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Some New BSDE Results for an Infinite-Horizon
Stochastic Control Problem

Ying Hu

Martin Schweizer

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Some new BSDE results for an infinite-horizon stochastic control problem

Ying Hu

Université Rennes 1

IRMAR

Campus de Beaulieu

F – 35042 Rennes Cedex

France

ying.hu@univ-rennes1.fr

Martin Schweizer *

ETH Zürich

Departement Mathematik

ETH-Zentrum, HG G 51.2

CH – 8092 Zürich

Switzerland

martin.schweizer@math.ethz.ch

Abstract: We study a quadratic BSDE in a continuous filtration with an unbounded generator and an infinite time horizon. This equation comes from a stochastic control problem in the context of robust utility maximisation. We prove existence and uniqueness, in a suitable class, of a solution to the BSDE, and we show that the BSDE characterises the dynamic value process of the stochastic control problem.

Key words: backward stochastic differential equations, infinite horizon, quadratic BSDE, unbounded solution, stochastic control, robust utility maximisation

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* corresponding author

0. Introduction

This paper studies a stochastic control problem arising in the context of robust utility maximisation, and proves new result via BSDE techniques. A particular feature is that the problem is formulated and solved for an infinite horizon and that we also obtain new results on a certain infinite-horizon BSDE with quadratic generator.

In loose terms, the basic problem we should like to tackle has the form

$$\text{find } \sup_{\pi} \inf_Q \mathbf{U}(\pi, Q),$$

where \mathbf{U} is some utility functional, π runs through a set of investment and consumption strategies, and Q through a set of models given by probability measures. In a first step, we focus only on the inner minimisation problem; thus we think of π as being fixed and look for a worst-case model Q . The functional $\mathbf{U}(\pi, Q)$ we consider has the form

$$\mathbf{U}(\pi, Q) = E_Q[\mathcal{U}_{0,\infty}^\delta + \beta \mathcal{R}_{0,\infty}^\delta(Q)],$$

where $\mathcal{U}_{0,\infty}^\delta = \alpha \int_0^\infty S_s^\delta U_s ds$ stands for a discounted utility term (whose dependence on π is suppressed) and $\mathcal{R}_{0,\infty}^\delta(Q) = \int_0^\infty S_s^\delta \log Z_s^Q ds$ is an entropic penalty term. A precise formulation is given later.

The finite-horizon version of this problem has been studied in Bordigoni/Matoussi/Schweizer (2007), who have characterised the dynamic value process $V = V^{(T)}$ of the resulting stochastic control problem as the unique solution of the BSDE

$$(0.1a) \quad dY_t = (\delta_t Y_t - \alpha U_t) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t$$

with a final condition at time T . We generalise these results here to an infinite-horizon setting with the terminal condition

$$(0.1b) \quad \lim_{t \rightarrow \infty} Y_t = 0.$$

In an unpublished PhD thesis, G. Bordigoni has already shown that V for the infinite-horizon problem is a solution of the BSDE (0.1), but uniqueness and the required integrability (and hence the characterisation of V by the BSDE) remained open. We close this gap here.

In contrast to Bordigoni (2007), our approach and main results here are on the side of BSDE theory. (0.1) is a quadratic BSDE in a continuous filtration which has (due to the presence of U) an unbounded generator and an infinite horizon. For the finite-horizon case, the classical results of Kobylanski (2000) on existence and uniqueness of a bounded solution

for quadratic BSDEs in a Brownian filtration have been extended to unbounded solutions in the Brownian setting by Briand/Hu (2006, 2008), and to bounded solutions in a continuous filtration by Morlais (2009). In infinite-horizon settings, Briand/Hu (1998) and later Royer (2004) have studied bounded solutions for BSDEs with a Lipschitz generator, and Briand/Confortola (2008) have extended these results to bounded solutions for a quadratic generator. The methods in all these papers rely on Girsanov techniques.

Our approach here is quite different. To prove existence of a solution to (0.1), we have adapted the localisation method from Briand/Hu (2006), while for uniqueness, we have applied the θ -difference method from Briand/Hu (2008). Both these techniques have so far only been used in finite-horizon settings.

Finally, let us also mention two closely related papers from the finance and economics literature. Schroder/Skiadas (1999) study the same BSDE as we do and obtain existence and uniqueness of unbounded solutions, but in a Brownian filtration and with a finite time horizon. Hansen/Sargent/Turmuhambetova/Williams (2006) study robustness aspects for infinite-horizon utility maximisation problems; their main ideas and problems are similar to ours, but the approach is rather heuristic, using Hamilton–Jacobi–Bellman equations and formal manipulations in a Markovian setting. For a more detailed discussion of additional references to the literature, we refer to Section 6 of Bordigoni/Matoussi/Schweizer (2007).

The paper is structured as follows. After some preliminaries and notations in Section 1, we study in Section 2 the BSDE on a finite horizon. This serves as preparation for the infinite-horizon BSDE studied in Section 3 and gives to that end fairly precise estimates for the solution Y . Section 3 establishes existence and uniqueness of a solution (Y, M) for the infinite-horizon BSDE (0.1) and gives a sufficient condition for $\mathcal{E}(-\frac{1}{\beta}M)$ to be a martingale. In Section 4, we prove by general arguments as in Bordigoni/Matoussi/Schweizer (2007) and Bordigoni (2007) the existence of a solution to our stochastic control problem and show that its value process V satisfies the boundary condition $\lim_{t \rightarrow \infty} V_t = 0$. Finally, Section 5 uses the BSDE results to characterise V as the unique solution, in a suitable space, for the BSDE (0.1), and in particular establishes that V has the required good integrability properties.

1. Preliminaries and overview

In this section, we introduce all required notations, the basic BSDEs and the basic optimisation problems. We start with a probability space (Ω, \mathcal{F}, P) and a time horizon $T \in (0, \infty]$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfies the usual conditions of right-continuity and P -completeness, \mathcal{F}_0 is P -trivial, and $\mathcal{F}_\infty := \bigvee_{t \geq 0} \mathcal{F}_t$. The basic ingredients for our optimisation problems are

- parameters $\alpha, \alpha' \in [0, \infty)$ and $\beta \in (0, \infty)$;
- progressively measurable processes $\delta = (\delta_t)_{t \geq 0}$ and $U = (U_t)_{t \geq 0}$;

– an \mathcal{F}_T -measurable random variable U'_T , with $U'_\infty = 0$ for $T = \infty$.

With these, we can formulate the BSDEs studied here. On the one hand, for a finite horizon $T < \infty$, we introduce the BSDE

$$(1.1) \quad dY_t = (\delta_t Y_t - \alpha U_t) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \quad Y_T = \alpha' U'_T.$$

On the other hand, for an infinite horizon $T = \infty$, the BSDE is

$$(1.2) \quad dY_t = (\delta_t Y_t - \alpha U_t) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \quad \lim_{t \rightarrow \infty} Y_t = 0.$$

A solution of (1.1) or (1.2) is a pair (Y, M) satisfying (1.1) or (1.2), respectively, where Y is a P -semimartingale and M is a locally P -square-integrable local P -martingale null at 0.

For the optimisation problems, we first define the discounting process

$$S_t^\delta := \exp\left(-\int_0^t \delta_s ds\right), \quad t \geq 0,$$

and for $T < \infty$ the auxiliary quantities, for $0 \leq t \leq T$,

$$(1.3) \quad \mathcal{U}_{t,T}^\delta := \alpha \int_t^T \frac{S_s^\delta}{S_t^\delta} U_s ds + \alpha' \frac{S_T^\delta}{S_t^\delta} U'_T = \int_t^T \alpha e^{-\int_t^s \delta_r dr} U_s ds + \alpha' e^{-\int_t^T \delta_r dr} U'_T,$$

$$(1.4) \quad \mathcal{R}_{t,T}^\delta(Q) := \int_t^T \delta_s \frac{S_s^\delta}{S_t^\delta} \log \frac{Z_s^Q}{Z_t^Q} ds + \frac{S_T^\delta}{S_t^\delta} \log \frac{Z_T^Q}{Z_t^Q},$$

for $Q \ll P$ on \mathcal{F}_T with density process Z^Q on $[0, T]$. We consider the cost functional

$$c_T(Q) := \mathcal{U}_{0,T}^\delta + \beta \mathcal{R}_{0,T}^\delta(Q),$$

and the basic stochastic control problem on a finite horizon is to minimise the functional

$$Q \mapsto \Gamma_T(Q) := E_Q[c_T(Q)]$$

over a suitable class of probability measures $Q \ll P$ on \mathcal{F}_T . In a classical way, we can choose an adapted RCLL process $V = (V_t)_{0 \leq t \leq T}$ such that

$$V_t = \operatorname{ess\,inf}_Q E_Q[\mathcal{U}_{t,T}^\delta + \beta \mathcal{R}_{t,T}^\delta(Q) \mid \mathcal{F}_t], \quad 0 \leq t \leq T.$$

For $T = \infty$, we define similarly, for $t \geq 0$,

$$\mathcal{U}_{t,\infty}^\delta := \alpha \int_t^\infty \frac{S_s^\delta}{S_t^\delta} U_s ds = \int_t^\infty \alpha e^{-\int_t^s \delta_r dr} U_s ds,$$

$$\mathcal{R}_{t,\infty}^\delta(Q) := \int_t^\infty \delta_s \frac{S_s^\delta}{S_t^\delta} \log \frac{Z_s^Q}{Z_t^Q} ds,$$

for $Q \lll^{loc} P$ with density process Z^Q . We consider the cost functional

$$c_\infty(Q) := \mathcal{U}_{0,\infty}^\delta + \beta \mathcal{R}_{0,\infty}^\delta(Q)$$

and in principle want to minimise the functional

$$Q \mapsto \Gamma_\infty(Q) := E_Q[c_\infty(Q)]$$

over a suitable class of probability measures $Q \lll^{loc} P$. (For the precise formulation, we refer to Section 4.) In a similar manner as for $T < \infty$, we can choose an adapted RCLL process $V = (V_t)_{t \geq 0}$ which is again called the dynamic value process of our stochastic control problem.

Our main results in this paper are

- a) existence and uniqueness results for the above BSDEs (both with finite and infinite horizon), and
- b) a characterisation of the value process V for the infinite-horizon setting as the solution of the BSDE (1.2).

2. The BSDE on a finite horizon

The main goal of this section is to prove the existence of a solution to the finite-horizon BSDE under weak conditions. This slightly extends previous work and above all serves as preparation for the infinite-horizon case. So we fix $T \in (0, \infty)$ and view U and δ as processes on $[0, T]$.

Hypothesis 2.1. Throughout this section, we impose the standing assumptions

(2.1a) \mathbb{F} is a *continuous filtration*, i.e. all local (P, \mathbb{F}) -martingales are continuous.

(2.1b) $\delta \geq 0$ is uniformly bounded (by $\bar{\delta}$, say).

Precise assumptions on U, U'_T will be specified below. Now we introduce the quantities

$$B := \alpha \int_0^T S_s^\delta U_s ds + \alpha' S_T^\delta U'_T = \mathcal{U}_{0,T}^\delta,$$

$$B_- := \alpha \int_0^T S_s^\delta U_s^- ds + \alpha' S_T^\delta (U'_T)^-,$$

$$B_+ := \alpha \int_0^T S_s^\delta U_s^+ ds + \alpha' S_T^\delta (U'_T)^+.$$

The BSDE (1.1) under study is

$$(2.2) \quad dY_t = (\delta_t Y_t - \alpha U_t) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \quad Y_T = \alpha' U'_T.$$

Definition 2.2. A *solution* of (2.2) is a pair of processes (Y, M) satisfying (2.2), where Y is a P -semimartingale and M is a locally P -square-integrable local P -martingale null at 0.

Due to the standing assumption (2.1a), M and then Y are continuous for any solution (Y, M) of (2.2). Our proof of existence applies the localisation method originally developed in a Brownian setting by Briand/Hu (2006). To that end, we need to establish precise a priori estimates in the bounded case. Note that $1/S^\delta = S^{-\delta}$.

Proposition 2.3 (A priori estimates). *Suppose that $\int_0^T |U_s| ds$ and U'_T are bounded random variables. Then there exists a unique solution (Y, M) to (2.2) such that Y is a bounded process. Moreover, we have the estimates*

$$(2.3) \quad S_t^{-\delta} \underline{Y}_t - \int_0^t \alpha e^{\int_s^t \delta_r dr} U_s ds \leq Y_t \leq S_t^{-\delta} \bar{Y}_t - \int_0^t \alpha e^{\int_s^t \delta_r dr} U_s ds, \quad 0 \leq t \leq T,$$

where

$$(2.4) \quad \underline{Y}_t := -\beta e^{-\delta T} \log E \left[e^{-\frac{1}{\beta} e^{\delta T} B} \middle| \mathcal{F}_t \right] \quad \text{and} \quad \bar{Y}_t := E[B | \mathcal{F}_t], \quad 0 \leq t \leq T.$$

Proof. Existence and uniqueness of a solution with Y bounded are immediate from Theorems 2.5 and 2.6 of Morlais (2009). From the definitions of \underline{Y} and \bar{Y} and Itô's formula, it is clear that there exist \underline{M} and \bar{M} such that

$$(2.5) \quad d\underline{Y}_t = \frac{1}{2\beta} e^{\delta T} d\langle \underline{M} \rangle_t + d\underline{M}_t, \quad \underline{Y}_T = B,$$

$$(2.6) \quad d\bar{Y}_t = d\bar{M}_t, \quad \bar{Y}_T = B.$$

If we first set $Y_t^1 := S_t^\delta Y_t$ and $M_t^1 := \int_0^t S_s^\delta dM_s$, then the BSDE (2.2) is transformed to

$$(2.7) \quad dY_t^1 = -\alpha S_t^\delta U_t dt + \frac{1}{2\beta} S_t^{-\delta} d\langle M^1 \rangle_t + dM_t^1, \quad Y_T^1 = \alpha' S_T^\delta U'_T.$$

If we next put $Y_t^2 := Y_t^1 + \int_0^t \alpha S_s^\delta U_s ds$ and $M^2 := M^1$, then the BSDE (2.7) becomes

$$(2.8) \quad dY_t^2 = \frac{1}{2\beta} S_t^{-\delta} d\langle M^2 \rangle_t + dM_t^2, \quad Y_T^2 = \alpha' S_T^\delta U'_T + \alpha \int_0^T S_s^\delta U_s ds = B.$$

Because $0 \leq S_t^{-\delta} \leq e^{\delta T}$, we deduce by comparison of (2.5), (2.8) and (2.6) that

$$\underline{Y}_t \leq Y_t^2 \leq \bar{Y}_t, \quad 0 \leq t \leq T.$$

Returning to Y by the formula $Y_t = S_t^{-\delta} Y_t^2 - \int_0^t \alpha e^{\int_s^t \delta_r dr} U_s ds$, we conclude the proof. **q.e.d.**

We now apply the localisation method to get the following existence result.

Theorem 2.4 (Existence of solution). *Let us suppose that*

$$(2.9) \quad E \left[e^{\frac{1}{2\beta} e^{\delta T} B_-} \right] + E[B_+] < \infty.$$

Then the BSDE (2.2) admits a solution (Y, M) which satisfies

$$(2.10) \quad S_t^{-\delta} \underline{Y}_t - \int_0^t \alpha e^{\int_s^t \delta_r dr} U_s ds \leq Y_t \leq S_t^{-\delta} \bar{Y}_t - \int_0^t \alpha e^{\int_s^t \delta_r dr} U_s ds, \quad 0 \leq t \leq T$$

with \underline{Y}, \bar{Y} given in (2.4).

Proof. 1) We first assume that U'_T and U are nonnegative; then \underline{Y} is also nonnegative. For each $n \in \mathbb{N}$, we consider $U'_T{}^{,n} := U'_T \wedge n$ and $U_t^n := U_t \mathbf{1}_{\{\int_0^t U_s ds \leq n\}}$, $0 \leq t \leq T$. According to Proposition 2.3, the BSDE

$$dY_t = (\delta_t Y_t - \alpha U_t^n) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \quad Y_T = \alpha' U'_T{}^{,n}$$

admits a unique solution (Y^n, M^n) such that Y^n is a bounded process, and by (2.3), extended from $U'_T{}^{,n}$ and U^n to U'_T and U thanks to nonnegativity,

$$- \int_0^t \alpha e^{\int_s^t \delta_r dr} U_s ds \leq Y_t^n \leq S_t^{-\delta} E[B|\mathcal{F}_t].$$

Since $U'_T{}^{,n} \leq U'_T{}^{,n+1}$ and $U^n \leq U^{n+1}$, the sequence (Y^n) is nondecreasing by a comparison result; this can be obtained similarly as in the proof of Theorem 8 in Mania/Schweizer (2005).

For $k \in \mathbb{N}$, define the stopping time

$$\tau_k := \inf \left\{ t \in [0, T] : \int_0^t \alpha e^{\int_s^t \delta_r dr} U_s ds + S_t^{-\delta} E[B|\mathcal{F}_t] \geq k \right\} \wedge T$$

and note that $(\tau_k)_{k \in \mathbb{N}}$ increases to T stationarily. By construction, the stopped processes $Y^{n;k} := (Y^n)^{\tau_k}$, $n \in \mathbb{N}$, are uniformly bounded by k . Setting $M^{n;k} := (M^n)^{\tau_k}$, we have

$$Y_t^{n;k} = Y_{\tau_k}^n - \int_t^{\tau_k} \mathbf{1}_{\{s \leq \tau_k\}} (\delta_s Y_s^{n;k} - \alpha U_s^n) ds - \int_t^{\tau_k} \frac{1}{2\beta} d\langle M^{n;k} \rangle_s - \int_t^{\tau_k} dM_s^{n;k}.$$

We now take the supremum over n and apply the monotonic stability theorem (see e.g. Lemma 3.3 in Morlais (2009)) to obtain, for each k , a solution (Y^k, M^k) to the BSDE

$$Y_t^k = \xi_k - \int_t^{\tau_k} (\delta_s Y_s^k - \alpha U_s) ds - \int_t^{\tau_k} \frac{1}{2\beta} d\langle M^k \rangle_s - \int_t^{\tau_k} dM_s^k, \quad \text{with } \xi_k := \sup_{n \in \mathbb{N}} Y_{\tau_k}^n.$$

More precisely, that result shows that $M_T^{n;k}$ converges to M_T^k in L^2 , so that both $M_t^{n;k}$ and $\langle M^{n;k} \rangle_t$ converge uniformly over $t \in [0, T]$ in probability to M_t^k and $\langle M^k \rangle_t$, respectively. Moreover, $\tau_k \leq \tau_{k+1}$ by construction; hence $Y_{t \wedge \tau_k}^{n;k+1} = Y_t^{n;k}$ and so we have the localisation property

$$Y_{t \wedge \tau_k}^{k+1} = Y_t^k \quad \text{and} \quad M_{t \wedge \tau_k}^{k+1} = M_t^k.$$

So if we set $\tau_0 := 0$ and define the processes Y and M on $[0, T]$ by

$$Y_t := Y_0^1 + \sum_{k=1}^{\infty} Y_t^k \mathbf{1}_{(\tau_{k-1}, \tau_k]}(t) \quad \text{and} \quad M_t := \sum_{k=1}^{\infty} M_t^k \mathbf{1}_{(\tau_{k-1}, \tau_k]}(t),$$

the last BSDE can be rewritten as

$$Y_t = \xi_k - \int_t^{\tau_k} (\delta_s Y_s - \alpha U_s) ds - \int_t^{\tau_k} \frac{1}{2\beta} d\langle M \rangle_s - \int_t^{\tau_k} dM_s.$$

Finally we observe that P -a.s., $\tau_k = T$ for k large enough. This allows us to send $k \rightarrow \infty$ in the previous equation and hence prove that (Y, M) is a solution to (2.2). The inequality (2.10) is satisfied by the process Y since it holds for each Y^n in view of Proposition 2.3.

2) If U_T' and U are not necessarily nonnegative, we use a double approximation by setting $U_T'^{n,p} := (U_T')^+ \wedge n - (U_T')^- \wedge p$ and $U_t^{n,p} := U_t^+ \mathbf{1}_{\{\int_0^t |U_s| ds \leq n\}} - U_t^- \mathbf{1}_{\{\int_0^t |U_s| ds \leq p\}}$. The condition (2.9) is used here to extend (2.3) from the truncated to the general case and to ensure that \bar{Y}, \underline{Y} remain well-defined. In some more detail, we define τ_k (with $|U|$ and $|B|$) and $Y^{n,p;k}$ and $M^{n,p;k}$ analogously as before. Then $Y^{n,p;k}$ is increasing in n and decreasing in p , and it remains bounded by k . Arguing as before, we set $Y^k := \inf_p \sup_n Y^{n,p;k}$ to get the existence of M^k such that $\lim_{p \rightarrow \infty} \lim_{n \rightarrow \infty} M_{s \wedge \tau_k}^{n,p;k} = M_s^k$ and (Y^k, M^k) still solves the BSDE. The rest of the proof is unchanged. **q.e.d.**

In connection with the stochastic control problem, it will be important to know when the stochastic exponential $\mathcal{E}(-\frac{1}{\beta}M)$ is a true martingale, where M comes from the solution of the BSDE (2.2).

Theorem 2.5. *Suppose that there exists a constant $\lambda > 1 + \frac{e^{\delta T} - 1}{e^{\delta T}}$ such that*

$$(2.11) \quad E \left[e^{\lambda \frac{1}{\beta} e^{\delta T} B_+} \right] + E \left[e^{\lambda \frac{1}{\beta} e^{\delta T} B_-} \right] < \infty.$$

Then the stochastic exponential $\mathcal{E}\left(-\frac{1}{\beta}M\right)$ is bounded in $L \log L(P)$ and hence a (uniformly integrable) martingale on $[0, T]$.

Proof. Since (2.11) clearly implies (2.9), existence of a solution is ensured by Theorem 2.4. For any stopping time τ with values in $[0, T]$, the BSDE (2.2) gives

$$\mathcal{E}\left(-\frac{1}{\beta}M\right)_\tau = \exp\left(-\frac{1}{\beta}\left(M_\tau + \frac{1}{2\beta}\langle M \rangle_\tau\right)\right) = \exp\left(-\frac{1}{\beta}\left(Y_\tau - Y_0 - \int_0^\tau \delta_s Y_s ds + \alpha \int_0^\tau U_s ds\right)\right).$$

Hence it suffices to prove that there exists a constant $\lambda_1 > 1$ such that

$$E\left[\exp\left(-\lambda_1 \frac{1}{\beta}\left(Y_\tau - \int_0^\tau \delta_s Y_s ds + \alpha \int_0^\tau U_s ds\right)\right)\right] \leq C,$$

where $C \in (0, \infty)$ is a constant which is independent of τ .

From the estimate (2.10), we have

$$\begin{aligned} & Y_\tau + \alpha \int_0^\tau U_s ds - \int_0^\tau \delta_t Y_t dt \\ & \geq S_\tau^{-\delta} \underline{Y}_\tau + \int_0^\tau \alpha \left(1 - e^{\int_s^\tau \delta_r dr}\right) U_s ds - \int_0^\tau \delta_t \left(S_t^{-\delta} \bar{Y}_t - \int_0^t \alpha e^{\int_s^t \delta_r dr} U_s ds\right) dt. \end{aligned}$$

Since Fubini's theorem gives $\int_0^\tau \delta_t \int_0^t \alpha e^{\int_s^t \delta_r dr} U_s ds dt = \int_0^\tau \alpha \left(e^{\int_s^\tau \delta_r dr} - 1\right) U_s ds$, we deduce that

$$(2.12) \quad Y_\tau - \int_0^\tau \delta_s Y_s ds + \alpha \int_0^\tau U_s ds \geq S_\tau^{-\delta} \underline{Y}_\tau - \int_0^\tau \delta_t S_t^{-\delta} \bar{Y}_t dt.$$

Now pick $p > 1$ with $\lambda > p > 1 + \frac{e^{\delta T} - 1}{e^{\delta T}}$, and set $\lambda_1 = \frac{\lambda}{p} > 1$. Using (2.12) and (2.4) and setting $L_t^* := \sup_{0 \leq s \leq t} E[B_+ | \mathcal{F}_s]$, we obtain

$$\begin{aligned} & E\left[\exp\left(-\lambda_1 \frac{1}{\beta}\left(Y_\tau - \int_0^\tau \delta_s Y_s ds + \alpha \int_0^\tau U_s ds\right)\right)\right] \\ & \leq E\left[\exp\left(-\lambda_1 \frac{1}{\beta}\left(S_\tau^{-\delta} \underline{Y}_\tau - \int_0^\tau \delta_t S_t^{-\delta} \bar{Y}_t dt\right)\right)\right] \\ & \leq E\left[\exp\left(\lambda_1 \log E\left[e^{\frac{1}{\beta} e^{\delta T} B_-} \mid \mathcal{F}_\tau\right]\right) \exp\left(\lambda_1 \frac{1}{\beta} L_T^* \int_0^T \delta_t S_t^{-\delta} dt\right)\right] \\ & \leq E\left[\left(E\left[e^{\frac{1}{\beta} e^{\delta T} B_-} \mid \mathcal{F}_\tau\right]\right)^{\lambda_1} e^{\frac{\lambda_1}{\beta} (e^{\delta T} - 1) L_T^*}\right]. \end{aligned}$$

Finally, we set $q := \frac{p}{p-1}$ and $r := (p-1)\frac{e^{\bar{\delta}T}}{e^{\bar{\delta}T}-1} > 1$ and use Hölder's inequality, $\lambda_1 p = \lambda$, Jensen's inequality and Doob's inequality in L^r to get

$$\begin{aligned} & E \left[\exp \left(-\lambda_1 \frac{1}{\beta} \left(Y_\tau - \int_0^\tau \delta_s Y_s ds + \alpha \int_0^\tau U_s ds \right) \right) \right] \\ & \leq E \left[\left(E \left[e^{\frac{1}{\beta} e^{\bar{\delta}T} B_-} \mid \mathcal{F}_\tau \right] \right)^{\lambda_1 p} \right]^{1/p} E \left[e^{\frac{\lambda_1 q}{\beta} (e^{\bar{\delta}T} - 1) L_\tau^*} \right]^{1/q} \\ & \leq C'_r E \left[e^{\lambda \frac{1}{\beta} e^{\bar{\delta}T} B_-} \right]^{1/p} E \left[e^{\frac{\lambda_1 q r}{\beta} (e^{\bar{\delta}T} - 1) B_+} \right]^{\frac{1}{qr}} = C < \infty, \end{aligned}$$

because $\lambda_1 q r (e^{\bar{\delta}T} - 1) = \lambda e^{\bar{\delta}T}$.

q.e.d.

3. The BSDE on an infinite horizon

In this section, we prove the existence and uniqueness of a solution to the infinite-horizon BSDE under suitable conditions via BSDE techniques.

Hypothesis 3.1. Throughout this section, we impose the standing assumptions

(3.1a) \mathbb{F} is a *continuous filtration*, i.e. all local (P, \mathbb{F}) -martingales are continuous.

(3.1b) $\delta \geq 0$ is uniformly bounded (by $\bar{\delta}$, say).

Again, the assumptions on $U = (U_t)_{t \geq 0}$ will be specified later; U'_T does not appear here. The BSDE (1.2) under study is now

$$(3.2) \quad dY_t = (\delta_t Y_t - \alpha U_t) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \quad \lim_{t \rightarrow \infty} Y_t = 0 \text{ } P\text{-a.s.},$$

and as before, a *solution* of (3.2) is a pair (Y, M) satisfying (3.2), where Y is a P -semimartingale and M is a locally P -square-integrable local P -martingale null at 0.

The first step in tackling (3.2) is to obtain a priori estimates for the finite-horizon version with terminal condition $Y_T = 0$. But in contrast to Section 3, we now need the bounds to be uniform in T , and so we need stronger assumptions on U .

Definition 3.2. We say that a random variable X is in D^{exp} if $E[e^{\lambda|X|}] < \infty$ for all $\lambda > 0$.

A progressively measurable process $U = (U_t)_{t \geq 0}$ is in $D_{1,T}^{\text{exp}}$, for $T \in (0, \infty]$, if $\int_0^T |U_s| ds$ is in D^{exp} , and an RCLL process $Y = (Y_t)_{t \geq 0}$ is in $D_{0,T}^{\text{exp}}$, for $T \in (0, \infty]$, if $Y_T^* := \sup_{0 \leq t \leq T} |Y_t|$ is in D^{exp} .

(By convention, $Y_\infty^* := \sup_{t \geq 0} |Y_t|$.)

Let us now consider the BSDE

$$(3.3) \quad dY_t = (\delta_t Y_t - \alpha U_t) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \quad Y_T = 0.$$

Proposition 3.3 (A priori estimates). *Suppose that $\int_0^T |U_s| ds$ is a bounded random variable. Then there exists a unique solution (Y, M) to (3.3) such that Y is a bounded process. Moreover, we have the estimate*

$$(3.4) \quad |Y_t| \leq \beta \log E \left[\exp \left(\frac{1}{\beta} \int_t^T \alpha |U_s| ds \right) \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T.$$

Proof. Existence and uniqueness of a solution with Y bounded follow as for Proposition 2.3 from Theorems 2.5 and 2.6 of Morlais (2009). Applying Tanaka's formula then first yields

$$d|Y_t| = \text{sign}(Y_t) dY_t + dL_t = \text{sign}(Y_t) \left((\delta_t Y_t - \alpha U_t) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t \right) + dL_t,$$

where L is the local time at 0 of the continuous semimartingale Y . Next, applying Itô's formula to the bounded process $Z_t := \exp \left(\frac{1}{\beta} \left(|Y_t| + \int_0^t \alpha |U_s| ds \right) \right)$, we obtain

$$\begin{aligned} dZ_t &= \frac{1}{\beta} Z_t \text{sign}(Y_t) \left((\delta_t Y_t - \alpha U_t) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t \right) \\ &\quad + \frac{1}{\beta} Z_t dL_t + \frac{1}{\beta} Z_t \alpha |U_t| dt + \frac{1}{2\beta^2} Z_t d\langle M \rangle_t \\ &\geq \frac{1}{\beta} Z_t \text{sign}(Y_t) dM_t, \end{aligned}$$

in the sense that the difference of the terms on the two sides of the inequality is an increasing process. So Z is a submartingale, which gives

$$\exp \left(\frac{1}{\beta} \left(|Y_t| + \int_0^t \alpha |U_s| ds \right) \right) \leq E \left[\exp \left(\frac{1}{\beta} \int_0^T \alpha |U_s| ds \right) \middle| \mathcal{F}_t \right]$$

since $Y_T = 0$, and (3.4) follows. **q.e.d.**

From this a priori estimate and the localisation method, we obtain the existence of a solution to the infinite-horizon BSDE (3.2).

Theorem 3.4 (Existence of solution). *Let us suppose that*

$$(3.5) \quad E \left[\exp \left(\frac{1}{\beta} \int_0^\infty \alpha |U_s| ds \right) \right] < \infty, \text{ i.e. } \exp \left(\frac{1}{\beta} \int_0^\infty \alpha |U_s| ds \right) \in L^1.$$

Then the BSDE (3.2) admits a solution (Y, M) which satisfies

$$(3.6) \quad |Y_t| \leq \beta \log E \left[\exp \left(\frac{1}{\beta} \int_t^\infty \alpha |U_s| ds \right) \middle| \mathcal{F}_t \right], \quad t \geq 0.$$

If $\exp \left(\frac{1}{\beta} \int_0^\infty \alpha |U_s| ds \right)$ is in L^r for some $r > 1$, then so is $\exp \left(\frac{1}{\beta} Y_\infty^* \right)$. If $U \in D_{1,\infty}^{\text{exp}}$, then $Y \in D_{0,\infty}^{\text{exp}}$.

Proof. 1) We first assume that U is nonnegative and set $U_t^n := U_t \mathbf{1}_{\{\int_0^t U_s ds \leq n\}}$ for each $n \in \mathbb{N}$. According to Proposition 3.3, the BSDE

$$dY_t = (\delta_t Y_t - \alpha U_t^n) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \quad Y_n = 0$$

on $[0, n]$ admits a unique solution (Y^n, M^n) such that Y^n is bounded, and by (3.4),

$$|Y_t^n| \leq \beta \log E \left[\exp \left(\frac{1}{\beta} \int_t^\infty \alpha |U_s| ds \right) \middle| \mathcal{F}_t \right], \quad t \in [0, n].$$

If we set $Y_t^n = 0$ and $M_t^n = M_n^n$ for $t > n$, then (Y^n, M^n) also satisfies on $[0, n+1]$ the BSDE

$$dY_t = (\delta_t Y_t - \alpha \mathbf{1}_{\{t \leq n\}} U_t^n) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \quad Y_{n+1} = 0.$$

Because U is nonnegative, $\mathbf{1}_{\{t \leq n\}} U_t^n \leq U_t^{n+1}$ for $t \in [0, n+1]$, and so the sequence (Y^n) is nondecreasing by the comparison theorem. For each $k \in \mathbb{N}$, we define the stopping time

$$\tau_k := \inf \left\{ t \geq 0 : \beta \log E \left[\exp \left(\frac{1}{\beta} \int_t^\infty \alpha |U_s| ds \right) \middle| \mathcal{F}_t \right] \geq k \right\} \wedge k.$$

Introducing the stopped processes $Y^{n;k} := (Y^n)^{\tau_k}$ and $M^{n;k} := (M^n)^{\tau_k}$, we can argue exactly as in the proof of Theorem 2.4 to construct processes Y and M , now on $[0, \infty)$, satisfying for each T the BSDE

$$Y_t = Y_T - \int_t^T (\delta_s Y_s - \alpha U_s) ds - \int_t^T \frac{1}{2\beta} d\langle M \rangle_s - \int_t^T dM_s.$$

Since each Y^n satisfies the estimate (3.6) by Proposition 3.3, it follows from the construction that so does Y , and this implies due to (3.5) that

$$\lim_{t \rightarrow \infty} Y_t = 0 \quad P\text{-a.s.}$$

2) In the general case where U is not necessarily nonnegative, we use the double approximation $U_t^{n,p} := U_t^+ \mathbf{1}_{\{\int_0^t |U_s| ds \leq n\}} \mathbf{1}_{\{t \leq n\}} - U_t^- \mathbf{1}_{\{\int_0^t |U_s| ds \leq p\}} \mathbf{1}_{\{t \leq p\}}$, $t \geq 0$, and denote by $(Y^{n,p}, M^{n,p})$ the solution to

$$dY_t = (\delta_t Y_t - \alpha U_t^{n,p}) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \quad Y_{n \vee p} = 0.$$

Then the proof goes like for Theorem 2.4, using that $Y^{n,p}$ increases in n and decreases in p .

3) The integrability assertions about Y follow from (3.6) and Doob's inequality. **q.e.d.**

To get a uniqueness result for the infinite-horizon BSDE (3.2), we need a stronger assumption.

Theorem 3.5 (Uniqueness of solution). *Suppose that U is in $D_{1,\infty}^{\text{exp}}$. Then the BSDE (3.2) admits a unique solution (Y, M) with $Y \in D_{0,\infty}^{\text{exp}}$.*

Proof. Existence is clear from Theorem 3.4. For uniqueness, let (Y, M) and (Y', M') be two such solutions and note that the martingale part is always unique by the uniqueness of the canonical decomposition of a special semimartingale. Fix $\theta \in (0, 1)$ and set $\hat{Y} := Y - \theta Y'$ and $\hat{M} := M - \theta M'$. Then

$$(3.7) \quad d\hat{Y}_t = (\delta_t \hat{Y}_t - \alpha(1-\theta)U_t) dt + \frac{1}{2\beta} d(\langle M \rangle_t - \theta \langle M' \rangle_t) + d\hat{M}_t.$$

Noting that convexity gives

$$(3.8) \quad d\langle M \rangle_t = d\left\langle \theta M' + (1-\theta) \frac{\hat{M}}{1-\theta} \right\rangle_t \leq \theta d\langle M' \rangle_t + \frac{1}{1-\theta} d\langle \hat{M} \rangle_t,$$

we rewrite (3.7) as

$$(3.9) \quad d\hat{Y}_t = (\delta_t \hat{Y}_t - \alpha(1-\theta)U_t) dt + d\hat{M}_t + \frac{1}{2\beta} d\left(\langle M \rangle_t - \theta \langle M' \rangle_t - \frac{1}{1-\theta} \langle \hat{M} \rangle_t\right) + \frac{1}{2\beta(1-\theta)} d\langle \hat{M} \rangle_t.$$

Now Tanaka's formula yields $d\hat{Y}_t^- = -\mathbf{1}_{\{\hat{Y}_t \leq 0\}} d\hat{Y}_t + \frac{1}{2} d\hat{L}_t$, where \hat{L} is the local time at 0 of the process \hat{Y} . Applying next Itô's formula to $Z_t := \exp\left(\frac{1}{\beta(1-\theta)}\left(\hat{Y}_t^- + \int_0^t \alpha(1-\theta)|U_s| ds\right)\right)$, $t \geq 0$, we get

$$\begin{aligned} dZ_t &= \frac{1}{\beta(1-\theta)} Z_t \left(-\mathbf{1}_{\{\hat{Y}_t \leq 0\}} d\hat{Y}_t + \frac{1}{2} d\hat{L}_t + \alpha(1-\theta)|U_t| dt \right) \\ &\quad + \frac{1}{2\beta^2(1-\theta)^2} Z_t \mathbf{1}_{\{\hat{Y}_t \leq 0\}} d\langle \hat{M} \rangle_t \\ &\geq -\frac{1}{\beta(1-\theta)} Z_t \mathbf{1}_{\{\hat{Y}_t \leq 0\}} d\hat{M}_t, \end{aligned}$$

where the last inequality uses (3.9) and (3.8). Thus Z is a local submartingale, and so there exists an increasing sequence of stopping times $\tau_n \nearrow \infty$ such that

$$\exp\left(\frac{1}{\beta(1-\theta)}\hat{Y}_{t\wedge\tau_n}^-\right) \leq E\left[\exp\left(\frac{1}{\beta(1-\theta)}\hat{Y}_{T\wedge\tau_n}^- + \frac{1}{\beta}\int_{t\wedge\tau_n}^{T\wedge\tau_n}\alpha|U_s|ds\right)\middle|\mathcal{F}_t\right].$$

Because $\hat{Y} \in D_{0,\infty}^{\text{exp}}$, $U \in D_{1,\infty}^{\text{exp}}$ and $\lim_{t \rightarrow \infty} \hat{Y}_t = 0$, we obtain for $n \rightarrow \infty$ and $T \rightarrow \infty$ that

$$\exp\left(\frac{1}{\beta(1-\theta)}\hat{Y}_t^-\right) \leq E\left[\exp\left(\frac{1}{\beta}\int_t^\infty\alpha|U_s|ds\right)\middle|\mathcal{F}_t\right],$$

which is equivalent to

$$(Y_t - \theta Y_t')^- = \hat{Y}_t^- \leq \beta(1-\theta) \log E\left[\exp\left(\frac{1}{\beta}\int_t^\infty\alpha|U_s|ds\right)\middle|\mathcal{F}_t\right].$$

Letting $\theta \rightarrow 1$, we deduce that $Y_t \geq Y_t'$, and since a symmetrical argument gives the reverse inequality, the proof is complete. **q.e.d.**

As in Section 3, we again want to know when the stochastic exponential $\mathcal{E}(-\frac{1}{\beta}M)$ is a true martingale, where M now comes from the solution of the BSDE (3.2). However, we only expect to obtain this here on the open interval $[0, \infty)$, and the proof below shows why $T = \infty$ causes a difficulty.

Theorem 3.6. *Suppose that U is in $D_{1,\infty}^{\text{exp}}$ and denote by (Y, M) the solution to (3.2) from Theorem 3.4. Then for every finite T and every $r < \infty$, the stochastic exponential $(\mathcal{E}(-\frac{1}{\beta}M)_t)_{0 \leq t \leq T}$ is bounded in L^r , and so $\mathcal{E}(-\frac{1}{\beta}M)$ is a martingale on $[0, \infty)$.*

Proof. Fix $T \in (0, \infty)$ and let τ be a stopping time with values in $[0, T]$. As in the proof of Theorem 2.5, the BSDE (3.2) gives

$$\begin{aligned} \mathcal{E}\left(-\frac{1}{\beta}M\right)_\tau &= \exp\left(-\frac{1}{\beta}\left(Y_\tau - Y_0 - \int_0^\tau \delta_s Y_s ds + \alpha \int_0^\tau U_s ds\right)\right) \\ &\leq \exp\left(\frac{1}{\beta}Y_0\right) \exp\left(\frac{1}{\beta}Y_\tau^*(1 + \bar{\delta}T) + \frac{1}{\beta}\int_0^\infty\alpha|U_s|ds\right), \end{aligned}$$

and the conclusion follows from Theorem 3.4 because $Y \in D_{0,\infty}^{\text{exp}}$.

q.e.d.

4. The stochastic control problem on an infinite horizon

In this section, we prove existence and uniqueness of a solution to the infinite-horizon stochastic control problem.

Let us first give a precise formulation. We recall from Section 1 the underlying filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ with $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq \infty}$, the parameters $\alpha \geq 0$, $\beta > 0$ and the processes $\delta = (\delta_t)_{t \geq 0}$ and $U = (U_t)_{t \geq 0}$. We denote by \mathcal{Q} the set of all probability measures $Q \ll^{\text{loc}} P$ and by $Z^Q = (Z_t^Q)_{t \geq 0}$ an RCLL version of the density process of Q with respect to P . Since \mathcal{F}_0 is trivial, we have

$$\{Z^Q \mid Q \in \mathcal{Q}\} \subseteq \{\text{all RCLL } P\text{-martingales } Z = (Z_t)_{t \geq 0} \text{ with } Z \geq 0 \text{ and } Z_0 = 1\} =: \mathcal{Z}.$$

Now define for any $t \geq 0$ and $Z \in \mathcal{Z}$ in analogy to (1.3) and (1.4)

$$\begin{aligned} \tilde{U}_{t,\infty}^\delta(Z) &:= \alpha \int_t^\infty \frac{S_s^\delta}{S_t^\delta} \frac{Z_s}{Z_t} U_s ds, \\ \tilde{\mathcal{R}}_{t,\infty}^\delta(Z) &:= \int_t^\infty \delta_s \frac{S_s^\delta}{S_t^\delta} \frac{Z_s}{Z_t} \log \frac{Z_s}{Z_t} ds \end{aligned}$$

and the cost functional

$$\tilde{c}_\infty(Z) := \tilde{U}_{0,\infty}^\delta(Z) + \beta \tilde{\mathcal{R}}_{0,\infty}^\delta(Z).$$

The stochastic control problem studied here is to minimise the functional

$$Z \mapsto \Gamma_\infty(Z) := E_P[\tilde{c}_\infty(Z)]$$

over a subset \mathcal{Z}_f of \mathcal{Z} , defined below. Note that by the minimum principle for supermartingales, $Z \in \mathcal{Z}$ remains 0 if it ever hits 0; so both summands of $\tilde{c}_\infty(Z)$ are well-defined.

Hypothesis 4.1. Throughout this section, we impose the standing assumptions

$$(4.1a) \quad \begin{aligned} &\text{There exists some } T_0 \in (0, \infty) \text{ such that for all } \gamma > 0, \\ &E_P \left[\exp \left(\gamma \int_0^{T_0} |U_s| ds \right) \right] + E_P \left[\int_{T_0}^\infty \exp(\gamma |U_s|) \mathbf{1}_{\{U_s \neq 0\}} ds \right] < \infty. \end{aligned}$$

$$(4.1b) \quad 0 < \underline{\delta} \leq \delta_t \leq \bar{\delta} < \infty, \text{ uniformly in } (t, \omega), \text{ for constants } \underline{\delta}, \bar{\delta}.$$

The first condition in (4.1a) says that U is in D_{1,T_0}^{exp} ; we remark that the indicator function in the second term fixes an obvious oversight in (4.4) of Bordigoni (2007). The condition (4.1b) is natural for an infinite-horizon problem. Note that we do not assume here that \mathbb{F} is a continuous filtration.

Definition 4.2. \mathcal{Z}_f denotes the set of all martingales $Z \in \mathcal{Z}$ with $E_P[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)] < \infty$.

Our stochastic control problem is slightly more general than the one studied in Chapter 4 of Bordigoni (2007) since we do not insist on working on the canonical (Skorohod) path space. The presentation here is linked to Bordigoni (2007) and to the slightly different formulation in Section 1 as follows. For any Q with $Z^Q \in \mathcal{Z}_f$, we have under (4.1a) for any $t \geq 0$

$$E_P[\tilde{\mathcal{R}}_{t,\infty}^\delta(Z^Q) | \mathcal{F}_t] = E_P\left[\int_t^\infty \delta_s \frac{S_s^\delta}{S_t^\delta} \frac{Z_s^Q}{Z_t^Q} \log \frac{Z_s^Q}{Z_t^Q} ds \middle| \mathcal{F}_t\right] = E_Q[\mathcal{R}_{t,\infty}^\delta(Q) | \mathcal{F}_t],$$

$$E_P[\tilde{\mathcal{U}}_{t,\infty}^\delta(Z^Q) | \mathcal{F}_t] = E_P\left[\int_t^\infty \frac{S_s^\delta}{S_t^\delta} \frac{Z_s^Q}{Z_t^Q} U_s ds \middle| \mathcal{F}_t\right] = E_Q[\mathcal{U}_{t,\infty}^\delta | \mathcal{F}_t];$$

this is proved in Lemma 4.6 and Remark 4.10 of Bordigoni (2007), essentially by using Bayes' rule. In particular, this shows that

$$\Gamma_\infty(Z^Q) := E_P[\tilde{c}_\infty(Z^Q)] = E_Q[c_\infty(Q)] =: \Gamma_\infty(Q).$$

Expressing everything under P and working with martingales $Z \in \mathcal{Z}_f$ turns out to be a bit more flexible than working with probability measures $Q \stackrel{\text{loc}}{\ll} P$. From now on, all expectations without subscript are under P .

Remark 4.3. Under (4.1b), $\int_t^\infty \delta_s S_s^\delta ds = S_t^\delta$ for every $t \geq 0$ and hence also

$$E\left[\int_t^\infty \delta_s \frac{S_s^\delta}{S_t^\delta} \frac{Z_s}{Z_t} ds \middle| \mathcal{F}_t\right] = 1 \quad \text{for every } t \geq 0 \text{ and any } Z \in \mathcal{Z}. \quad \diamond$$

We start with some auxiliary estimates. These are true for any $Z \in \mathcal{Z}$, with the understanding that we set $E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)] = +\infty$ for $Z \in \mathcal{Z} \setminus \mathcal{Z}_f$.

Lemma 4.4. For every $Z \in \mathcal{Z}$ and every $T \in (0, \infty)$,

$$E[Z_T \log Z_T] \leq \frac{1}{\underline{\delta}} e^{\bar{\delta}(T+1)} (E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)] + e^{-1}).$$

Proof. Since $t \mapsto E[Z_t \log Z_t]$ is increasing and $z \log z \geq -e^{-1}$, Fubini and (4.1b) give

$$\begin{aligned} E[Z_T \log Z_T] &\leq E\left[\int_T^{T+1} \frac{\delta_s}{\underline{\delta}} S_s^\delta e^{\bar{\delta}s} (Z_s \log Z_s + e^{-1}) ds\right] \\ &\leq \frac{1}{\underline{\delta}} e^{\bar{\delta}(T+1)} E\left[\int_0^\infty \delta_s S_s^\delta (Z_s \log Z_s + e^{-1}) ds\right] \\ &= \frac{1}{\underline{\delta}} e^{\bar{\delta}(T+1)} (E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)] + e^{-1}) \end{aligned}$$

by Remark 4.3.

q.e.d.

Proposition 4.5. *There is a constant $C < \infty$ such that for every $Z \in \mathcal{Z}$,*

$$(4.2) \quad E \left[\int_0^\infty S_s^\delta Z_s |U_s| ds \right] \leq C(1 + E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)]) \leq C(1 + \Gamma_\infty(Z)).$$

Proof. This is a simplified version of the proofs for Lemma 4.9 and Proposition 4.11 in Bordigoni (2007). We use several times the elementary inequality $xy \leq e^x + y \log y$ for $x \in \mathbb{R}, y \geq 0$, typically applied to $xy = \gamma x \frac{1}{\gamma} y$. Starting with $\tilde{c}_\infty(Z)$, we split $\tilde{\mathcal{U}}_{0,\infty}^\delta(Z)$ into an integral from 0 to T_0 and another from T_0 to ∞ to write first

$$(4.3) \quad E \left[\int_0^{T_0} S_s^\delta Z_s |U_s| ds \right] \leq E \left[Z_{T_0} \int_0^{T_0} |U_s| ds \right] \\ \leq E \left[\exp \left(\gamma \int_0^{T_0} |U_s| ds \right) \right] + \frac{1}{\gamma} E [Z_{T_0} (\log Z_{T_0} + |\log \gamma|)].$$

Next, we have due to $S^\delta \leq 1$ and Remark 4.3 that

$$(4.4) \quad E \left[\int_{T_0}^\infty S_s^\delta Z_s |U_s| ds \right] \leq E \left[\int_{T_0}^\infty e^{\gamma |U_s|} \mathbf{1}_{\{U_s \neq 0\}} ds \right] + E \left[\int_{T_0}^\infty \frac{1}{\gamma} S_s^\delta Z_s \log \left(\frac{1}{\gamma} S_s^\delta Z_s \right) \mathbf{1}_{\{U_s \neq 0\}} ds \right] \\ \leq E \left[\int_{T_0}^\infty e^{\gamma |U_s|} \mathbf{1}_{\{U_s \neq 0\}} ds \right] \\ + \frac{1}{\gamma} E \left[\int_0^\infty \frac{\delta_s}{\underline{\delta}} S_s^\delta (Z_s \log Z_s + e^{-1}) ds \right] + \frac{1}{\gamma} |\log \gamma| E \left[\int_0^\infty \frac{\delta_s}{\underline{\delta}} S_s^\delta Z_s ds \right] \\ = E \left[\int_{T_0}^\infty e^{\gamma |U_s|} \mathbf{1}_{\{U_s \neq 0\}} ds \right] + \frac{1}{\gamma \underline{\delta}} (E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)] + e^{-1} + |\log \gamma|).$$

Combining (4.3) with Lemma 4.4 and (4.4) gives the left inequality in (4.2) in the form

$$(4.5) \quad E \left[\int_0^\infty S_s^\delta Z_s |U_s| ds \right] \leq C + E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)] \frac{1}{\gamma \underline{\delta}} (e^{\bar{\delta}(T_0+1)} + 1),$$

where the constant C depends on γ and also on U via (4.1a). By definition, then,

$$\Gamma_\infty(Z) = E[\tilde{\mathcal{U}}_{0,\infty}^\delta(Z)] + \beta E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)] \\ \geq -\alpha E \left[\int_0^\infty S_s^\delta Z_s |U_s| ds \right] + \beta E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)] \\ \geq -\alpha C + E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)] \left(\beta - \frac{1}{\gamma \underline{\delta}} (1 + e^{\bar{\delta}(T_0+1)}) \right),$$

and so the right inequality in (4.2) follows by taking γ large enough and choosing a new constant appropriately. **q.e.d.**

The above argument also shows that $\Gamma_\infty(Z) \leq C(1 + E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)])$ for all $Z \in \mathcal{Z}$, with a suitable constant. Another direct consequence is the following result that will be used later.

Corollary 4.6. *For every $T \geq T_0$, every $Z \in \mathcal{Z}$ and every $\gamma > 0$,*

$$E\left[\int_T^\infty S_s^\delta Z_s U_s ds\right] \leq E\left[\int_T^\infty e^{\gamma|U_s|} \mathbf{1}_{\{U_s \neq 0\}} ds\right] + \frac{1}{\underline{\gamma}\delta} (E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)] + e^{-1} + |\log \gamma|).$$

With these preparations, we are ready to prove existence and uniqueness of a solution to our stochastic control problem.

Theorem 4.7. *Under Hypothesis 4.1, there exists a unique $Z^* \in \mathcal{Z}_f$ that minimises the cost functional $Z \mapsto \Gamma_\infty(Z)$ over all $Z \in \mathcal{Z}_f$.*

Proof. This again follows closely the arguments in Bordigoni (2007); see there the proof of Theorem 4.15. Since we optimise over Z instead of Q , we need not work on path space and can simplify some arguments.

First of all, uniqueness is clear because $Z \mapsto \Gamma_\infty(Z)$ is strictly convex like $z \mapsto z \log z$. Existence is proved in several steps.

1) Since $Z \equiv 1$ is in \mathcal{Z}_f and $E[\tilde{\mathcal{R}}_{0,\infty}^\delta(1)] = 0$, (4.1a) and Proposition 4.5 imply that $-\infty < \inf_{Z \in \mathcal{Z}_f} \Gamma_\infty(Z) < \infty$. So we can take a sequence $(Z^n)_{n \in \mathbb{N}}$ in \mathcal{Z}_f such that $\Gamma_\infty(Z^n)$ decreases to $\inf_{Z \in \mathcal{Z}_f} \Gamma_\infty(Z)$ as $n \rightarrow \infty$. Combining the well-known Komlós-type result in Lemma A1.1 in Delbaen/Schachermayer (1994) with a diagonalisation argument produces a sequence $(\bar{Z}^n)_{n \in \mathbb{N}}$ with $\bar{Z}^n \in \text{conv}(Z^n, Z^{n+1}, \dots)$ for all n and such that with probability 1,

$$\lim_{n \rightarrow \infty} \bar{Z}_r^n =: \bar{Z}_r^\infty \quad \text{exists in } [0, \infty] \text{ for all } r \in \mathcal{Q}^+.$$

Since each \bar{Z}^n is like the Z^n a martingale ≥ 0 with expectation 1, Fatou's lemma yields that each \bar{Z}_r^∞ is integrable and $(\bar{Z}_r^\infty)_{r \in \mathcal{Q}^+}$ is a supermartingale. By a standard argument (see Dellacherie/Meyer (1982), Theorem VI.2), we can therefore extend $(\bar{Z}_r^\infty)_{r \in \mathcal{Q}^+}$ to a process $Z^* = (Z_t^*)_{t \geq 0}$ with RCLL trajectories and such that Z^* is a supermartingale ≥ 0 (now over $[0, \infty)$ instead of \mathcal{Q}^+). In fact, we can take $Z_t^* := \lim_{r \searrow t, r \in \mathcal{Q}^+} \bar{Z}_r^\infty$.

2) In order to show that Z^* is even a martingale and in \mathcal{Z}_f , we first use Lemma 4.4 to obtain for each $r \in \mathcal{Q}^+$ that

$$\sup_{n \in \mathbb{N}} E[\bar{Z}_r^n \log \bar{Z}_r^n] \leq \frac{1}{\underline{\delta}} e^{\delta(r+1)} \left(\sup_{n \in \mathbb{N}} E[\tilde{\mathcal{R}}_{0,\infty}^\delta(\bar{Z}^n)] + e^{-1} \right) < \infty,$$

because by Proposition 4.5 and convexity of $Z \mapsto \Gamma_\infty(Z)$,

$$E[\tilde{\mathcal{R}}_{0,\infty}^\delta(\bar{Z}^n)] \leq C(1 + \Gamma_\infty(\bar{Z}^n)) \leq C\left(1 + \sup_{m \geq n} \Gamma_\infty(Z^m)\right) \leq C(1 + \Gamma_\infty(Z^1)) < \infty.$$

So $(\bar{Z}_r^n)_{n \in \mathbb{N}}$ is uniformly integrable for each $r \in \mathcal{Q}^+$, and this implies

$$E[\bar{Z}_r^\infty] = \lim_{n \rightarrow \infty} E[\bar{Z}_r^n] = 1 \quad \text{for all } r \in \mathcal{Q}^+,$$

which means that the supermartingale \bar{Z}^∞ is a martingale (over \mathcal{Q}^+). Using Doob's maximal inequality and the fact that $(\bar{Z}_m^n)_{n \in \mathbb{N}}$ converges to Z_m^* in L^1 for every $m \in \mathbb{N}$ next shows that with probability 1, $(\bar{Z}^n)_{n \in \mathbb{N}}$ converges to Z^* uniformly on compact subsets of $[0, \infty)$, and the same uniform integrability argument as above then yields that also Z^* is a martingale (over $[0, \infty)$), hence in \mathcal{Z} . Finally, Fatou's lemma, Remark 4.3 and Proposition 4.5 give

$$E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z^*)] = E\left[\int_0^\infty \delta_s S_s^\delta Z_s^* \log Z_s^* ds\right] \leq \liminf_{n \rightarrow \infty} E[\tilde{\mathcal{R}}_{0,\infty}^\delta(\bar{Z}^n)] \leq \sup_{n \in \mathbb{N}} C(1 + \Gamma_\infty(\bar{Z}^n)) < \infty$$

so that Z^* is in \mathcal{Z}_f .

3) To show that Z^* is optimal, we want to prove that $Z \mapsto \Gamma_\infty(Z)$ is lower semicontinuous along the sequence $(\bar{Z}^n)_{n \in \mathbb{N}}$, because we then get by convexity

$$\Gamma_\infty(Z^*) \leq \liminf_{n \rightarrow \infty} \Gamma_\infty(\bar{Z}^n) \leq \liminf_{n \rightarrow \infty} \Gamma_\infty(Z^n) = \inf_{Z \in \mathcal{Z}_f} \Gamma_\infty(Z).$$

We have just seen in step 2) that $E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z^*)] \leq \liminf_{n \rightarrow \infty} E[\tilde{\mathcal{R}}_{0,\infty}^\delta(\bar{Z}^n)]$ so that it only remains to prove the analogous inequality for the part with $\tilde{\mathcal{U}}_{0,\infty}^\delta(Z)$. Now $U \in D_{1,T_0}^{\text{exp}}$ by (4.1a), and so we can use the finite-horizon results in Bordigoni/Matoussi/Schweizer (2007) to obtain

$$\begin{aligned} E\left[\int_0^{T_0} S_s^\delta Z_s^* U_s ds\right] &= E\left[Z_{T_0}^* \int_0^{T_0} S_s^\delta U_s ds\right] \\ &\leq \liminf_{n \rightarrow \infty} E\left[\bar{Z}_{T_0}^n \int_0^{T_0} S_s^\delta U_s ds\right] \\ &= \liminf_{n \rightarrow \infty} E\left[\int_0^{T_0} S_s^\delta \bar{Z}_s^n U_s ds\right]; \end{aligned}$$

see step 4) in the proof of Theorem 9 in Bordigoni/Matoussi/Schweizer (2007). We then split the remaining integral from T_0 to ∞ into one integral from T_0 to $T \geq T_0$ and another from T to ∞ . The integral over the finite interval $(T_0, T]$ is again treated as above, using that (4.1a) also gives $U \in D_{1,T}^{\text{exp}}$ by Jensen's inequality. Finally, Corollary 4.6 gives

$$E\left[\int_T^\infty S_s^\delta \bar{Z}_s^n U_s ds\right] \leq E\left[\int_T^\infty e^{\gamma|U_s|} \mathbf{1}_{\{U_s \neq 0\}} ds\right] + \frac{1}{\gamma \underline{\delta}} \left(\sup_{n \in \mathbb{N}} E[\tilde{\mathcal{R}}_{0,\infty}^\delta(\bar{Z}^n)] + e^{-1} + |\log \gamma| \right)$$

for all $\gamma > 0$ and all $n \in \mathbb{N}$, and the same estimate holds for Z^* instead of \bar{Z}^n as well. Choosing first γ large to make the second summand above small and then, using (4.1a), T large to get the first summand small as well, we deduce that

$$\lim_{T \rightarrow \infty} \sup \left\{ E\left[\int_T^\infty S_s^\delta Z_s U_s ds\right] : Z = Z^* \text{ or } Z = \bar{Z}^n \text{ for some } n \in \mathbb{N} \right\} = 0,$$

and so we obtain after putting everything together that

$$E \left[\int_0^\infty S_s^\delta Z_s^* U_s ds \right] \leq \liminf_{n \rightarrow \infty} E \left[\int_0^\infty S_s^\delta \bar{Z}_s^n U_s ds \right].$$

This completes the proof. **q.e.d.**

As in the finite-horizon case treated in Bordigoni/Matoussi/Schweizer (2007), one can show that the optimal Z^* from Theorem 4.7 is strictly positive. If there exists Q^* with density process Z^* (e.g. as in Bordigoni (2007) if one works on path space), this translates into saying that $Q^* \stackrel{\text{loc}}{\approx} P$. The proof of positivity can be found in Bordigoni (2007), Theorem 4.18, and largely parallels that of Theorem 12 in Bordigoni/Matoussi/Schweizer (2007).

Also as in Bordigoni/Matoussi/Schweizer (2007), one can show that the *martingale optimality principle* holds in our setting; see Proposition 4.19 and Corollary 4.20 in Bordigoni (2007) for details of this standard argument. As a consequence, the optimal Z^* from Theorem 4.7 is also conditionally optimal at any time t or even stopping time τ . To properly formulate this, we denote by $V = (V_t)_{t \geq 0}$ an RCLL version of the process

$$(4.6) \quad V_t := \operatorname{ess\,inf}_{Z \in \mathcal{Z}_f} E \left[\tilde{\mathcal{U}}_{t,\infty}^\delta(Z) + \beta \tilde{\mathcal{R}}_{t,\infty}^\delta(Z) \mid \mathcal{F}_t \right] =: \operatorname{ess\,inf}_{Z \in \mathcal{Z}_f} J_t(Z), \quad t \geq 0.$$

Then conditional optimality says that

$$V_t = J_t(Z^*) \quad P\text{-a.s. for all } t \geq 0,$$

and we now use this to describe the behaviour of V_t as $t \rightarrow \infty$.

Proposition 4.8. *Under Hypothesis 4.1,*

$$\lim_{t \rightarrow \infty} V_t = 0 \quad P\text{-a.s.}$$

Proof. This is analogous to the proof of Lemma 4.22 in Bordigoni (2007). Since $Z \equiv 1$ is in \mathcal{Z}_f , S^δ is decreasing and $E[\tilde{\mathcal{R}}_{t,\infty}^\delta(Z) \mid \mathcal{F}_t] \geq 0$, we have

$$V_t = J_t(Z^*) \leq J_t(1) \leq \alpha E \left[\int_t^\infty |U_s| ds \mid \mathcal{F}_t \right]$$

which yields $\limsup_{t \rightarrow \infty} V_t \leq 0$ due to (4.1a). To get a lower bound for V_t , we use analogous arguments as in the proof of Proposition 4.5 to first obtain

$$\begin{aligned} \left| E \left[\int_t^\infty \frac{S_s^\delta}{S_t^\delta} \frac{Z_s^*}{Z_t^*} U_s ds \mid \mathcal{F}_t \right] \right| &\leq E \left[\int_t^\infty e^{\gamma |U_s|} \mathbf{1}_{\{U_s \neq 0\}} ds \mid \mathcal{F}_t \right] \\ &+ \frac{1}{\gamma \delta} (E[\tilde{\mathcal{R}}_{t,\infty}^\delta(Z^*) \mid \mathcal{F}_t] + e^{-1} + |\log \gamma|). \end{aligned}$$

Hence

$$\begin{aligned} V_t &= J_t(Z^*) \\ &\geq -\alpha E \left[\int_t^\infty e^{\gamma|U_s|} \mathbf{1}_{\{U_s \neq 0\}} ds \mid \mathcal{F}_t \right] - \frac{\alpha}{\gamma \underline{\delta}} (e^{-1} + |\log \gamma|) + \left(\beta - \frac{\alpha}{\gamma \underline{\delta}} \right) E[\tilde{\mathcal{R}}_{t,\infty}^\delta(Z^*) \mid \mathcal{F}_t], \end{aligned}$$

and taking γ so large that $\beta - \frac{\alpha}{\gamma \underline{\delta}} \geq 0$, we get from $E[\tilde{\mathcal{R}}_{t,\infty}^\delta(Z^*) \mid \mathcal{F}_t] \geq 0$ and (4.1a) that

$$\liminf_{t \rightarrow \infty} V_t \geq -\frac{\alpha}{\gamma \underline{\delta}} (e^{-1} + |\log \gamma|) \quad P\text{-a.s.}$$

Since γ is arbitrary, we conclude that $\liminf_{t \rightarrow \infty} V_t \geq 0$ P -a.s., which completes the proof. **q.e.d.**

All results in this section so far hold for a general filtration. If \mathcal{F} is continuous, one can in addition show as in Bordigoni/Matoussi/Schweizer (2007) that V obeys the dynamics

$$dV_t = (\delta_t V_t - \alpha U_t) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t$$

for some (continuous) local martingale M ; see Theorem 4.27 in Bordigoni (2007) for a detailed proof. Together with Proposition 4.8, this explains where the infinite-horizon BSDE (1.2) comes from. Since the above derivation uses no essential new ideas in comparison with Bordigoni/Matoussi/Schweizer (2007), we refrain from giving more details.

5. Solving the stochastic control problem via the BSDE

Our goal in this section is to use the results on the infinite-horizon BSDE (3.2) for a characterisation of the dynamic value process V for the stochastic control problem from Section 4. As just mentioned, we could have shown that V solves (3.2), but this is not enough: The uniqueness result in Theorem 3.5 only holds for solutions (Y, M) with $Y \in D_{0,\infty}^{\text{exp}}$, and we do not know at this point how to argue directly that V from the control problem is in $D_{0,\infty}^{\text{exp}}$. The BSDE techniques developed so far will enable us to prove this. To that end, we now show how one can construct from a (particular) solution to the BSDE (3.2) a (and actually the, by uniqueness) solution for the infinite-horizon stochastic control problem.

Hypothesis 5.1. Throughout this section, we impose the standing assumptions

(5.1a) \mathcal{F} is a *continuous filtration*, i.e. all local (P, \mathcal{F}) -martingales are continuous.

(5.1b) $0 < \underline{\delta} \leq \delta_t \leq \bar{\delta} < \infty$, uniformly in (t, ω) , for constants $\underline{\delta}, \bar{\delta}$.

Conditions on U will be specified below, when we successively treat three cases.

Our arguments rely substantially on the finite-horizon results proved in Bordigoni/Matoussi/Schweizer (2007) so that we very briefly recall this here. Fix $T < \infty$ and consider on $[0, T]$ the BSDE

$$(2.2) \quad dY_t = (\delta_t Y_t - \alpha U_t) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \quad Y_T = \alpha' U'_T.$$

Recall from (1.3) and (1.4) the definitions of $\mathcal{U}_{t,T}^\delta$ and $\mathcal{R}_{t,T}^\delta$, and assume that U (as a process on $[0, T]$) is in $D_{1,T}^{\text{exp}}$ and U'_T is in D^{exp} . Then Theorem 17 of Bordigoni/Matoussi/Schweizer (2007) states that (2.2) has a unique solution (Y, M) in $D_{0,T}^{\text{exp}} \times \mathcal{M}_{0,\text{loc}}(P)$, that $\bar{Z} := \mathcal{E}(-\frac{1}{\beta}M)$ is a martingale on $[0, T]$ with $E[\bar{Z}_T \log \bar{Z}_T] < \infty$, and that for any martingale $Z \geq 0$ on $[0, T]$ with $Z_0 = 1$ and $E[Z_T \log Z_T] < \infty$, we have for any stopping time $\tau \leq T$ that

$$Y_\tau = E \left[\frac{\bar{Z}_T}{\bar{Z}_\tau} \mathcal{U}_{\tau,T}^\delta + \beta \tilde{\mathcal{R}}_{\tau,T}^\delta(\bar{Z}) \mid \mathcal{F}_\tau \right] \leq E \left[\frac{Z_T}{Z_\tau} \mathcal{U}_{\tau,T}^\delta + \beta \tilde{\mathcal{R}}_{\tau,T}^\delta(Z) \mid \mathcal{F}_\tau \right].$$

(This reformulates the fact that the dynamic value process of the finite-horizon stochastic control problem is the unique solution of (2.2).) For $\tau \equiv 0$, this reduces to

$$(5.2) \quad Y_0 \leq E \left[Z_T \alpha \int_0^T S_s^\delta U_s ds + Z_T \alpha' S_T^\delta U'_T \right] + \beta E \left[\int_0^T \delta_s S_s^\delta Z_s \log Z_s ds + S_T^\delta Z_T \log Z_T \right],$$

with equality for $Z = \bar{Z}$.

5.1. The bounded case

Let us now first study the case where $\int_0^\infty |U_s| ds$ is bounded. This is of course a restrictive assumption, but it allows fairly simple arguments and provides a basic building block. We shall see that the proofs in more general cases follow the same scheme.

Proposition 5.2. *Suppose that $\int_0^\infty |U_s| ds$ is a bounded random variable. For any solution (Y, M) to the infinite-horizon BSDE (3.2) with Y bounded, we then have for any $t \geq 0$*

$$Y_t = \text{ess inf}_{Z \in \mathcal{Z}_f} E[\tilde{\mathcal{U}}_{t,\infty}^\delta(Z) + \beta \tilde{\mathcal{R}}_{t,\infty}^\delta(Z) \mid \mathcal{F}_t].$$

Proof. Without loss of generality, we prove the result for $t = 0$.

1) Let us start by arguing that for any $Z \in \mathcal{Z}_f$, we have

$$(5.3) \quad Y_0 \leq E[\tilde{\mathcal{U}}_{0,\infty}^\delta(Z) + \beta \tilde{\mathcal{R}}_{0,\infty}^\delta(Z)].$$

We first note that by Remark 4.3, the nonnegative function $g(s) := E[\delta_s S_s^\delta (Z_s \log Z_s + e^{-1})]$ satisfies by Fubini $\int_0^\infty g(s) ds = E\left[\int_0^\infty \delta_s S_s^\delta (Z_s \log Z_s + e^{-1}) ds\right] = E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)] + e^{-1} < \infty$. This implies that there exists a sequence of deterministic times $T_n \nearrow \infty$ such that

$$\lim_{n \rightarrow \infty} E[S_{T_n}^\delta (Z_{T_n} \log Z_{T_n} + e^{-1})] \leq \lim_{n \rightarrow \infty} \frac{1}{\underline{\delta}} g(T_n) = 0$$

and therefore also

$$(5.4) \quad \lim_{n \rightarrow \infty} E[S_{T_n}^\delta Z_{T_n} \log Z_{T_n}] = 0,$$

since $S_{T_n}^\delta \leq e^{-\delta T_n} \rightarrow 0$. Moreover, because Y is bounded and Z is a martingale, we have

$$(5.5) \quad \lim_{n \rightarrow \infty} |E[S_{T_n}^\delta Z_{T_n} Y_{T_n}]| \leq \lim_{n \rightarrow \infty} e^{-\delta T_n} \|Y\|_\infty E[Z_{T_n}] = 0.$$

Now Y is bounded, hence in D_{0,T_n}^{exp} , and it satisfies the finite-horizon BSDE (2.2) with final value $\alpha' U'_{T_n} := Y_{T_n}$. Moreover, $Z \in \mathcal{Z}_f$ has $E[Z_{T_n} \log Z_{T_n}] < \infty$ due to Lemma 4.4, and so the finite-horizon results tell us that

$$(5.6) \quad Y_0 \leq E\left[\alpha \int_0^{T_n} S_s^\delta Z_s U_s ds + S_{T_n}^\delta Z_{T_n} Y_{T_n}\right] + \beta E\left[\int_0^{T_n} \delta_s S_s^\delta Z_s \log Z_s ds + S_{T_n}^\delta Z_{T_n} \log Z_{T_n}\right].$$

On the right-hand side, the second and the fourth summand tend to 0 as $n \rightarrow \infty$ by (5.5) and (5.4), respectively. Next, Fatou's lemma yields

$$E\left[\int_0^\infty S_s^\delta Z_s |U_s| ds\right] \leq \liminf_{n \rightarrow \infty} E\left[Z_{T_n} \int_0^{T_n} S_s^\delta |U_s| ds\right] \leq \liminf_{n \rightarrow \infty} E\left[Z_{T_n} \int_0^\infty |U_s| ds\right] < \infty$$

since $\int_0^\infty |U_s| ds$ is bounded and Z is a martingale. From dominated convergence, we thus

deduce that the first summand in (5.6) converges to $E\left[\alpha \int_0^\infty S_s^\delta Z_s U_s ds\right] = E[\tilde{\mathcal{U}}_{0,\infty}^\delta(Z)]$ as $n \rightarrow \infty$. Finally,

$$\int_0^{T_n} \delta_s S_s^\delta Z_s \log Z_s ds = \int_0^{T_n} \delta_s S_s^\delta (Z_s \log Z_s + e^{-1}) ds - e^{-1}(1 - S_{T_n}^\delta)$$

implies via monotone integration and by using $\int_0^\infty \delta_s S_s^\delta ds = 1$ that the third summand in

(5.6) converges to $\beta E\left[\int_0^\infty \delta_s S_s^\delta Z_s \log Z_s ds\right] = \beta E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)]$ as $n \rightarrow \infty$. Putting everything together gives (5.3).

2) Now define $\bar{Z} := \mathcal{E}\left(-\frac{1}{\beta}M\right)$. Since $\int_0^\infty |U_s| ds$ is bounded, U is in $D_{1,\infty}^{\text{exp}}$; and since (Y, M) solves (3.2), Theorem 3.6 tells us that \bar{Z} is a martingale on $[0, \infty)$ so that $\bar{Z} \in \mathcal{Z}$. We want to prove that \bar{Z} is even in \mathcal{Z}_f . To that end, we apply the finite-horizon results to write

$$Y_0 = E\left[\bar{Z}_{T_n} \alpha \int_0^{T_n} S_s^\delta U_s ds + S_{T_n}^\delta \bar{Z}_{T_n} Y_{T_n}\right] + \beta E\left[\int_0^{T_n} \delta_s S_s^\delta \bar{Z}_s \log \bar{Z}_s ds + S_{T_n}^\delta \bar{Z}_{T_n} \log \bar{Z}_{T_n}\right].$$

Using Fatou's lemma and $z \log z \geq -e^{-1}$ therefore gives

$$\begin{aligned} \beta E[\tilde{\mathcal{R}}_{0,\infty}^\delta(\bar{Z})] &= \beta E\left[\int_0^\infty \delta_s S_s^\delta \bar{Z}_s \log \bar{Z}_s ds\right] \\ &\leq \liminf_{n \rightarrow \infty} \beta E\left[\int_0^{T_n} \delta_s S_s^\delta \bar{Z}_s \log \bar{Z}_s ds\right] \\ &= \liminf_{n \rightarrow \infty} \left(Y_0 - \beta E[S_{T_n}^\delta \bar{Z}_{T_n} \log \bar{Z}_{T_n}] - E\left[\bar{Z}_{T_n} \alpha \int_0^{T_n} S_s^\delta U_s ds + S_{T_n}^\delta \bar{Z}_{T_n} Y_{T_n}\right]\right) \\ &\leq \liminf_{n \rightarrow \infty} \left(Y_0 + \beta e^{-1} + \alpha \left\| \int_0^\infty |U_s| ds \right\|_{L^\infty} E[\bar{Z}_{T_n}] + \|Y_\infty^*\|_{L^\infty} E[\bar{Z}_{T_n}]\right) < \infty \end{aligned}$$

because Y and $\int_0^\infty |U_s| ds$ are both bounded and \bar{Z} is a martingale. Hence \bar{Z} is indeed in \mathcal{Z}_f .

3) Since \bar{Z} is in \mathcal{Z}_f by step 2) and satisfies (5.2) with equality, the same argument as in step 1) shows that the inequality in (5.3) becomes an equality for $Z = \bar{Z}$. Hence \bar{Z} attains the infimum, and the proof is complete. **q.e.d.**

5.2. The positive case

We now turn to the case where U is nonnegative and satisfies some integrability condition. Note that $U \geq 0$ is a fairly natural assumption. Indeed, if we think of a full-fledged robust control problem for utility maximisation, then U_t represents the utility $\mathbf{U}(c_t)$ from consumption at time t , where we still optimise over c in a second step. As a consumption rate, $c_t \geq 0$; so $U_t = \mathbf{U}(c_t) \geq 0$ for any nonnegative utility function \mathbf{U} on $[0, \infty)$, like e.g. power utility $\mathbf{U}(x) = \frac{1}{\gamma} x^\gamma$ for $\gamma \in (0, 1)$.

Theorem 5.3. *Suppose that $U \geq 0$ and U is in $D_{1,\infty}^{\text{exp}}$. For the solution (Y, M) to (3.2) from Theorem 3.4, we then have for any $t \geq 0$*

$$Y_t = \operatorname{ess\,inf}_{Z \in \mathcal{Z}_f} E[\tilde{\mathcal{U}}_{t,\infty}^\delta(Z) + \beta \tilde{\mathcal{R}}_{t,\infty}^\delta(Z) \mid \mathcal{F}_t].$$

Proof. Without loss of generality, we again argue for $t = 0$. The overall structure of the proof is like for Proposition 5.2, but we first need to recall the construction of (Y, M) . For each

$n \in \mathbb{N}$, set $U_t^n := U_t \mathbf{1}_{\{\int_0^t U_s ds \leq n\}}$ and denote by (Y^n, M^n) with Y^n bounded the solution to the BSDE

$$dY_t = (\delta_t Y_t - \alpha U_t^n) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \quad Y_n = 0$$

on $[0, n]$. Extending (Y^n, M^n) to $[0, \infty)$ by setting $Y_t^n = 0$, $M_t^n = M_n^n$ for $t > n$, we then get on $[0, \infty)$ a solution (Y^n, M^n) to the BSDE

$$(5.7) \quad dY_t = (\delta_t Y_t - \alpha \mathbf{1}_{\{t \leq n\}} U_t^n) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \quad \lim_{t \rightarrow \infty} Y_t = 0.$$

We first prove that for any $Z \in \mathcal{Z}_f$,

$$(5.8) \quad Y_0 \leq E[\tilde{\mathcal{U}}_{0,\infty}^\delta(Z) + \beta \tilde{\mathcal{R}}_{0,\infty}^\delta(Z)].$$

Indeed, applying Proposition 5.2 to the process $(\mathbf{1}_{\{t \leq n\}} U_t^n)_{t \geq 0}$ and the solution to (5.7) gives

$$Y_0^n \leq E\left[\alpha \int_0^\infty S_s^\delta Z_s \mathbf{1}_{\{s \leq n\}} U_s^n ds + \beta \int_0^\infty \delta_s S_s^\delta Z_s \log Z_s ds\right],$$

and (5.8) follows by monotone integration since $U \geq 0$.

Now set $\bar{Z} := \mathcal{E}(-\frac{1}{\beta}M)$ so that $\bar{Z} \in \mathcal{Z}$ by Theorem 3.6; this uses the integrability assumption on U . To prove that \bar{Z} is even in \mathcal{Z}_f , we first note that by Proposition 5.2 and its proof, we have equality in (5.8) for $Z = \bar{Z}^n := \mathcal{E}(-\frac{1}{\beta}M^n)$ so that

$$Y_0^n = E\left[\alpha \int_0^\infty S_s^\delta \bar{Z}_s^n \mathbf{1}_{\{s \leq n\}} U_s^n ds + \beta \int_0^\infty \delta_s S_s^\delta \bar{Z}_s^n \log \bar{Z}_s^n ds\right].$$

But by construction, $(Y_0^n)_{n \in \mathbb{N}}$ increases to Y_0 , and $(M^n)_{n \in \mathbb{N}}$ and $(\langle M^n \rangle)_{n \in \mathbb{N}}$ converge to M and $\langle M \rangle$ locally uniformly in probability so that also $\bar{Z}^n \rightarrow \bar{Z}$ locally uniformly in probability as $n \rightarrow \infty$. Hence Fatou's lemma yields

$$Y_0 \geq E\left[\alpha \int_0^\infty S_s^\delta \bar{Z}_s U_s ds + \beta \int_0^\infty \delta_s S_s^\delta \bar{Z}_s \log \bar{Z}_s ds\right] = E[\tilde{\mathcal{U}}_{0,\infty}^\delta(\bar{Z}) + \beta \tilde{\mathcal{R}}_{0,\infty}^\delta(\bar{Z})],$$

and so $\bar{Z} \in \mathcal{Z}_f$ since $U \geq 0$. Since (5.8) gives the converse inequality, we actually have equality in (5.8) for $Z = \bar{Z}$, and this completes the proof. **q.e.d.**

5.3. The general case

Finally, we study a situation where U can be real-valued. Then we need slightly stronger integrability assumptions.

Theorem 5.4. Suppose that U is in $D_{1,\infty}^{\text{exp}}$ and that U also satisfies (4.1a), i.e. there exists some $T_0 \in (0, \infty)$ such that for all $\gamma > 0$,

$$E_P \left[\exp \left(\gamma \int_0^{T_0} |U_s| ds \right) \right] + E_P \left[\int_{T_0}^{\infty} \exp(\gamma |U_s|) \mathbf{1}_{\{U_s \neq 0\}} ds \right] < \infty.$$

For the solution (Y, M) to (3.2) from Theorem 3.4, we then have for any $t \geq 0$

$$Y_t = \text{ess inf}_{Z \in \mathcal{Z}_f} E[\tilde{\mathcal{U}}_{t,\infty}^\delta(Z) + \beta \tilde{\mathcal{R}}_{t,\infty}^\delta(Z) \mid \mathcal{F}_t].$$

Proof. As already in the last proof, we argue for $t = 0$ without loss of generality and again first recall from the proof of Theorem 3.4 the construction of (Y, M) . For $n, p \in \mathbb{N}$, set $U_t^{n,p} := U_t^+ \mathbf{1}_{\{\int_0^t |U_s| ds \leq n\}} \mathbf{1}_{\{t \leq n\}} - U_t^- \mathbf{1}_{\{\int_0^t |U_s| ds \leq p\}} \mathbf{1}_{\{t \leq p\}}$ and denote by $(Y^{n,p}, M^{n,p})$ with $Y^{n,p}$ bounded the solution to the BSDE

$$dY_t = (\delta_t Y_t - \alpha U_t) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \quad Y_{n \vee p} = 0$$

on $[0, n \vee p]$. We extend $(Y^{n,p}, M^{n,p})$ to $[0, \infty)$ by setting $Y_t^{n,p} = 0$, $M_t^{n,p} = M_{n \vee p}^{n,p}$ for $t > n \vee p$ to get on $[0, \infty)$ a solution to the BSDE

$$(5.9) \quad dY_t = (\delta_t Y_t - \alpha \mathbf{1}_{\{t \leq n \vee p\}} U_t^{n,p}) dt + \frac{1}{2\beta} d\langle M \rangle_t + dM_t, \quad \lim_{t \rightarrow \infty} Y_t = 0.$$

Then $Y_t = \nearrow - \lim_{n \rightarrow \infty} \searrow - \lim_{p \rightarrow \infty} Y_t^{n,p}$ and $M_t = \lim_{n \rightarrow \infty} \lim_{p \rightarrow \infty} M_t^{n,p}$. We proceed on a familiar path.

1) First we prove that for any $Z \in \mathcal{Z}_f$,

$$(5.10) \quad Y_0 \leq E[\tilde{\mathcal{U}}_{0,\infty}^\delta(Z) + \beta \tilde{\mathcal{R}}_{0,\infty}^\delta(Z)].$$

Like in Section 5.2, using Proposition 5.2 gives

$$Y_0^{n,p} \leq E \left[\alpha \int_0^\infty S_s^\delta Z_s \mathbf{1}_{\{s \leq n \vee p\}} U_s^{n,p} ds + \beta \int_0^\infty \delta_s S_s^\delta Z_s \log Z_s ds \right],$$

and (5.10) follows by letting $p \rightarrow \infty$ and then $n \rightarrow \infty$, provided we can use dominated convergence. But this is ensured by the first estimate in Proposition 4.5; indeed, (4.1a) yields

$$E \left[\int_0^\infty S_s^\delta Z_s |U_s| ds \right] < \infty \text{ for } Z \in \mathcal{Z}_f.$$

2) Thanks to the first integrability assumption on U , the process $\bar{Z} := \mathcal{E}(-\frac{1}{\beta} M)$ is in \mathcal{Z} by Theorem 3.6. To show it is in \mathcal{Z}_f , we use Proposition 5.2 for $\bar{Z}^{n,p} := \mathcal{E}(-\frac{1}{\beta} M^{n,p})$ to get

$$(5.11) \quad Y_0^{n,p} = E \left[\alpha \int_0^\infty S_s^\delta \bar{Z}_s^{n,p} \mathbf{1}_{\{s \leq n \vee p\}} U_s^{n,p} ds + \beta \int_0^\infty \delta_s S_s^\delta \bar{Z}_s^{n,p} \log \bar{Z}_s^{n,p} ds \right].$$

But (4.5) in the proof of Proposition 4.5 gives

$$\begin{aligned} \left| E \left[\alpha \int_0^\infty S_s^\delta \bar{Z}_s^{n,p} \mathbf{1}_{\{s \leq n \vee p\}} U_s^{n,p} ds \right] \right| &\leq E \left[\alpha \int_0^\infty S_s^\delta \bar{Z}_s^{n,p} |U_s| ds \right] \\ &\leq C_{\gamma,U} + E[\tilde{\mathcal{R}}_{0,\infty}^\delta(\bar{Z}^{n,p})] \frac{1}{\gamma \delta} (e^{\delta(T_0+1)} + 1), \end{aligned}$$

where the constant $C_{\gamma,U}$ depends on γ and U via (4.1a), but not on n and p . Plugging this estimate with a minus sign into (5.11) and taking γ big enough yields

$$(5.12) \quad \sup_{n,p \in \mathbb{N}} E[\tilde{\mathcal{R}}_{0,\infty}^\delta(\bar{Z}^{n,p})] \leq C \left(1 + \sup_{n,p \in \mathbb{N}} Y_0^{n,p} \right) < \infty$$

because applying the a priori estimate (3.6) from Theorem 3.4 to (5.9) tells us that

$$|Y_0^{n,p}| \leq \beta \log E \left[\exp \left(\frac{1}{\beta} \int_0^\infty \alpha |U_s| ds \right) \right] < \infty$$

for all n and p , by using the definition of $U^{n,p}$. As $n \rightarrow \infty$ and $p \rightarrow \infty$, we have locally uniformly in probability $M^{n,p} \rightarrow M$ and $\langle M^{n,p} \rangle \rightarrow \langle M \rangle$, hence also $\bar{Z}^{n,p} \rightarrow \bar{Z}$, and so $\bar{Z} \in \mathcal{Z}_f$ because $E[\tilde{\mathcal{R}}_{0,\infty}^\delta(\bar{Z})] < \infty$ by Fatou's lemma and (5.12).

3) To prove that we have equality in (5.10) for $Z = \bar{Z}$, we start from the equality in (5.11). As $n \rightarrow \infty$ and $p \rightarrow \infty$, $Y_0^{n,p}$ tends to Y_0 and $\liminf_{n \rightarrow \infty} \liminf_{p \rightarrow \infty} E[\tilde{\mathcal{R}}_{0,\infty}^\delta(\bar{Z}^{n,p})] \geq E[\tilde{\mathcal{R}}_{0,\infty}^\delta(\bar{Z})]$ by Fatou's lemma. Because $Z \mapsto E[\tilde{\mathcal{R}}_{0,\infty}^\delta(Z)]$ is bounded along the sequence $(\bar{Z}^{n,p})_{n,p \in \mathbb{N}}$ by (5.12), almost the same argument as in step 3) of the proof of Theorem 4.7 gives

$$\liminf_{n \rightarrow \infty} \liminf_{p \rightarrow \infty} E \left[\alpha \int_0^\infty S_s^\delta \bar{Z}_s^{n,p} \mathbf{1}_{\{s \leq n \vee p\}} U_s^{n,p} ds \right] \geq E \left[\alpha \int_0^\infty S_s^\delta \bar{Z}_s U_s ds \right] = E[\tilde{\mathcal{U}}_{0,\infty}^\delta(\bar{Z})].$$

Note that this exploits the integrability assumption (4.1a). Therefore (5.11) implies that $Y_0 \geq E[\tilde{\mathcal{U}}_{0,\infty}^\delta(\bar{Z}) + \beta \tilde{\mathcal{R}}_{0,\infty}^\delta(\bar{Z})]$, and so we must have equality due to (5.10) since $\bar{Z} \in \mathcal{Z}_f$. This completes the proof. **q.e.d.**

5.4. Consequences for the stochastic control problem

As in (4.6), denote by $V = (V_t)_{t \geq 0}$ the dynamic value process of the infinite-horizon stochastic control problem. We already mentioned at the end of Section 4 that if \mathbb{F} is continuous, then V satisfies the infinite-horizon BSDE (3.2). For the proof, we referred to Bordigoni (2007); let us just note here that the required assumptions are Hypothesis 4.1 plus continuity of \mathbb{F} , i.e. Hypothesis 5.1 plus (4.1a). Under a slightly stronger condition, we can now even prove a BSDE characterisation for V .

Theorem 5.5. *Assume Hypothesis 5.1 and that U is in $D_{1,\infty}^{\text{exp}}$. If in addition either $U \geq 0$ or U satisfies (4.1a), then V is the first component of the unique solution in $D_{0,\infty}^{\text{exp}} \times \mathcal{M}_{0,\text{loc}}(P)$ to the infinite-horizon BSDE (3.2). In particular, $V \in D_{0,\infty}^{\text{exp}}$.*

Proof. By Theorem 3.5, (3.2) has a unique solution (Y, M) with $Y \in D_{0,\infty}^{\text{exp}}$; and by the definition of V in (4.6) and either Theorem 5.3 or Theorem 5.4, Y coincides with V . **q.e.d.**

Remark 5.6. 1) The second case of Theorem 5.5 is the infinite-horizon analogue to the finite-horizon Theorem 17 in Bordigoni/Matoussi/Schweizer (2007), with assumptions and conclusions almost exactly parallel. The only difference lies in the conditions on U : In (4.1a), we need $U \in D_{1,T_0}^{\text{exp}}$, but also an exponential moment control over U on the infinite time interval $[T_0, \infty)$. See Remark 4.28 in Bordigoni (2007) for a more detailed comment on this point. The result for $U \geq 0$ has no precedent.

2) Our approach for $T = \infty$ here is different from Bordigoni (2007) in that we show for the solution of the BSDE (3.2) that it satisfies the defining property (4.6) of the value process V . As a bonus, we are able to deduce that V is indeed in $D_{0,\infty}^{\text{exp}}$; this was conjectured, but not proved in Bordigoni (2007).

3) Again in remarkable analogy to the finite-horizon results in Bordigoni/Matoussi/Schweizer (2007), we obtain existence of a solution to the stochastic control problem for a general filtration \mathcal{F} . But the integrability property $V \in D_{0,\infty}^{\text{exp}}$ is only known for continuous \mathcal{F} , since its proof exploits the BSDE results. Like in Bordigoni/Matoussi/Schweizer (2007), we do not know if $V \in D_{0,\infty}^{\text{exp}}$ also holds for general \mathcal{F} . \diamond

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