1$, $r = 6\%$, $\sigma = 30\%$)

	B75	RGW	Num	HHL(5)
t		$X = 100$		
0.0001	10.5805	10.5805	10.5829	10.5806
0.5000	10.6932	11.1971	11.6601	11.6564
0.9999	14.7162	13.9468	14.7053	14.7162
		$X = 130$		
0.0001	3.0976	3.0976	3.0987	3.0977
0.5000	3.1437	3.1586	3.4578	3.4595
0.9999	4.9189	4.3007	4.9071	4.9189
		$X = 70$		
0.0001	30.0004	30.0004	30.0000	30.0004
0.5000	32.3034	32.3365	32.4604	32.4608
0.9999	34.9839	34.7065	34.9737	34.9839

Table 3 shows that the RGW model works reasonably well when the divided is in the very beginning of the option lifetime. The RGW model exhibits the same problems as the simpler M73 or escrowed dividend method used for European options. The pricing error is particularly large when the dividend occurs at the end of the option's lifetime. The B75 approximation also significantly misprices options.

Table 4: American calls with dividend of 30
($D = 30, S = 100$, $T = 1$, $r = 6\%$, $\sigma = 30\%$)

	B75	RGW	Num	HHL(5)
t		$X = 100$		
0.0001	2.0579	2.0579	2.0574	2.0583
0.5000	9.8827	7.5202	9.9296	9.9283
0.9999	14.7162	11.4406	14.7053	14.7162
		$X = 130$		
0.0001	0.3345	0.3345	0.3322	0.3346
0.5000	1.6439	0.6742	1.7851	1.7855
0.9999	4.9189	2.4289	4.9071	4.9189
		$X = 70$		
0.0001	30.0004	30.0004	30.0000	30.0004
0.5000	32.3034	32.0762	32.3033	32.3037
0.9999	34.9839	34.1637	34.9737	34.9839

Table 5: American calls with dividend of 50
($D = 50, S = 100$, $T = 1$, $r = 6\%$, $\sigma = 30\%$)

	B75	RGW	Num	HHL(5)
t		$X = 100$		
0.0001	0.1282	0.1437	0.1273	0.1922
0.5000	9.8827	5.8639	9.8745	9.8828
0.9999	14.7162	9.3137	14.7053	14.7162
		$X = 130$		
0.0001	0.0094	0.0094	0.0092	0.0094
0.5000	1.6439	0.1375	0.5112	1.6492
0.9999	4.9189	1.1029	4.9071	4.9189
		$X = 70$		
0.0001	30.0004	30.0004	30.0000	30.0004
0.5000	32.3034	32.0762	32.6600	32.3034
0.9999	34.9839	34.1637	34.9737	34.9839

For very high dividend, as in Table 5, the mispricing in the RGW formula is even more clear; the values are significantly off compared with both non-recombining tree (Num) and our exact solution (HHL(5)). The simple B75 approximation is remarkably accurate. The intuition behind this is naturally that a very high dividend makes it very likely to be optimal to exercise just before the dividend date—a situation where the B75 approximation for good reasons should be accurate.

Multiple dividend approximation

We showed in Section 2 that it is necessary to evaluate an n-fold integral when there are multiple dividends. It is therefore useful to have a fast, accurate approximation. We now show how to approximate the option value in the case of a call option on a stock whose cum-dividend price follows a GBM, using the liquidator dividend policy.

First, let's write the exact answer on date t with a sequence of n dividends prior to T as $C_n(S, X, t, T)$, where X is the strike and T is the expiration date. Then, the first iteration of (4) in an exact treatment becomes

$$C_1(S, X, t_{n-1}, T) = \mathrm{e}^{-r(t_n - t_{n-1})} \int_{D_n}^{\infty} C_{\mathrm{BSM}}(S_1 - D_n, X, t_n, T)\phi(S, S_1, t_n - t_{n-1})\,\mathrm{d}S_1, \quad (10)$$

where $C_{\mathrm{BSM}}(\cdot)$ is the BSM model. This integral is quick to evaluate, just as in the single dividend cases tabulated above. The second iteration becomes

$$C_2(S, X, t_{n-2}, T) = \mathrm{e}^{-r(t_{n-1} - t_{n-2})} \int_{D_{n-1}}^{\infty} C_1(S_1 - D_{n-1}, X, t_{n-1}, T)\phi(S, S_1, t_{n-1} - t_{n-2})\,\mathrm{d}S_1. \quad (11)$$

Notice that we now integrate not over the BSM model, but rather the option price derived in the first iteration (10). Evaluation of (11) therefore involves a double integral. We know, however, that $C_1(\cdot)$ will look like an option solution and hence will have many of the characteristics of the BSM formula. If we can effectively parametrize $C_1(\cdot)$ with a BSM formula then it will be quick to evaluate (11).

Some key characteristics of $C_1(S, X, t_{n-1}, T)$ are as follows. First, it vanishes as $S \to 0$.

Second, because (standard) put-call parity becomes asymptotically exact for large S,

$$C_1(S, X, t_{n-1}, T) \approx S - \mathrm{e}^{-r(T-t_{n-1})}X - \mathrm{e}^{-r(t_n-t_{n-1})}D_n.$$

This suggests the BSM parametrization

$$C_1(S, X, t_{n-1}, T) \approx C_{\mathrm{BSM}}(S, X_{\mathrm{adj}}, t_{n-1}, T), \tag{12}$$

where $X_{\mathrm{adj}} = X + D_n\mathrm{e}^{-r(t_n-T)}$. The strike adjustment ensures correct large-S behavior.

A little experimentation will show that the approximating BSM formula just suggested is inaccurate for S near the money. Still, we have another degree of freedom in our ability to adjust the volatility in the right-hand-side of (12). By choosing σ_{adj} so that $C_1(S_0, X, t_{n-1}, T) \equiv C_{\mathrm{BSM}}(S_0, X_{\mathrm{adj}}, \sigma_{\mathrm{adj}}, t_{n-1}, T)$, where S_0 is the original stock price of the problem, we obtain an accurate approximation

$$C_1(S, X, t_{n-1}, T) \approx C_{\mathrm{BSM}}(S, X_{\mathrm{adj}}, \sigma_{\mathrm{adj}}, t_{n-1}, T)$$

that often differs by less than a penny over the full range of S on $(0, \infty)$.

This same scheme is used at successive iterations of the exact integration. That is, the "previous" iteration will always be fast because it uses the BSM formula. Then, after you get the answer, you approximate that answer by a BSM formula parameterization. In that parameterization, you choose an adjusted strike price and an adjusted volatility to fit the large-S behavior and the S_0 value. This enables you to move on to the next iteration.

Table 6 reports call option values when there is a dividend payment of 4 in the middle of each year. The first column shows the years to expiration for the contracts we consider. The models Vol2, Vol3, BV, and Num are identical to the ones described earlier. HHL is our closed form solution from Section 2 evaluated by numerical quadrature. As we have already mentioned, this approach is computer intensive. We have therefore limited ourself to value options with this method with up to three dividend payments. An efficient implementation in for instance C++ will naturally make this approach viable for any practical number of dividend payments. Non-recombining trees are even more computer intensive, especially for multiple dividends. They also entail problems with propagation of errors when the number of time steps is increased, so we limited ourself to compute option values for three dividends (3 years to maturity), with 500 time steps for $T = 1, 2$, and 1000 time steps for $T = 3$. The column Appr is the approximation just described above. The two rightmost columns report the adjusted strike and volatility used in this approximation method.

Table 6: European calls with multiple dividends of 4
($S = 100$, $X = 100$, $r = 6\%$, $\sigma = 25\%$, $D = 4$)

T	Num	Vol2	Vol3	BV	HHL	Appr	Adjusted strike	Adjusted volatility
1	10.6615	10.6585	10.6530	10.6596	10.6606	10.6606	104.122	0.2467
2	15.2024	15.1780	15.1673	15.1992	15.1989	15.1996	108.499	0.2421
3	18.5798	18.5348	18.5241	18.5981	18.5984	18.5998	113.146	0.2375
4	–	21.2297	21.2304	21.3592	–	21.3644	118.081	0.2328
5	–	23.4666	23.4941	23.6868	–	23.6978	123.320	0.2282
6	–	23.3556	25.4279	25.6907	–	25.7100	128.884	0.2237
7	–	26.9661	27.1023	27.4395	–	27.4695	–	–

The approximation we suggest above (Appr) is clearly very accurate, when compared to our exact integration (HHL). Also the non-recombining binomial implementation (Num) of the spot process yields results very close to our exact integration. Vol2 and Vol3 seems to give rise to significant mispricing with multiple dividends. The BV approximation seems somewhat more accurate. However, as we already know, it significantly misprices options when the dividend is very high. From a trader's perspective, our approach seems to be a clear choice—at least if you care about having a robust and accurate model that will work in "any" situation. Remember also that our method is valid for any price process, including stochastic volatility, jumps, and other factors that can have a significant impact on pricing and hedging.

Exotic and real options: Several exotic options trade in the OTC equity market, and many are embedded in warrants and other complex equity derivatives. The exact model treatment of options on dividend paying stocks presented in this paper holds also in these cases. Many exotic options, in particular barrier options, are known to be very sensitive to stochastic volatility. Luckily the model described above also holds for stochastic volatility, jumps, volatility term structure, as well as other factors that can be of vital importance when pricing exotic options. The model we have suggested should also be relevant to real options pricing, when the underlying asset offers known discrete payouts (of generic nature) during the lifetime of the real option.

Appendix A

The following is a volatility adjustment that has been suggested used in combination with the escrowed dividend model. The adjustment seems to have been discovered independently by Haug and Haug (1998) (unpublished working paper), as well as by Beneder and Vorst (2001). σ in the BSM formula is replaced with σ_{adj}, and the stock price minus the present value of the dividends until expiration is substituted for the stock price.

$$\begin{aligned}\sigma_{\text{adj}}^2 &= \left(\frac{S\sigma}{S-\sum_{i=1}^{n} D_i \mathrm{e}^{rt_i}}\right)^2 (t_1 - t_0) + \left(\frac{S\sigma}{S-\sum_{i=2}^{n} D_i \mathrm{e}^{rt_i}}\right)^2 (t_2 - t_1) + \cdots + \sigma^2 (T - t_n) \\ &= \sum_{j=1}^{n} \left(\frac{S\sigma}{S-\sum_{i=j}^{n} D_i \mathrm{e}^{rt_i}}\right)^2 (t_j - t_{j-1}) + \sigma^2 (T - t_n)\end{aligned}$$

This method seems to work better than for instance the volatility adjustment discussed by Chriss (1997), among others. However this is still simply a rough approximation, without much of a theory behind it. For this reason, there is no guarantee for it to be accurate in all circumstances. Any such model could be dangerous for a trader to use.

Appendix B

Bos et al. (2003) suggest the following volatility adjustment to be used in combination with the escrowed dividend adjustment:

$$\sigma(S,X,T)^2 = \sigma^2 + \sigma\sqrt{\frac{\pi}{2T}}\left\{4\mathrm{e}^{\frac{z_1^2}{2}-s}\sum_{i=1}^{n} D_i \mathrm{e}^{-rt_i}\left[N(z_1) - N\left(z_1 - \sigma\frac{t_i}{\sqrt{T}}\right)\right]\right.$$
$$\left. + \mathrm{e}^{\frac{z_2^2}{2}-2s}\sum_{i}^{n}\sum_{j}^{n} D_i D_j \mathrm{e}^{-r(t_i+t_j)}\left[N(z_2) - N\left(z_2 - \frac{2\sigma\min(t_i,t_j)}{\sqrt{T}}\right)\right]\right\},$$

where n is the number of dividends in the option's lifetime, $s = \ln(S)$, $x = \ln[(X + D_T)\mathrm{e}^{-rT}]$, where $D_T = \sum_i^n D_i \mathrm{e}^{-rt_i}$, and

$$z_1 = \frac{s-x}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}, \qquad z_2 = z_1 + \frac{\sigma\sqrt{T}}{2}.$$

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